Additive Decompositions in Primitive Extensions

Shaoshi Chen

Key Laboratory of Mathematics Mechanization
AMSS, Chinese Academy of Sciences

ICMS'18, University of Notre Dame
24–27 July 2018

Additive Decompositions in Primitive Extensions

Shaoshi Chen

Key Laboratory of Mathematics Mechanization
AMSS, Chinese Academy of Sciences

ICMS'18, University of Notre Dame
24–27 July 2018

Joint work with Hao Du and Ziming Li

Outline

- Additive decomposition problem
- Previous results
- Additive decompositions in primitive extensions
  - Hermite reduction
  - Polynomial reduction
- Applications

Terminologies

Let F be a field of characteristic zero.

- A derivation on F is a map $' : F \to F$ s.t. for all $a, b \in F$,

$$(a+b)' = a' + b' \quad \text{and} \quad (ab)' = ab' + a'b.$$

- $(F, ')$ is a differential field.
- $C_F = \{a \in F \mid a' = 0\}$ is the subfield of constants.
- A differential field (E, D) is a differential extension of F if

$$F \subseteq E \quad \text{and} \quad D\mid_F = '.$$

Terminologies

Let F be a field of characteristic zero.

- A derivation on F is a map $' : F \to F$ s.t. for all $a, b \in F$,

$$(a+b)' = a' + b' \quad \text{and} \quad (ab)' = ab' + a'b.$$

- $(F, \ ')$ is a differential field.
- $C_F = \{a \in F \mid a' = 0\}$ is the subfield of constants.
- A differential field (E, D) is a differential extension of F if

$$F \subseteq E \quad \text{and} \quad D\mid_F = \ '.$$

Example. Set $' = d/dx$.

$$\mathbb{C}(x), \ \mathbb{C}(x, \log(x)), \ \mathbb{C}(x, e^x), \ \mathbb{C}(x, \sqrt{x}), \ \ldots$$

are differential fields.

Additive decomposition problem

Notation. $F' := \{f' \mid f \in F\}$.

Problem. Given $f \in F$, find $g, r \in F$ s.t.

$$f = g' + r$$

with the properties that

- $f \in F' \iff r = 0$,
- r is minimal in some sense.

Previous results

- Rational functions in $\mathbb{C}(x)$ (Ostrogradsky 1845, Hermite 1872)
- Rational functions in $\mathbb{C}(x_1, \ldots, x_n)$ (Bostan, Lairez and Salvy 2013)
- Hyperexponential functions over $\mathbb{C}(x)$ (Bostan, Chen, Chyzak, Li and Xin 2013)
- Algebraic functions over $\mathbb{C}(x)$ (Chen, Kauers, Koutschan 2016)
- Fuchsian D-finite functions over $\mathbb{C}(x)$ (Chen, van Hoeij, Kauers, Koutschan 2017)
- D-finite functions over $\mathbb{C}(x)$ (van der Hoeven 2017, 2018, Bostan, Chyzak, Lairez and Salvy 2018)

Primitive towers

Definition. Let $(F, ') \subset (E, ')$. $t \in E$ is a primitive monomial if $t' \in F$, t is transcendental over F and $C_{F(t)} = C_F$.

Examples.

- $\log(x)$ and $\arctan(x)$ are primitive monomials over $\mathbb{C}(x)$,
- $\mathrm{Li}(x) := \int \frac{dx}{\log(x)}$ is a primitive monomial over $\mathbb{C}(x, \log(x))$.

Primitive towers

Definition. Let $(F, ') \subset (E, ')$. $t \in E$ is a primitive monomial if $t' \in F$, t is transcendental over F and $C_{F(t)} = C_F$.

Examples.

- $\log(x)$ and $\arctan(x)$ are primitive monomials over $\mathbb{C}(x)$,
- $\mathrm{Li}(x) := \int \frac{dx}{\log(x)}$ is a primitive monomial over $\mathbb{C}(x, \log(x))$.

