Complexity of Integration, Special Values, and Recent Developments

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Abstract. Two questions often come up when the author discusses integration: what is the complexity of the integration process, and for what special values of parameters is an unintegrable function actually integrable. These questions have not been much considered in the formal literature, and where they have been, there is one recent development indicating that the question is more delicate than had been supposed.

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1 Introduction

The author is often asked two questions about integration.

1. "What is the complexity of integration?".
2. "My integrand $f(x, a)$ is unintegrable. For what special a is it integrable?"

These questions have rather different answers for purely transcendental integrands and for algebraic function (or mixed) integrands. In fact, they are essentially unexplored for mixed integrands, given the difficulties of the two special cases.

Integration of $f(x)$, in the sense of determining a formula $F(x)$ such that $F'(x) = f(x)$, is a process of differential algebra. There is then a question of whether this formula actually corresponds to a continuous function $\mathbf{R} \to \mathbf{R}$. This is an important question in terms of usability of the results, but a rather different one than we wish to consider here: see [7].

2 Transcendental Integration

In order to use differential algebra, the integrand f is written (itself a non-trivial procedure: see [9], generally known as the **Risch Structure Theorem**) in a suitable field $K(x, \theta_1, \ldots, \theta_n)$ where each θ_i is transcendental over $K(x, \theta_1, \ldots, \theta_{i-1})$ with $K(x, \theta_1, \ldots, \theta_i)$ having the same field of constants as $K(x, \theta_1, \ldots, \theta_{i-1})$ and each θ_i being either:

1) a *logarithm* over $K(x, \theta_1, \ldots, \theta_{i-1})$, i.e. $\theta'_i = \eta'_i/\eta_i$ for $\eta_i \in K(x, \theta_1, \ldots, \theta_{i-1})$;

e) an *exponential* over $K(x, \theta_1, \ldots, \theta_{i-1})$, i.e. $\theta'_i = \eta'_i \theta_i$ for $\eta_i \in K(x, \theta_1, \ldots, \theta_{i-1})$.

This process may generate special cases: for example $\exp(a \log x)$ lives in such a $K(x, \theta_1, \theta_2)$ with $\theta'_1 = \frac{1}{x}$ (θ_1 corresponds to $\log x$) and $\theta'_2 = \frac{a}{x}\theta_2$ (θ_2 corresponds to $\exp(a \log x)$), *except* when a is rational, when in fact we have x^a. However, this is generally not what is meant by the "special values" question, and in general we assume that parameters are not in exponents.

2.1 Elementary Transcendental Functions

Here we have a decision procedure, as outlined in [8]. The proof of the procedure proceeds by induction on n, the ingenuity lying in the induction hypothesis: we suppose that we can:

a) "integrate in $K(x, \theta_1, \ldots, \theta_{n-1})$", i.e. given $g \in K(x, \theta_1, \ldots, \theta_{n-1})$, either write $\int g \mathrm{d}x$ as an elementary function over $K(x, \theta_1, \ldots, \theta_{n-1})$, or prove that no such elementary function exists;
b) "solve Risch differential equations in $K(x, \theta_1, \ldots, \theta_{n-1})$", i.e. given elements $F, g \in K(x, \theta_1, \ldots, \theta_{n-1})$ such that $\exp(F)$ is transcendental over $K(x, \theta_1, \ldots, \theta_{n-1})$ (with the same field of constants), solve $y' + F'y = g$ for $y \in K(x, \theta_1, \ldots, \theta_{n-1})$, or prove that no such y exists.

We then prove that (a) and (b) hold for $K(x, \theta_1, \ldots, \theta_n)$.

2.2 Logarithmic θ_n

If θ_n is logarithmic, the proof of part (a) is a straightforward exercise building on part (a) for $K(x, \theta_1, \ldots, \theta_{n-1})$: see, e.g. [3, §5.1]. Unintegrability manifests itself as the insolubility of certain equations, and any special values of the parameters will be found as special values rendering these equations soluble.

It is also straightforward (though as far as the author knows, not done) to prove that, if all θ_i are logarithmic, then the degree in each θ_i of the integral is no more than it is in the integrand, and that the denominator of the integral is a divisor of the denominator of the integrand. Hence, in the dense model, the integral is, apart from coefficient growth, not much larger than the integrand, and the compute cost is certainly polynomial.

In a sparse model, the situation is very different.

$$\int \log^n x \mathrm{d}x = x \log^n x - nx \log^{n-1} x + \cdots \pm n!x,$$

so an integrand requiring $\Theta(\log n)$ bits can require $\Omega(n)$ bits for the integral. The same is true for $\int x^n \log^n x \mathrm{d}x$, but $\int x^n \log^n(x+1) \mathrm{d}x$ shows that $\Omega(n^2)$ bits can be required. As far as the author knows, it is an open question whether the problem is even in EXPSPACE, though it probably is.

