Combinatorics of truncated partition theorems

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Integer partitions

- $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$: partition of n if

$$n = \lambda_1 + \lambda_2 + \lambda_3 + \cdots \quad \text{and} \quad \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots .$$

Example. $n = 4$: $(4), (3,1), (2,2), (2,1,1), (1,1,1,1)$.

$$\begin{aligned} 4 &= 4 \\ &= 3 + 1 \\ &= 2 + 2 \\ &= 2 + 1 + 1 \\ &= 1 + 1 + 1 + 1 \end{aligned}$$

- $p(n)$: total number of partitions of n.

$$p(4) = 5.$$

The generating function of $p(n)$

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(1-q)(1-q^2)(1-q^3)\cdots}.$$

Euler's pentagonal number theorem:

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2}.$$

Recurrence of $p(n)$:

$$p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) - p(n-12) - \cdots = 0.$$

Observation of Andrews–Merca

$$p(n) - p(n-1) \geq 0$$

$$p(n) - p(n-1) - p(n-2) + p(n-5) \leq 0$$

$$p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) - p(n-12) \geq 0$$

$$\vdots$$

$$(-1)^k \sum_{j=0}^{k} (-1)^j \left(p\big(n - \frac{j(3j+1)}{2}\big) - p\big(n - \frac{j(3j+1)}{2} - 2j - 1\big) \right) \geq 0$$

(Andrews (1971), Bressoud (1980) – Partition Sieves – Connection to partition rank)

Truncated pentagonal number theorem

Theorem (Andrews-Merca (2012))

$$\frac{1}{(q;q)_\infty}\sum_{j=0}^{k}(-1)^j q^{\frac{j(3j+1)}{2}}(1-q^{2j+1}) = 1+(-1)^k\sum_{n=k+1}^{\infty}\frac{q^{\binom{k+1}{2}+(k+2)n}}{(q;q)_n}\begin{bmatrix} n-1 \\ k \end{bmatrix}.$$

Notation

$$\begin{aligned}
(a;q)_n &:= (1-a)(1-aq)\cdots(1-aq^{n-1}),\\
(a;q)_\infty &:= \lim_{n\to\infty}(a;q)_n,
\end{aligned}$$

$$\begin{bmatrix} n \\ k \end{bmatrix} := \begin{bmatrix} n \\ k \end{bmatrix}_q := \begin{cases} \dfrac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, & \text{if } n \geq k \geq 0,\\ 0, & \text{otherwise.} \end{cases}$$

Two identities of Gauss

$$\frac{(q;q)_\infty}{(-q;q)_\infty} = \sum_{j=-\infty}^{\infty} (-1)^j q^{j^2},$$

$$\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(2j+1)}.$$

$$\frac{(-q;q)_\infty}{(q;q)_\infty} = \sum_{n=0}^{\infty} \overline{p}(n) q^n$$

$$\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} = \sum_{n=0}^{\infty} pod(n) q^n.$$

Truncated theorems on Gauss' identities

Theorem (Guo-Zeng (2012))

$$
\begin{aligned}
&\frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{j=-k}^{k} (-1)^j q^{j^2} \\
&= 1 + (-1)^k \sum_{n=k+1}^{\infty} \frac{(-q;q)_k(-1;q)_{n-k}\, q^{(k+1)n}}{(q;q)_n} \begin{bmatrix} n-1 \\ k \end{bmatrix},
\end{aligned}
$$

$$
\begin{aligned}
&\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{j=0}^{k} (-1)^j q^{j(2j+1)} (1-q^{2j+1}) \\
&= 1 + (-1)^k \sum_{n=k+1}^{\infty} \frac{(-q;q^2)_{k+1}(-q;q^2)_{n-(k+1)}\, q^{2(k+2)n-(k+1)}}{(q^2;q^2)_n} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{q^2}.
\end{aligned}
$$

Conjecture of Andrews-Merca and Guo-Zeng

For $1 \le S \le R/2$,

$$\frac{(-1)^k}{(q^R, q^S, q^{R-S}; q^R)_\infty} \sum_{n=0}^{k} (-1)^n q^{\binom{n+1}{2}R - nS} \left(1 - q^{(2n+1)S}\right) + (-1)^{k-1}$$

has nonnegative coefficients.