A primitive tower is

$$\begin{array}{ccccccc} F_0 & \subset & F_1 & \subset & \cdots & \subset & F_n \\ \shortparallel & & \shortparallel & & & & \shortparallel \\ \mathbb{C}(x) & & F_0(t_1) & & & & F_{n-1}(t_n) \end{array}$$

where t_i is a primitive monomial over F_{i-1} for all $1 \leq i \leq n$.

Hermite reduction

Definition. Given a primitive tower $F_0 \subset \cdots \subset F_n$,

- $p \in F_{n-1}[t_n]$ is t_n-normal if $\gcd(p, p') \in F_{n-1}$;
- $f \in F_n$ is t_n-simple if f is proper and $\mathrm{den}(f)$ is t_n-normal.

Hermite reduction

Definition. Given a primitive tower $F_0 \subset \cdots \subset F_n$,

- $p \in F_{n-1}[t_n]$ is t_n-normal if $\gcd(p, p') \in F_{n-1}$;
- $f \in F_n$ is t_n-simple if f is proper and $\mathrm{den}(f)$ is t_n-normal.

Lemma. For $f \in F_n$, there exist $g, h \in F_n$ and $p \in F_{n-1}[t_n]$ s.t.

$$f = g' + h + p.$$

where h is t_n-simple. Moreover,

$$f \in F_n' \implies h = 0.$$

Polynomial reduction

Problem P. For $p \in F_{n-1}[t_n]$, find $g, q \in F_{n-1}[t_n]$ s.t.

$$p = g' + q,$$

with the properties that

- $p \in F_n' \;\Leftrightarrow\; q = 0,$
- q is minimal in some sense.

Polynomial reduction

Problem P. For $p \in F_{n-1}[t_n]$, find $g, q \in F_{n-1}[t_n]$ s.t.

$$p = g' + q,$$

with the properties that

- $p \in F_n' \;\Leftrightarrow\; q = 0$,
- q is minimal in some sense.

Main idea. For $a \in F_{n-1}$ and $d \in \mathbb{N}$,

$$a\, t_n^d = g' + q \quad \text{with } \deg_{t_n}(q) < d.$$
$$\Updownarrow$$
$$a - c\, t_n' \in F_{n-1}' \quad \text{for some } c \in \mathbb{C}.$$

Hermitian parts

By Hermite reduction, for $f \in F_i$, $\exists!$ t_i-simple $h \in F_i$ s.t.

$$f = g' + h + p,$$

where $g \in F_i$ and $p \in F_{i-1}[t_i]$ for $1 \leq i \leq n$.

Definition. Call h the Hermitian part of f, denoted by $\mathrm{hp}_{t_i}(f)$.

Hermitian parts

By Hermite reduction, for $f \in F_i$, $\exists!$ t_i-simple $h \in F_i$ s.t.

$$f = g' + h + p,$$

where $g \in F_i$ and $p \in F_{i-1}[t_i]$ for $1 \leq i \leq n$.

Definition. Call h the Hermitian part of f, denoted by $\mathrm{hp}_{t_i}(f)$.

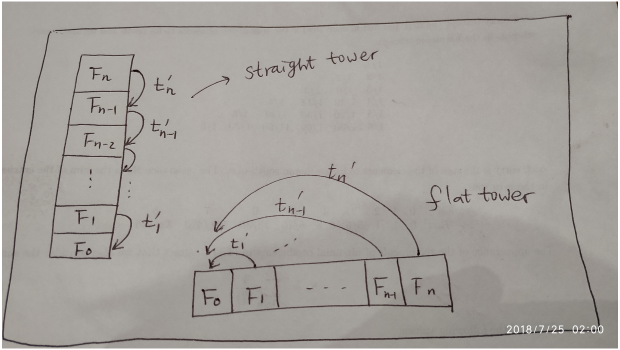
If $a - c\, t_n' \in F_{n-1}'$ and $\mathrm{hp}_{t_{n-1}}(t_n') \neq 0$, then

$$c = \frac{\mathrm{hp}_{t_{n-1}}(a)}{\mathrm{hp}_{t_{n-1}}(t_n')}.$$

Straight and flat towers

Definition. A primitive tower $F_0 \subset \cdots \subset F_n$ with $F_0 = \mathbb{C}(t_0)$ is

- straight if $\mathrm{hp}_{t_{i-1}}(t_i') \neq 0$ for all $1 \leq i \leq n$.
- flat if $t_i' \in F_0$ for all $1 \leq i \leq n$.