2.3 Exponential θ_n

Here the problem is different. Suppose $\theta_n = \exp(F)$. $\int g \exp(F) \mathrm{d}x = y \exp(F)$ where $y' + F'y = g$ (and can be nothing else if it is to be an elementary function). Hence solving (a) in $K(x, \theta_1, \ldots, \theta_n)$ reduces (among other things) to solving (b) in $K(x, \theta_1, \ldots, \theta_{n-1})$. In general, the solution to (b) proceeds essentially by undetermined coefficients, which is feasible as $y' + F'y$ is linear in the unknown coefficients. Before we can start this, we need to answer two questions: what is the denominator of y, and what is the degree (number of unknown coefficients)? In general, the answers are obvious: if the denominator of g has an irreducible factor p of multiplicity k, y' will have the same, so the denominator of y will have a factor of (at most) p^{k-1}, and F' can only reduce this. Similarly, if g has degree d, y' will have degree at most d, so y will have degree $d+1$, and again F' can only reduce this. The complication is when there is cancellation in $y' + F'y$, so that this has lower degree, or smaller denominator, than its summands. [8] shows how to resolve this problem, and does not pay it much attention, not being interested in the complexity question.

In [2] it is noted that these complications come from what one might loosely call "eccentric" integrands. For example

$$y' + \left(1 + \frac{5}{x}\right) y = 1 \tag{1}$$

has solution

$$y = \frac{x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120}{x^5}, \tag{2}$$

(and in general $y' + \left(1 + \frac{n}{x}\right) y = 1$ will have a solution with denominator x^n) but this comes from

$$\int \exp(x + 5 \log x) \mathrm{d}x, \tag{3}$$

which might be more clearly expressed as

$$\int x^5 \exp(x) \mathrm{d}x. \tag{4}$$

However, the integrand in (3) has total degree 1, whereas that in (4) has total degree 6, consistent with the degrees in (2). Ultimately, the point is that the dense model is not applicable when we can move things into/out of the exponents at will.

We do have a result [2, Theorem 4] which says that, provided $K(x, \theta_1, \ldots, \theta_n)$ is *exponentially reduced* (loosely speaking, doesn't allow "eccentric" integrands) then we have natural degree bounds on the solutions of (b) equations. As stated there, "this is far from being a complete bounds on integrals, but it does indicate that the worst anomalies cannot take place" here.

Again, the complexity is still an open question, but the author is inclined to conjecture that it is no worse than EXPSPACE.

What of special values of parameters? These come in two kinds.

1. As in the logarithmic case, we can get proofs of unintegrability because certain equations are insoluble. For example $(x+a)\exp(-bx^2+cx)$ is integrable if, and only if, $c=-2ab$, and this equation arises during the undetermined coefficients process.
2. More complicated are those that change the "exponentially reduced" nature of the integrand. For example, $\int \exp(x+a\log x)\mathrm{d}x$ does not have an elementary expression *except* when a is a non-negative integer, when we are in a similar position to (3). These values are similar to those that change the Risch Structure Theorem expression of the integrand.

3 Algebraic Functions

The integration of algebraic functions [1, 11] is a more complex process. If $f \in K(x,y)$ where y is algebraic over $K(x)$, the integral, if it is elementary, has to have the form $v_0 + \sum c_i \log(v_i)$, where $v_0 \in K(x,y)$, the c_i are algebraic over K, and the $v_i \in L(x,y)$ where L is the extension of K by the c_i (and possibly more algebraic numbers added by the algorithm, though these should be irrelevant). So far, this is the same as the integration of rational functions, and the challenge is to determine the c_i and v_i.

3.1 The logarithmic part

Looked at from the point of view of analysis, the $\sum c_i \log(v_i)$ term is to represent the logarithmic singularities in $\int f\mathrm{d}x$, which come from the simple poles of f: in a power series world c_i would be the residue at the pole corresponding to v_i. Hence an obvious algorithm would be

1. Compute all the residues r_j at all the corresponding poles p_j (which might include infinity, and which might be ramified: the technical term would be "place"). Assume $1 \le j \le m$.
2. Let c_i be a **Z**-basis for the r_j, so that $r_j = \sum \alpha_{i,j} c_i$.
3. For each c_i, let v_i be a function $\in L(x,y)$ with residue $\alpha_{i,j}$ at p_j for $1 \le j \le m$ (and nowhere else). The technical term for this residue/place combination is "divisor", and a divisor with a corresponding function v_i is termed a "principal divisor".
* Returning "unintegrable" if we can't find such v_i.
4. Having determined the logarithms this way, find v_0 by undetermined coefficients.

The problem with the correctness of this algorithm is a major feature of algebraic geometry. It is possible that D_i is not a principal divisor, but that $2D_i$, or $3D_i$ or ... is principal. In this case, we say that D_i is a torsion divisor, and the corresponding order is referred to as the torsion of the divisor. If, say, $3D_i$ is principal with corresponding function v_i, then, although not in $L(x,y)$, $\sqrt[3]{v_i}$ corresponds to the divisor D_i, and we can use $c_i \log \sqrt[3]{v_i}$, or, more conveniently and fitting in with general theory, $\frac{c_i}{3} \log v_i$ as a contribution to the logarithmic part.

3.2 Complexity

There are three main challenges with complexity theory for algebraic function integration.

