Computer algebra tools for integrals

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Outline

Introduction

Indefinite integration

Antiderivatives

$$\int f(x)\,dx = g(x)$$

Examples

$$\int \frac{\mathrm{Li}_3(x) - x\mathrm{Li}_2(x)}{(1-x)^2}dx = \frac{x}{1-x}(\mathrm{Li}_3(x) - \mathrm{Li}_2(x)) + \frac{\ln(1-x)^2}{2}$$

$$\int \mathrm{Ai}'(x)^2\,dx = \tfrac{1}{3}\left(x\mathrm{Ai}'(x)^2 + 2\mathrm{Ai}(x)\mathrm{Ai}'(x) - x^2\mathrm{Ai}(x)^2\right)$$

$$\int \frac{1}{xJ_n(x)Y_n(x)}\,dx = \frac{\pi}{2}\ln\left(\frac{Y_n(x)}{J_n(x)}\right)$$

Definite integration

Integrals depending on parameters

$$\int_a^b f(x,y)\,dx = g(y)$$

Examples

$$\begin{aligned}\int_0^\infty \frac{zx}{e^x - z}\,dx &= \mathrm{Li}_2(z)\\ \int_0^\infty e^{-sx}\gamma(a,x)\,dx &= \frac{\Gamma(a)}{s(s+1)^a}\\ \int_0^1 xP_n(1-2x^2)J_0(yx)\,dx &= \frac{J_{2n+1}(y)}{y}\end{aligned}$$

Parametric integration

Compute linear relation of integrals

Given $f(x)$, find $g(x)$ s.t.

$$f(x) \qquad = g'(x)$$

Lift this to a relation of corresponding integrals

$$\int_a^b f(x)dx \qquad = g(b) - g(a)$$

Parametric integration

Compute linear relation of integrals

Given $f_0(x), \ldots, f_m(x)$, find $g(x)$ s.t.

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Parametric integration

Compute linear relation of integrals

Given $f_0(x), \dots, f_m(x)$, find $g(x)$ and $c_0, \dots, c_m$ const. w.r.t. x s.t.

$$f\,(x) \qquad = g'(x)$$

Lift this to a relation of corresponding integrals

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Given $f_0(x), \dots, f_m(x)$, find $g(x)$ and $c_0, \dots, c_m$ const. w.r.t. x s.t.

$$c_0 f_0(x) + \cdots + c_m f_m(x) = g'(x)$$

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Compute linear relation of integrals

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Certificate

$$c_0 \int_a^b f_0(x)dx + \cdots + c_m \int_a^b f_m(x)dx = r$$

Parametric integration

Compute linear relation of integrals

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Lift this to a relation of corresponding integrals

$$c_0 \int_a^b f_0(x)dx + \dots + c_m \int_a^b f_m(x)dx = g(b) - g(a)$$

Certificate

$g(x)$ is a certificate for the relation

$$c_0 \int_a^b f_0(x)dx + \dots + c_m \int_a^b f_m(x)dx = r$$

It is easy to verify

$$c_0 f_0(x) + \dots + c_m f_m(x) = g'(x) \quad \text{and} \quad r = g(b) - g(a)$$

Linear Relations

Integrals depending on one parameter

- $c_0(y)f(x,y) + \cdots + c_m(y)\frac{\partial^m f}{\partial y^m}(x,y) = \frac{d}{dx}g(x,y)$ yields an ODE for
$$I(y) := \int_a^b f(x,y)dx$$
- $c_0(n)f(x,n) + \cdots + c_m(n)f(x,n+m) = \frac{d}{dx}g(x,n)$ yields a recurrence for
$$I(n) := \int_a^b f(x,n)dx$$

Zeilberger-type algorithms

History

Ansatz for integrals of special functions [Sonine 1880]

$$J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x)$$
$$J_n'(x) = -\frac{n}{x}J_n(x) + J_{n-1}(x)$$

$$\int f(x)x^{n+1}J_n(x)\,dx =$$

History

Ansatz for integrals of special functions [Sonine 1880]

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$$\int f(x) x^{n+1} J_n(x)\, dx = x^{n+1} \left(g(x) J_{n+1}(x) + h(x) J_n(x)\right)$$