For convenience, we will say a truncated series satisfies **positivity property**.

Note.

$$\frac{(-1)^k}{(q;q)_\infty} \sum_{n=0}^{k} (-1)^n q^{\binom{n+1}{2}R - nS} \left(1 - q^{(2n+1)S}\right)$$

satisfies positivity property. (Andrews and Bressoud – Partition Sieves)

Note. This conjecture was proved by R. Mao and Y. independently in 2015, and reproved by C. Wang and Y. in 2019.

Papers on this topic

- Andrews, Merca: The truncated pentagonal number theorem (JCTA, 119 (2012), 1639–1643)
- Guo, Zeng: Two truncated identities of Gauss (JCTA, 120 (2013), 700–707)
- Mao: Proofs of two conjectures on truncated series (JCTA, 130 (2015), 15–25)
- Y. : A truncated Jacobi triple product theorem (JCTA, 130 (2015), 1–14)
- Kolitsch: Another approach to the truncated pentagonal number theorem (Int. J. Number Theory 11 (2015) 1563–1569)
- He, Ji, Zang: Bilateral truncated Jacobi's identity (European J. Combin., 51 (2016), 255–267)
- Chan, Ho, Mao: Truncated series from the quintuple product identity (J. Number Theory, 169 (2016), 420–438)
- Andrews, Merca: Truncated theta series and a problem of Guo and Zeng (JCTA, 154 (2018), 610–619)
- Ballantine, Merca, Passary, Y.: Combinatorial proofs of two truncated theta series theorems (JCTA, 160 (2018), 168–185)
- Merca, Wang, Y. : A truncated theta identity of Gauss and overpartitions into odd parts (Ann. Comb. 23 (2019), 907–915)
- Wang, Y. : Truncated Jacobi triple product series (JCTA, 166 (2019), 382–392)
- Wang, Y. : Truncated Hecke–Rogers type series (Adv. Math, 365 (2020), 107051, 19 pp)

Analytic approach

Transformation formulas are main tools.

- Andrews' formula for the truncated pentagonal number theorem:

$$\sum_{n=0}^{m}(-1)^n q^{n(3n+1)/2}(1-q^{2n+1}) = (-1)^m q^{\binom{m+1}{2}} \sum_{n=0}^{m} \frac{(q^{-m};q)_n (q^{m+1};q)_{n+1}}{(q;q)_n}.$$

- Shank's formula for the work of Guo and Zeng:

$$\sum_{n=0}^{m}(-1)^n q^{n(2n+1)}(1-q^{2n+1}) = (-1)^m q^{2\binom{m+1}{2}} \sum_{n=0}^{m} \frac{(q^{-2m};q^2)_n (q^{2m+2};q^2)_{n+1}}{(q^2;q^2)_n(-q;q^2)_{n+1}}.$$

- Liu's formula: For an arbitrary sequence $\{A_n\}$,

$$\sum_{n=0}^{m}(-1)^n q^{\binom{n}{2}}(1-q^{2n+1}) \sum_{j=0}^{n}(q^{-n},q^{n+1};q)_j\, q^j A_j$$

$$= (-1)^m q^{\binom{m+1}{2}} \sum_{n=0}^{m}(q^{-m};q)_n (q^{m+1};q)_{n+1}\, A_n.$$

