The Method of Brackets (MoB)

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The 5th International Congress on Mathematical Software
@ Zuse Institute Berlin

July 13th 2016

Acknowledgement

Joint Work with:

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Karen Kohl

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Outlines

Rules

Idea

MoB evaluates the definite integral

$$\int_0^\infty f(x)\,dx$$

(most of the time) in terms of SERIES, with *ONLY SIX* rules:

Defintion [Indicator]

$$\phi_n := \frac{(-1)^n}{n!} = \frac{(-1)^n}{\Gamma(n+1)}$$

and

$$\phi_{1,\ldots,r} := \phi_{n_1,\ldots,n_r} = \phi_{n_1}\phi_{n_2}\cdots\phi_{n_r} = \prod_{i=1}^{r}\phi_{n_i}.$$

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Rules (P-Production; E-Evaluation) $I = \int_0^\infty f(x)\,dx$

P_1: $f(x) = \sum_{n=0}^{\infty} a_n x^{\alpha n+\beta-1} \Rightarrow \int_0^\infty f(x)\,dx \mapsto \sum_n a_n \langle \alpha n + \beta \rangle$—Bracket Series;

P_2: For $\alpha < 0$, $(a_1 + \cdots + a_r)^\alpha \mapsto \sum_{n_1,\ldots,n_r} \phi_{1,\ldots,r} a_1^{n_1} \cdots a_r^{n_r} \frac{\langle -\alpha+n_1+\cdots+n_r \rangle}{\Gamma(-\alpha)}$;

P_3: For each bracket series, we assign *index=# of sums− # of brackets*;

E_1: $\sum_n \phi_n f(n) \langle \alpha n + \beta \rangle = \frac{1}{|\alpha|} f(n^*)\Gamma(-n^*)$, where n^* solves $\alpha n + \beta = 0$;

E_2: $$\sum_{n_1,\ldots,n_r} \phi_{1,\ldots,r} f(n_1,\ldots,n_r) \prod_{i=1}^{r} \langle a_{i1} n_1 + \cdots + a_{ir} n_r + c_i \rangle = \frac{f(n_1^*,\ldots,n_r^*) \prod_{i=1}^{r} \Gamma(-n_i^*)}{|\det A|},$$

$$(n_1^*,\ldots,n_r^*) \text{ solves } \begin{cases} a_{11} n_1 + \cdots + a_{1r} n_r + c_1 & = 0 \\ \ldots & \cdots \\ a_{r1} n_1 + \cdots + a_{rr} n_r + c_r & = 0 \end{cases};$$

E_3: The value of a multi-dimensional bracket series of ***POSITIVE*** index is obtained by computing all the contributions of maximal rank by Rule E_2. These contributions to the integral appear as series in the free parameters. Series converging in a common region are added and divergent series are discarded. Any series producing a non-real contribution is also discarded.

Ramanujan's Master Theorem[RMT]

Theorem[RMT]

$$\int_0^\infty x^{s-1}\left\{a(0) - \frac{a(1)}{1!}x + \frac{a(2)}{2!}x^2 - \cdots\right\}dx = a(-s)\Gamma(s)$$

(1)

$$\int_0^\infty x^{s-1}\left(\sum_{n=0}^\infty \phi_n a(n) x^n\right)dx = a(-s)\Gamma(s)$$

(2) [Hardy]

•$H(\delta) := \{s = \sigma + \iota t : \sigma \geq -\delta, 0 < \delta < 1\}$;

•$\psi(x) \in C^\infty(H(\delta))$; $\exists C, P, A,\ A < \pi$ such that $|\psi(s)| \leq Ce^{P\delta + A|t|}$, $\forall s \in H(\delta)$;

•$0 < c < \delta,\ \Psi(x) := \frac{1}{2\pi\iota}\int_{c-\iota\infty}^{c+\iota\infty}\frac{\pi}{\sin(\pi s)}\psi(-s)x^{-s}dx \overset{0<x<e^{-P}}{=\!=\!=\!=} \sum_{k=0}^\infty \psi(k)(-x)^k$;

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(2) Keep Track of s;
(3) Apply the Formula;
(4) Multiple Integrals;

$$\int_0^\infty \int_0^\infty \sum_{n,m} a(m,n) x^m y^n dxdy = ?$$

(5) More Sums than Integrals (brackets);

$$\int_0^\infty f_1(x) f_2(x) dx = \int_0^\infty \sum_{m,n} a(m,n) x^{m+n} dx = \sum_{m,n} a(m,n) \langle m+n+1\rangle = ?$$