History

Ansatz for integrals of special functions [Sonine 1880]

$$
\begin{aligned}
J_{n+1}(x) &= \tfrac{2n}{x} J_n(x) - J_{n-1}(x) \\
J_n'(x) &= -\tfrac{n}{x} J_n(x) + J_{n-1}(x)
\end{aligned}
$$

If $g''(x) + \frac{2n+1}{x} g'(x) + g(x) = f(x)$, then

$$\int f(x) x^{n+1} J_n(x)\,dx = x^{n+1} \left(g(x) J_{n+1}(x) + g'(x) J_n(x) \right)$$

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Generalization

- Ansatz for product of special functions [Piquette, Van Buren 1984]
- Holonomic systems approach, creative telescoping [Zeilberger 1990]

Algebraic formulation

Definition

Let $A = C(x, \vec{y}, \vec{n})[\frac{\partial}{\partial x}][\partial_{\vec{y}}][S_{\vec{n}}]$ an Ore algebra, a function $f(x, \vec{y}, \vec{n})$ is D-finite if

$$_A\langle f\rangle = \{L(f) \mid L \in A\}$$

is finite-dimensional over $C(x, \vec{y}, \vec{n})$

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Creative telescoping

- Given $f(x, \vec{y}, \vec{n})$ D-finite, $M := {}_A\langle f \rangle$
- Find $T \in C(\vec{y}, \vec{n})[\partial_{\vec{y}}][S_{\vec{n}}]$ and $g \in M$ s.t.

$$T(f) = \frac{\partial}{\partial x} g$$

Or with $g = L(f)$: $\quad T - \frac{\partial}{\partial x} L \in \mathrm{Ann}_A(f)$

Algorithms for creative telescoping

Hyperexponential functions

Almkvist, Zeilberger 1990, Geddes, Le, Li 2004,
Bostan, Chen, Chyzak, Li, Xin, 2013

Algebraic functions

Chen, Kauers, Singer 2012, Chen, Kauers, Koutschan 2016

D-finite functions

Zeilberger 1990, Chyzak 2000, Koutschan 2009,
Chen, Kauers, Koutschan 2014

Non-D-finite functions

Chyzak, Kauers, Salvy 2009

Demo

- `HolonomicFunctions`

  Mathematica package by Christoph Koutschan available at
  `http://www.risc.jku.at/research/combinat/software/ergosum/RISC/HolonomicFunctions.html`

- `Mgfun`

  Maple package by Frédéric Chyzak available at
  `http://algo.inria.fr/chyzak/mgfun.html`

Risch-type algorithms

History

Liouville's Theorem

[Liouville 1833/35, Rosenlicht 1968]

If $f \in F$ has an integral in an elementary extension of F, then there exist constants $c_1, \dots, c_j$ and $u_0, \dots, u_j \in F$ s.t.

$$\int f = u_0 + \sum_{i=1}^{j} c_i \log(u_i)$$

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Liouville's Theorem

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Risch's decision procedure

[Risch 1969, Bronstein 1990]

- Given an elementary function f
- Decide whether there is an elementary function g with

$$f = Dg$$

  and compute such a g if it exists

Algebraic formulation

Indefinite integration

- Given $F = C(t_1, \ldots, t_n)$ a differential field and
$$f \in F$$
- Find field extension $E \supseteq F$ of certain type and $g \in E$ s.t.
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Main variants

- $E = F$ (in-field integration)
- E any elementary extension of F (elementary integration)
- …

Algebraic formulation

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Elementary extensions

$E = F(\theta_1, \dots, \theta_n)$ is called an elementary extension of F if each θ_i is elementary over $F_{i-1} := F(\theta_1, \dots, \theta_{i-1})$, i.e.