Sketch of the proof of Andrews–Merca

$$
\begin{aligned}
&\frac{1}{(q;q)_\infty}\sum_{j=0}^{k-1}(-1)^j q^{\frac{j(3j+1)}{2}}(1-q^{2j+1})\\
&= \frac{(-1)^{k-1}q^{\binom{k}{2}}}{(q;q)_\infty}\sum_{n=0}^{k-1}\frac{(q^{-k+1};q)_n(q^k;q)_{n+1}}{(q;q)_n} && \text{(by Andrews' formula)}\\
&= \frac{(-1)^{k-1}q^{\binom{k}{2}}}{(q;q)_{k-1}}\sum_{n=0}^{k-1}\frac{(q^{-k+1};q)_n}{(q;q)_n(q^{k+n+1};q)_\infty}\\
&= \frac{(-1)^{k-1}q^{\binom{k}{2}}}{(q;q)_{k-1}}\sum_{n=0}^{k-1}\frac{(q^{-k+1};q)_n}{(q;q)_n}\sum_{j=0}^{\infty}\frac{q^{(k+n+1)j}}{(q;q)_j} && \text{(by the q-binomial theorem)}\\
&= \frac{(-1)^{k-1}q^{\binom{k}{2}}}{(q;q)_{k-1}}\sum_{j=0}^{\infty}\frac{q^{(k+1)j}}{(q;q)_j}\sum_{n=0}^{k-1}\frac{(q^{-k+1};q)_n\, q^{jn}}{(q;q)_n}\\
&= \frac{(-1)^{k-1}q^{\binom{k}{2}}}{(q;q)_{k-1}}\sum_{j=0}^{\infty}\frac{q^{(k+1)j}}{(q;q)_j}(q^{-k+1+j};q)_{k-1} && \text{(by the q-Chu Vandermonde)}\\
&= 1+\frac{(-1)^{k-1}q^{\binom{k}{2}}}{(q;q)_{k-1}}\sum_{j=k}^{\infty}\frac{q^{(k+1)j}}{(q;q)_j}(q^{j-k+1};q)_{k-1}\\
&= 1+(-1)^{k-1}q^{\binom{k}{2}}\sum_{j=k}^{\infty}\frac{q^{(k+1)j}}{(q;q)_j}\frac{(q;q)_{j-1}}{(q;q)_{k-1}(q;q)_{j-k}}.
\end{aligned}
$$

Combinatorial approach

Papers with more combinatorial flavors:

- Y. : A truncated Jacobi triple product theorem (JCTA, 130 (2015), 1–14)
- L. Kolitsch: Another approach to the truncated pentagonal number theorem (Int. J. Number Theory 11 (2015) 1563–1569)
- He, Ji, Zang: Bilateral truncated Jacobi's identity (European J. Combin., 51 (2016), 255–267)
- Ballantine, Merca, Passary, Y.: Combinatorial proofs of two truncated theta series theorems (JCTA, 160 (2018), 168–185)
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There exists no unified treatment.

Xia's new truncated series

Recently, Ernest Xia found several new truncated series identities and asked for their combinatorial proofs.

- Xia's truncated series:

$$(q;q)_\infty \longrightarrow \sum_{j=0}^{k-1}(-1)^j q^{3j(j+1)/2}(1-q^{j+1})(1-q^{2j+2}),$$

$$\frac{(q;q)_\infty}{(-q;q)_\infty} \longrightarrow \sum_{j=0}^{k-1}(-1)^j q^{j(j+1)}(1-q^{j+1})^2,$$

$$\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} \longrightarrow \sum_{j=0}^{k-1}(-1)^j q^{2j^2+j}(1-q^{2j+2})(1-q^{4j+4}).$$

Xia's identities

1. $$\frac{1}{(q;q)_\infty}\sum_{j=0}^{k-1}(-1)^j q^{3j(j+1)/2}(1-q^{j+1})(1-q^{2j+2})$$
$$=1+(-1)^{k-1}q^{\binom{k}{2}}\sum_{n=k}^{\infty}\frac{q^{n(k+2)}}{(q;q)_n}\begin{bmatrix}n-1\\k-1\end{bmatrix}.$$