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Rule P_2

$$
\begin{aligned}
& \frac{\Gamma(-\alpha)}{(a_1+\cdots+a_r)^{-\alpha}}. \\
= & \int_0^\infty x^{-\alpha-1} e^{-(a_1+\cdots+a_r)x} dx \\
= & \int_0^\infty x^{-\alpha-1} e^{-a_1 x} e^{-a_2 x} \cdots e^{-a_r x} dx \\
= & \int_0^\infty x^{-\alpha-1} \prod_{i=1}^{r} \left(\sum_{n_i=0}^{\infty} \phi_{n_i} (ax)^{n_i} \right) dx \\
= & \int_0^\infty \sum_{n_1,\ldots,n_r} \phi_{1,\ldots,r} a_1^{n_1} \cdots a_r^{n_r} x^{n_1+\cdots+n_r-\alpha-1} dx \\
= & \sum_{n_1,\ldots,n_r} \phi_{1,\ldots,r} a_1^{n_1} \cdots a_r^{n_r} \langle -\alpha + n_1 + \cdots + n_r \rangle
\end{aligned}
$$

Examples

$$I:=\int_0^\infty xJ_0(xy)\frac{dx}{\sqrt{a^2+x^2}}=y^{-1}e^{-ay}\quad [y>0\ Re(a)>0]$$

Rule P_2:

$$\frac{1}{\sqrt{a^2+x^2}}=\left(a^2+x^2\right)^{-\frac{1}{2}}=\sum_{n_1,n_2}\phi_{1,2}a^{2n_1}x^{2n_2}\frac{\left\langle\frac{1}{2}+n_1+n_2\right\rangle}{\Gamma\left(\frac{1}{2}\right)}$$

$J_0\left(xy\right)$

$$J_0\left(xy\right)=\sum_{n_3}\phi_{n_3}\frac{y^{2n_3}}{\Gamma\left(n_3+1\right)2^{2n_3}}x^{2n_3}$$

Rule P_1

$$\begin{aligned}I &= \int_0^\infty\sum_{n_1,n_2,n_3}\phi_{1,2,3}\frac{y^{2n_3}a^{2n_1}}{\Gamma\left(n_3+1\right)\Gamma\left(\frac{1}{2}\right)2^{2n_3}}\left\langle n_1+n_2+\frac{1}{2}\right\rangle x^{2n_2+2n_3+1}dx\\ &= \sum_{n_1,n_2,n_3}\phi_{1,2,3}\frac{y^{2n_3}a^{2n_1}}{\Gamma\left(n_3+1\right)\Gamma\left(\frac{1}{2}\right)2^{2n_3}}\left\langle n_1+n_2+\frac{1}{2}\right\rangle\left\langle 2n_2+2n_3+2\right\rangle\end{aligned}$$

Examples

$$I := \int_0^\infty x J_0(xy) \frac{dx}{\sqrt{a^2+x^2}} = y^{-1} e^{-ay} \quad [y>0 \ Re(a)>0]$$

Rule P_2:

$$\frac{1}{\sqrt{a^2+x^2}} = \left(a^2+x^2\right)^{-\frac{1}{2}} = \sum_{n_1,n_2} \phi_{1,2} a^{2n_1} x^{2n_2} \frac{\left\langle \frac{1}{2}+n_1+n_2 \right\rangle}{\Gamma\left(\frac{1}{2}\right)}$$

$J_0(xy)$

$$J_0(xy) = \sum_{n_3} \phi_{n_3} \frac{y^{2n_3}}{\Gamma(n_3+1) 2^{2n_3}} x^{2n_3}$$

Rule P_1

$$\begin{aligned} I &= \int_0^\infty \sum_{n_1,n_2,n_3} \phi_{1,2,3} \frac{y^{2n_3} a^{2n_1}}{\Gamma(n_3+1)\Gamma\left(\frac{1}{2}\right) 2^{2n_3}} \left\langle n_1+n_2+\frac{1}{2} \right\rangle x^{2n_2+2n_3+1} dx \\ &= \sum_{n_1,n_2,n_3} \phi_{1,2,3} \frac{y^{2n_3} a^{2n_1}}{\Gamma(n_3+1)\Gamma\left(\frac{1}{2}\right) 2^{2n_3}} \left\langle n_1+n_2+\frac{1}{2} \right\rangle \langle 2n_2+2n_3+2 \rangle \end{aligned}$$

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Examples

$$I := \int_0^\infty x J_0(xy) \frac{dx}{\sqrt{a^2+x^2}} = y^{-1} e^{-ay} \quad [y > 0 \ Re(a) > 0]$$

Rule P_2:

$$\frac{1}{\sqrt{a^2+x^2}} = \left(a^2+x^2\right)^{-\frac{1}{2}} = \sum_{n_1,n_2} \phi_{1,2} a^{2n_1} x^{2n_2} \frac{\left\langle \frac{1}{2}+n_1+n_2 \right\rangle}{\Gamma\left(\frac{1}{2}\right)}$$