Straight polynomials

Definition. A polynomial $q \in F_{n-1}[t_n]$ is t_n-straight if

- q is t_0-straight if $q = 0$,
- q is t_n-straight if $\mathrm{lc}_{t_n}(q) = u + v$ s.t.
  - $u \in F_{n-1}$ is t_{n-1}-simple,
  - $u \neq c\,\mathrm{hp}_{t_{n-1}}(t_n')$ for any nonzero $c \in \mathbb{C}$,
  - $v \in F_{n-2}[t_{n-1}]$ is t_{n-1}-straight.

Straight polynomials

Definition. A polynomial $q \in F_{n-1}[t_n]$ is t_n-straight if

- q is t_0-straight if $q = 0$,
- q is t_n-straight if $\mathrm{lc}_{t_n}(q) = u + v$ s.t.
  - $u \in F_{n-1}$ is t_{n-1}-simple,
  - $u \neq c\,\mathrm{hp}_{t_{n-1}}(t_n')$ for any nonzero $c \in \mathbb{C}$,
  - $v \in F_{n-2}[t_{n-1}]$ is t_{n-1}-straight.

Prop. Let $q \in F_{n-1}[t_n]$ be t_n-straight. Then

$$q \in F_n' \quad \Leftrightarrow \quad q = 0.$$

Associated sequences

Definition. Let $p \in F_{n-1}[t_n]$. A sequence $(p_0, p_1, \ldots, p_{n-1})$ is called a associated sequence of p if

$$p = p_0 + p_1 + \cdots + p_{n-1},$$

where $p_i \in F_i[t_{i+1}, \ldots, t_n]$ for all i with $0 \leq i \leq n-1$.

Example. Consider the flat tower

$$\mathbb{C}(t_0) \subset \mathbb{C}(t_0, t_1) \subset \mathbb{C}(t_0, t_1, t_2) \subset \mathbb{C}(t_0, t_1, t_2, t_3),$$

where $t_0 = x$, $t_1 = \log(x)$, $t_2 = \log(x+1)$ and $t_3 = \log(x+2)$.

$$p = \underbrace{\frac{1}{x+1} t_3^3 + x t_2 t_3}_{p_0} + \underbrace{\frac{1}{t_1} t_2 t_3}_{p_1} + \underbrace{\frac{1}{t_2} t_3^2}_{p_2},$$

Flat polynomials

Let $q \in F_{n-1}[t_n]$ and $(q_0, \ldots, q_{n-1})$ be the associated sequence.

Definition. q is t_n-flat if:

- $\mathrm{hc}_i(q_i)$ is t_{i-1}-simple for $1 \leq i \leq n-1$,
- $q_1 = 0$ or $\mathrm{hc}_0(q_1) \notin \mathrm{span}_{\mathbb{C}}\{t_1', \ldots, t_m'\}$ where

$$m = \begin{cases} n & \text{if } \mathrm{hm}_0(q_1) = 1, \\ s & \text{if } \mathrm{hm}_0(q_1) = t_s^{e_s} \cdots t_n^{e_n} \text{ with } e_s > 0 \end{cases}$$

Flat polynomials

Let $q \in F_{n-1}[t_n]$ and $(q_0, \ldots, q_{n-1})$ be the associated sequence.

Definition. q is t_n-flat if:

- $\mathrm{hc}_i(q_i)$ is t_{i-1}-simple for $1 \leq i \leq n-1$,
- $q_1 = 0$ or $\mathrm{hc}_0(q_1) \notin \mathrm{span}_{\mathbb{C}}\{t_1', \ldots, t_m'\}$ where

$$m = \begin{cases} n & \text{if } \mathrm{hm}_0(q_1) = 1, \\ s & \text{if } \mathrm{hm}_0(q_1) = t_s^{e_s} \cdots t_n^{e_n} \text{ with } e_s > 0 \end{cases}$$

Prop. Let $q \in F_{n-1}[t_n]$ be t_n-flat. Then

$$q \in F_n' \quad \Leftrightarrow \quad q = 0.$$

The main result

Theorem. Given a straight (flat) tower $F_0 \subset \cdots \subset F_n$ and $f \in F_n$, there are $g \in F_n$ and $q \in F_{n-1}[t_n]$ s.t.

$$f = \underbrace{g'}_{\text{integrable}} + \underbrace{\mathrm{hp}_{t_n}(f) + q}_{\text{non-integrable}},$$

where q is t_n-straight (t_n-flat).