1. The first is that it is far from clear what the "simplest" form of an integral of this form is. The choice of c_i is far from unique, and a "bad" choice of c_i may lead to large $\alpha_{i,j}$ and complicated v_i.
2. The second is that the r_j are algebraic numbers, and there are no known non-trivial bounds for the r_j, or the $\alpha_{i,j}$.
3. The third is that there is very little known about the torsion. This might seem surprising to those who know some algebraic geometry, and have heard of, say, Mazur's bound [6]. This does indeed show that, if the algebraic curve defined by y is elliptic (has genus 1) *and* the divisor is defined over $\mathbf{Q}$, then the torsion is at most 12. The trouble is that this requires the divisor to be defined over $\mathbf{Q}$, and not just f. For elliptic curves, a recent survey of the known bounds is given in [10].

Hence it appears unrealistic to think of complexity bounds in the current state of knowledge.

4 Two meis culpis about algebraic integration and parameters

In the author's thesis (see the expanded version in [1]) we considered the question of whether $f(x,u)\mathrm{d}x$, an algebraic function of x, could have an elementary integral for specific values of u, even if the uninstantiated integral were not elementary.

4.1 The claim

We began [1, pp. 89–90] with a rehearsal of the ways in which substituting a value for u could change the working of the integration algorithm, and how these could be detected, i.e. given such an unintegrable $f(x,u)$ how one might determine the specific u values for which the integrand *might* have an elementary integral.

1. The curve can change genus: look at the canonical divisor.
2. The [geometry of the] places at which residues occur can change: look at values of u for which numerator/denominator cancel, or roots coincide.
3. The dimension of the space of residues can collapse.
4. A divisor may be a torsion divisor for a particular value of u, even though it is not a torsion divisor in general. These cases can be detected by looking at the roots of `SUM` in `FIND_ORDER_MANIN`.
5. the algebraic part may be integrable for a particular u, though not in general. Hence the contradicting equation in `FIND_ALGEBRAIC_PART` collapses.

As a potential example of case 3, consider

$$\frac{1}{x\sqrt{x^2+1}}+\frac{1}{x\sqrt{x^2+u^2}}$$

whose residues are $\pm 1, \pm u$ and therefore every rational u is a special case.

Lemma 1 ([1, Lemma 6, page 90]). *Let the* $\mathbf{Z}$*-module of residues* r_i *of* $f(x,u)$ *have dimension* k*, and suppose there are values* $(u_1,\ldots,u_k)$ *such that* $f(x,u_i)$ *has an elementary logarithmic part (not in cases 1,2,4,5) and such that the set of vectors* $\{(r_i(u_a) : 1 \le i \le k) 1 \le a \le k\}$ *is of dimension* k*. Then* $f(x,u)\mathrm{d}x$ *has an elementary logarithmic part.*

Proof: some $(n, 0 \ldots, 0)$ can be expressed as a linear combination with integer coefficients of the $(r_i(u_a))$. Hence the divisor d_1 must be a torsion divisor, as nd_1 is a sum of torsion divisors. Similarly the other d_i.

We suppose $f(x,u)$ depends algebraically on u (else it's a new transcendental).

Theorem 1 ([1, Theorem 7, page 90]). *If* $f(x,u)\mathrm{d}x$ *is not elementarily integrable, then there are only finitely many values* u_i *of* u *for which* $f(x,u_i)\mathrm{d}x$ *has an elementary integral.*

Proof. Case 3 is the hard one. Lemma 6 disposes of the case where k values generate a full-dimensional space, so there is a linear relationship between the $r_i(u_a)$ which is not true in general, but which is true infinitely often. But the $r_i(u_a)$ are algebraic in u (proposition 5) and this means we have an algebraic expression which is not identically zero, but which has infinitely many roots, and this establishes the required contradiction.

4.2 The first problem

[4] observes that $\frac{x\mathrm{d}x}{(x^2-u^2)\sqrt{x^3-x}}$ is not elementarily integrable, but is integrable whenever the point $(u, ?)$ is of order at least three on the curve $y^2 = x^3 - x$, and this can be achieved infinitely often, at the cost of extending the number field. The simplest example is $u = i$, when $(i, 1-i)$ is of order 4 and we have

$$\int \frac{x\mathrm{d}x}{(x^2+1)\sqrt{x^3-x}} = \frac{1+i}{16}\ln\left(\frac{x^2+(2+2\,i)\sqrt{x^3-x}+2\,ix-1}{x^2-(2+2\,i)\sqrt{x^3-x}+2\,ix-1}\right) + \frac{1-i}{16}\ln\left(\frac{x^2+(2-2\,i)\sqrt{x^3-x}-2\,ix-1}{x^2-(2-2\,i)\sqrt{x^3-x}-2\,ix-1}\right)$$

Unfortunately neither Maple (2016) nor Mathematica (10.0) nor Reduce (build 3562) can actually integrate this elementarily.

The full problem is treated in [5]. It seems that the arguments in [1] are implicitly assuming a fixed number field, but a full analysis awaits the publication of [5].

4.3 The second problem

The assertion that the case of transcendental u is trivial, if true at all, is certainly not trivial, and probably false, if we also allow transcendental constants in f, for they and u can then "collide". [4].

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