2. $$\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{j=0}^{k}(-1)^j q^{j(j+1)}(1-q^{j+1})^2$$
$$=1+(-1)^k\frac{(-q;q)_{k+1}}{(q;q)_k}\sum_{n=k}^{\infty}\frac{q^{(n+2)(k+1)}(-q^{n+3};q)_\infty}{(q^{n+3};q)_\infty}.$$

3. $$\frac{(-q^3;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{j=0}^{k}(-1)^j q^{2j^2+j}(1-q^{2j+2})(1-q^{4j+4})$$
$$=1+(-1)^k\frac{(-q^3;q^2)_{k+1}}{(q^2;q^2)_k}\sum_{n=k}^{\infty}\frac{q^{(2n+3)(k+1)}(-q^{2n+5};q^2)_\infty}{(q^{2n+6};q^2)_\infty}.$$

A simple version of Chen's combinatorial telescoping method

$$\sum_{j=0}^{k} (-1)^j F_j(x) = (-1)^k G_k(x).$$

$$F_k(x) = G_k(x) + G_{k-1}(x).$$

$$G_k(x) \longrightarrow G_k'(x).$$

$$F_k(x) = G_k(x) + G_{k-1}'(x).$$

$M_k(n)$ (Andrews and Merca)

Theorem

$$\frac{(-1)^{k-1}}{(q;q)_\infty}\sum_{j=0}^{k-1}(-1)^j q^{j(3j+1)/2}(1-q^{2j+1}) = (-1)^{k-1} + \sum_{n=k}^{\infty}\frac{q^{\binom{k}{2}+(k+1)n}}{(q;q)_n}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$

- $M_k(n) :=$ # partitions of n where k is the least positive integer that is not a part and there are more parts $> k$ than there are parts $< k$.

$$\sum_{n=0}^{\infty} M_k(n)q^n = \sum_{n=k}^{\infty}\frac{q^{\binom{k}{2}+(k+1)n}}{(q;q)_n}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$

- The theorem above can be rewritten as follows:

$$\frac{q^{k(3k+1)/2}(1-q^{2k+1})}{(q;q)_\infty} = \sum_{n=0}^{\infty}(M_k(n)+M_{k+1}(n))q^n.$$

- Idea: Define two partition functions $m_k(n)$ and $m'_{k+1}(n)$, which are equal to $M_k(n)$ and $M_{k+1}(n)$, respectively, and then show

$$\sum_{n=0}^{\infty}(m_k(n)+m'_{k+1}(n))q^n = \frac{q^{k(3k+1)/2}(1-q^{2k+1})}{(q;q)_\infty}.$$

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- $M_k(n) := \#$ partitions of n where k is the least positive integer that is not a part

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Combinatorial proof of the truncated pentagonal number theorem

Notation: f_i counts the number of parts of size i.

- $m_k(n) := \#$ partitions of n satisfying the following conditions:
  - i) $f_i \geq 1$ for $i = 1, \ldots, k-1$;
  - ii) $k+1 \leq f_k \leq x$, where x is the smallest part $> k$; if there are no parts $> k$, $x = \infty$.

Then,

$$\sum_{n\geq 0} m_k(n)q^n = \frac{q^{\binom{k}{2}+k(k+1)}}{(q;q)_{k-1}} \sum_{n\geq 0} \frac{q^{(k+1)n}}{(q;q)_n(1-q^{n+k})}.$$

Recall

$$\sum_{n\geq 0} M_k(n)q^n = \sum_{n\geq 0} \frac{q^{\binom{k}{2}+(k+1)(n+k)}}{(q;q)_{n+k}} \begin{bmatrix} n+k-1 \\ k-1 \end{bmatrix}.$$

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- $m'_{k+1}(n) := \#$ partitions of n satisfying the following conditions:
  - i) $f_i \geq 1$ for $i = 1, \ldots, k-1$;
  - ii) $f_k > x$, where x is the smallest part $> k+1$ and x exists.