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E_3 : $$I = \frac{1}{y}\cosh(ay) - \frac{\sinh(ay)}{y} = y^{-1} e^{-ay}.$$

Examples

$$I := \int_0^\infty xJ_0(xy)\frac{dx}{\sqrt{a^2+x^2}} = y^{-1}e^{-ay}$$

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Example

$I = \int_0^\infty e^{-x} dx = 1$

$$I = \int_0^\infty \sum_n \phi_n x^n dx = \sum_n \phi_n \langle n+1 \rangle = \Gamma(-(-1)) = 1.$$

On the other hand

$$e^{-x} = e^{-\frac{x}{3}} e^{-\frac{2x}{3}} \quad \left(e^{-ax} e^{-bx},\ a+b=1\right)$$

$$I = \int_0^\infty \left(\sum_{n_1} \phi_{n_1} \frac{x^{n_1}}{3^{n_1}}\right) \left(\sum_{n_2} \phi_{n_2} \frac{2^{n_2} x^{n_2}}{3^{n_2}}\right) dx = \sum_{n_1, n_2} \phi_{1,2} \frac{2^{n_2}}{3^{n_1+n_2}} \langle n_1 + n_2 + 1 \rangle$$

$$I = \begin{cases} n_2^* = -1 - n_1 : & \sum_{n_1} \phi_{n_1} \frac{3}{2^{n_1+1}} \Gamma(n_1+1) = \frac{3}{2} \cdot \sum_{n_1} \left(-\frac{1}{2}\right)^{n_1} = 1; \\ n_1^* = -1 - n_2 : & \sum_{n_2} \phi_{n_2} 3 \cdot 2^{n_2} \Gamma(n_2+1) = 3 \cdot \sum_{n_2} (-2)^{n_2} \overset{AC}{=} 1. \end{cases}$$

Example

$I=\int_0^\infty e^{-x}dx=1$

$$I=\int_0^\infty \sum_n \phi_n x^n dx=\sum_n \phi_n \langle n+1\rangle=\Gamma(-(-1))=1.$$

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$$e^{-x}=e^{-\frac{x}{3}}e^{-\frac{2x}{3}}\quad \left(e^{-ax}e^{-bx},\ a+b=1\right)$$

$$I=\int_0^\infty \left(\sum_{n_1}\phi_{n_1}\frac{x^{n_1}}{3^{n_1}}\right)\left(\sum_{n_2}\phi_{n_2}\frac{2^{n_2}x^{n_2}}{3^{n_2}}\right)dx=\sum_{n_1,n_2}\phi_{1,2}\frac{2^{n_2}}{3^{n_1+n_2}}\langle n_1+n_2+1\rangle$$

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Theorem (L. J.)

Assume that $f(x)$ admits a representation of the form

$$f(x) = \prod_{i=1}^{r} f_i(x).$$

Then, the values of the following two integrals

$$I_1 = \int_0^\infty f(x)\,dx \text{ and } I_2 = \int_0^\infty \prod_{i=1}^{r} f_i(x)\,dx,$$

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Implementation

Karen Kohl–***Sage+Mathematica***

Ivan Gonzalez–***Maple***

Mathematica

Implementation

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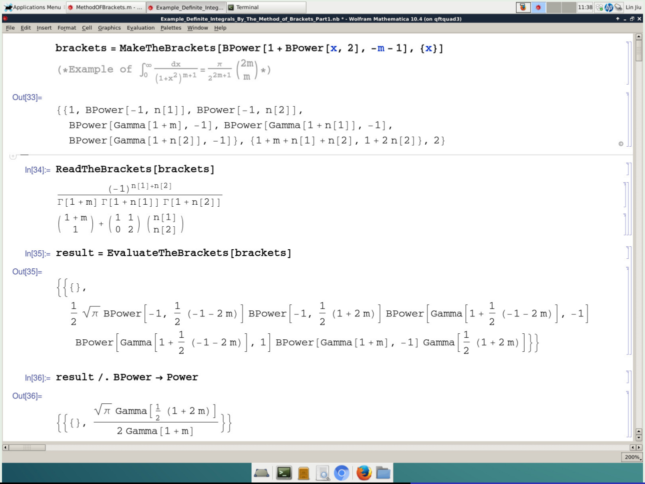
$$\int_0^\infty \frac{dx}{(1+x^2)^{m+1}} = \frac{\pi}{2^{2m+1}}\binom{2m}{m}$$



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Future Work

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$$I := \int_0^\infty xJ_0(xy)\frac{dx}{\sqrt{a^2+x^2}} = \frac{1}{y}\cosh(ay) - \frac{\sinh(ay)}{y} = y^{-1}e^{-ay}.$$

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End

Thank You!