- θ_i is algebraic over F_{i-1}, or
- $D\theta_i = \frac{Da}{a}$ for some $a \in F_{i-1}$, i.e. $\theta_i = \log(a)$, or
- $\frac{D\theta_i}{\theta_i} = Da$ for some $a \in F_{i-1}$, i.e. $\theta_i = \exp(a)$

Algorithms for parametric integration

Elementary integrands

Risch 1969, Mack 1976

Liouvillian integrands

Singer, Saunders, Caviness 1985

non-Liouvillian integrands

Bronstein 1990/97, CGR 2012

Decision procedure

Recall: parametric integration

- Given a differential field (F, D) and $f_0, \ldots, f_m \in F$
- Find all $c_0, \ldots, c_m \in \mathrm{Const}(F)$ s.t.

$$c_0 f_0 + \cdots + c_m f_m = Dg$$

has an elementary integral over (F, D) and compute such g

Recursive reduction algorithm

Exploit tower structure: focus on topmost generator only

1. integrands from $K(t_n) = C(t_1, \ldots, t_n)$

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3. subtract its derivative $\Rightarrow$ remaining integrands are from $K = C(t_1, \dots, t_{n-1})$

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2. compute parts of the integral involving t_n
3. subtract its derivative $\Rightarrow$ remaining integrands are from $K = C(t_1, \dots, t_{n-1})$
4. proceed recursively with the smaller tower

Avoid recursive computations

Principle

Treat the generators of $F = C(t_1, \dots, t_n)$ all at once

1. Ansatz with $c_1, \dots, c_m \in C$ and $u_0, v_0, \dots, v_m \in C[t_1, \dots, t_n]$

$$\int f = \frac{u_0}{v_0} + \sum_i c_i \log(v_i)$$

where only u_0 has undetermined coefficients

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$$\int f = \frac{u_0}{v_0} + \sum_i c_i \log(v_i)$$

   where only u_0 has undetermined coefficients

2. Differentiate

$$f = \frac{Du_0}{v_0} + \frac{u_0 Dv_0}{v_0^2} + \sum_i c_i \frac{Dv_i}{v_i}$$

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2. Differentiate

$$f = \frac{Du_0}{v_0} + \frac{u_0 Dv_0}{v_0^2} + \sum_i c_i \frac{Dv_i}{v_i}$$

3. Compare coefficients to obtain linear algebraic system over C for c_i and coefficients of u_0

Algorithms

Elementary functions

Risch, Norman 1976, Norman, Moore 1977, Davenport 1982

Special functions

Fitch 1981, Bronstein 2007, Boettner 2010

Demo

- `Integrator`

  Mathematica package by CGR not released yet

- `pmint`

  Maple code by Manuel Bronstein available at
  `http://www-sop.inria.fr/cafe/Manuel.Bronstein/pmint/`

Rule-based algorithms

Overview

Integral tables and other standard references

Hirsch 1810, Whittaker, Watson, 1902/27,
Gradshteyn, Ryzhik 1943/2014, Magnus, Oberhettinger 1943/66,
Gröbner, Hofreiter 1949/75, Erdélyi et al. 1953/55,
Abramowitz, Stegun 1964/72,
Prudnikov, Brychkov, Marichev 1981/2003, Olver et al. 2010, ...

Reduction rules

Rich, Jeffrey 2009

Demo

- RuBI

  Mathematica code by Albert Rich available at
  http://www.apmaths.uwo.ca/~arich/

Examples: reduction rules for definite integrals

- $$\int_x^1 \frac{f(t)}{(t-c)\sqrt{t-x}}\,dt = \\ = \frac{1}{\sqrt{x-c}}\int_x^1 \frac{1}{\sqrt{t-c}}\left(\frac{f(1)}{\sqrt{1-t}} - \int_t^1 \frac{f'(u)}{\sqrt{u-t}}\,du\right)dt$$

- $$\int_0^1 x^n \frac{f(x)}{\sqrt{x-a}}\,dx = \\ = \frac{(4a)^n}{(2n+1)\binom{2n}{n}}\left(\int_0^1 \frac{f(x)}{\sqrt{x-a}}\,dx + 2\sum_{i=1}^{n} \frac{\binom{2i}{i}}{(4a)^i}\left(\sqrt{1-a}f(1) - \int_0^1 x^i\sqrt{x-a}f'(x)\,dx\right)\right)$$

[Ablinger, Blümlein, CGR, Schneider 2014]