Then,

$$\sum_{n\geq 0} m'_{k+1}(n)q^n = \frac{q^{\binom{k}{2}+k(k+3)}}{(q;q)_{k-1}} \sum_{n\geq 1} \frac{q^{(k+2)n}}{(1-q^k)(q;q)_{n-1}(1-q^{n+k})}.$$

Recall

$$\sum_{n\geq 0} M_{k+1}(n)q^n = \sum_{n\geq 0} \frac{q^{\binom{k}{2}+k+(k+2)(n+k+1)}}{(q;q)_{n+k+1}} \begin{bmatrix} n+k \\ k \end{bmatrix}.$$

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Recall

$$\sum_{n\geq 0} M_{k+1}(n)q^n = \sum_{n\geq 0} \frac{q^{\binom{k}{2}+k+(k+2)(n+k+1)}}{(q;q)_{n+k+1}} \begin{bmatrix} n+k \\ k \end{bmatrix}.$$

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  i) $f_i \geq 1$ for $i = 1, \ldots, k-1$;
  ii) $k+1 \leq f_k \leq x$, where x is the smallest part $> k$; if there are no parts $> k$, $x = \infty$.
- $m'_{k+1}(n) := \#$ partitions of n satisfying the following conditions:
  i) $f_i \geq 1$ for $i = 1, \ldots, k-1$;
  ii) $f_k > x$, where x is the smallest part $> k+1$ and x exists.
- $m_k(n) + m'_{k+1}(n)$ counts the number of partitions of n satisfying the following:
  i) $f_i \geq 1$ for $i = 1, \ldots, k-1$;
  ii) $f_k \geq k+1$;
  iii) if $f_{k+1} \geq 1$, then $f_k = k+1$.

Then

$$\sum_{n \geq 0} \big(m_k(n) + m'_{k+1}(n)\big) q^n = \frac{q^{\binom{k}{2} + k(k+1)}(1 - q^{2k+1})}{(q;q)_\infty}.$$

Xia's identity on overpartitions

Recall

$$\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{j=0}^{k}(-1)^j q^{j(j+1)}(1-q^{j+1})^2 = 1+(-1)^k\frac{(-q;q)_{k+1}}{(q;q)_k}\sum_{n=k}^{\infty}\frac{q^{(n+2)(k+1)}(-q^{n+3};q)_\infty}{(q^{n+3};q)_\infty}.$$

Define $C_k(n)$ as follows:

$$\sum_{n\geq 0} C_k(n)q^n = \frac{(-q;q)_{k+1}}{(q;q)_k}\sum_{n=k}^{\infty}\frac{q^{(n+2)(k+1)}(-q^{n+3};q)_\infty}{(q^{n+3};q)_\infty}.$$

Then the identity above is equivalent to

$$\sum_{n\geq 0}\big(C_k(n)+C_{k-1}(n)\big)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}(1-q^{k+1})^2 q^{k(k+1)}.$$

We will define ovepartition functions $c_k(n)$ and $c'_{k-1}(n)$ which equal $C_k(n)$ and $C_{k-1}(n)$, respectively, and then we prove combinatorially the following identity

$$\sum_{n\geq 0}\big(c_k(n)+c'_{k-1}(n)\big)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}(1-q^{k+1})^2 q^{k(k+1)}.$$

Sketch of Proof

Notation: $f_{\overline{i}}$ counts the number of overlined parts of size i.

- $c_k(n) :=$ # overpartitions of n satisfying the following conditions:
  - i) $f_{k+1} = f_{\overline{k+1}} = 0$;
  - ii) $f_{k+2} = 0$;
  - iii) $f_k \geq x$, where x is the smallest part $\geq k+2$ and x exists;

Case 1: x is overlined and unique. Then the generating function is

$$\frac{(-q;q)_k}{(q;q)_k} \sum_{n \geq 1} \frac{(-1;q)_{n-1}\, q^{k(k+2)+(k+2)n+n-1}}{(q;q)_{n-1}(1-q^{n+k})}.$$

Case 2: x is non-overlined but unique. Then the generating function is

$$\frac{(-q;q)_k}{(q;q)_k} \sum_{n \geq 1} \frac{(-1;q)_{n-1}\, q^{(k+3)(k+n)+n-1}}{(q;q)_{n-1}(1-q^{n+k})}.$$

Case 3: x is not unique. Then the generating function is

$$\frac{(-q;q)_k}{(q;q)_k} \sum_{n \geq 1} \frac{(-1;q)_{n-1}\, q^{(k+3)(k+n)}}{(q;q)_{n-2}(1-q^{n+k})}.$$

- $c_k(n) := \#$ overpartitions of n satisfying the following conditions:
  i) $f_{k+1} = f_{\overline{k+1}} = 0$;
  ii) $f_{k+2} = 0$;
  iii) $f_k \geq x$, where x is the smallest part $\geq k+2$ and x exists;

Thus

$$\sum_{n\geq 0} c_k(n)q^n = \frac{(-q;q)_{k+1}}{(q;q)_k} \sum_{n\geq 1} \frac{(-1;q)_{n-1}\, q^{(k+2)(k+n)+n-1}}{(q;q)_{n-1}(1-q^{n+k})} = \sum_{n\geq 0} C_k(n)q^n.$$

- $c'_{k-1}(n) := \#$ overpartitions of n satisfying the following conditions:
    i) $f_{k+1} = f_{\overline{k+1}} = 0$;
    ii) $f_k \geq k+1$;
    iii) $f_k < x$, where x is the smallest part $\geq k+2$; if there are no parts $\geq k+2$, then $x = \infty$.

Case 1: $x = \infty$. Then the generating function is

$$\frac{(-q;q)_k\, q^{k(k+1)}}{(q;q)_k}.$$

Case 2: $x < \infty$. Then the generating function is

$$\frac{(-q;q)_k}{(q;q)_{k-1}} \sum_{n\geq 1} \frac{(-1;q)_n q^{k(k+1)+(k+2)n}}{(q;q)_n(1-q^{n+k})}.$$

Thus,

$$\sum_{n\geq 0} c'_{k-1}(n)q^n = \frac{(-q;q)_k\, q^{k(k+1)}}{(q;q)_{k-1}} \sum_{n\geq 0} \frac{(-1;q)_n\, q^{(k+2)n}}{(q;q)_n(1-q^{n+k})} = \sum_{n\geq 0} C_{k-1}(n)q^n.$$

It follows from the definitions that $c_k(n) + c'_{k-1}(n)$ counts the number of partitions of n satisfying the following:

i) $f_{k+1} = f_{\overline{k+1}} = 0$;

ii) $f_k \geq k+1$;

iii) $f_{k+2} \geq 1$, then $f_k = k+1$.

Thus,

$$\sum_{n\geq 0} \big(c_k(n) + c'_{k-1}(n)\big)q^n = \frac{q^{k(k+1)}(1-q^{k+1})(-q;q)_\infty}{(1+q^{k+1})(q;q)_\infty} - \frac{q^{k(k+2)+k+2}(1-q^{k+1})(-q;q)_\infty}{(1+q^{k+1})(q;q)_\infty}$$

$$= \frac{q^{k(k+1)}(1-q^{k+1})^2(-q;q)_\infty}{(1+q^{k+1})(q;q)_\infty}.$$

Remarks

1. Can we prove the truncated Jacobi triple product theorem?

   For $1 \leq S \leq R/2$,

   $$\frac{(-1)^k}{(q^R, q^S, q^{R-S}; q^R)_\infty} \sum_{n=0}^{k} (-1)^n q^{\binom{n+1}{2}R - nS} \left(1 - q^{(2n+1)S}\right) + (-1)^{k-1}$$

   has nonnegative coefficients.

2. Does this method work for other truncated theorems?

Thank you!