Moreover,

- $f \in F_n' \quad \Leftrightarrow \quad \mathrm{hp}_{t_n}(f) = q = 0,$
- if $f = \tilde{g}' + \tilde{h} + \tilde{q}$ for t_n-proper $\tilde{h}$ and $\tilde{q} \in F_{n-1}[t_n]$, then

$$\mathrm{den}(\mathrm{hp}_{t_n}(f)) \mid \mathrm{den}(\tilde{h}) \quad \text{and} \quad \begin{cases} \deg_{t_n}(q) \le \deg_{t_n}(\tilde{q}) & \text{(straight)} \\ q \preceq_{\mathrm{plex}} \tilde{q} & \text{(flat).} \end{cases}$$

Examples

1. Straight:

$$f_1 = \frac{1}{\log(x)\mathrm{Li}(x)} + \left(\log(x) + \frac{1}{\log(x)}\right)\mathrm{Li}(x) - \frac{x}{\log(x)} \in \mathbb{C}(x, \log(x), \mathrm{Li}(x))$$

2. Flat:

$$f_2 = \left(\frac{\arctan(x)}{x^2+1}\right)^3 - \frac{\log(x)\arctan(x)^2}{x} + \log(x)^2 \in \mathbb{C}(x, \log(x), \arctan(x))$$

Examples

1. Straight:

$$f_1 = \frac{1}{\log(x)\mathrm{Li}(x)} + \left(\log(x) + \frac{1}{\log(x)}\right)\mathrm{Li}(x) - \frac{x}{\log(x)} \in \mathbb{C}(x, \log(x), \mathrm{Li}(x))$$

$$= (\cdots)' + \frac{1}{\log(x)\mathrm{Li}(x)}$$

2. Flat:

$$f_2 = \left(\frac{\arctan(x)}{x^2+1}\right)^3 - \frac{\log(x)\arctan(x)^2}{x} + \log(x)^2 \in \mathbb{C}(x, \log(x), \arctan(x))$$

$$= (\ldots)' + \frac{1}{x^2+1}\log(x)^2\arctan(x)$$

Application I: elementary integrability

Given a straight (flat) tower $F_0 \subset \cdots \subset F_n$ and $f \in F_n$, we have

$$f = g' + \underbrace{\mathrm{hp}_{t_n}(f) + q}_{r}.$$

Application I: elementary integrability

Given a straight (flat) tower $F_0 \subset \cdots \subset F_n$ and $f \in F_n$, we have

$$f = g' + \underbrace{\mathrm{hp}_{t_n}(f) + q}_{r}.$$

If t_i is logarithmic over F_{i-1} for all $1 \leq i \leq n$, then

$$f \text{ is elementary integrable over } F_n$$

$$\Updownarrow$$

$$r \in \mathrm{span}_{\mathbb{C}}\{a'/a \mid a \in F_n\}.$$

Application I: elementary integrability

Given a straight (flat) tower $F_0 \subset \cdots \subset F_n$ and $f \in F_n$, we have

$$f = g' + \underbrace{\mathrm{hp}_{t_n}(f) + q}_{r}.$$

If t_i is logarithmic over F_{i-1} for all $1 \le i \le n$, then

$$f \text{ is elementary integrable over } F_n$$

$$\Updownarrow$$

$$r \in \mathrm{span}_{\mathbb{C}}\{a'/a \mid a \in F_n\}.$$

Example.

$$f_1 = (\cdots)' + \frac{1}{\log(x)\mathrm{Li}(x)} = (\cdots)' + \frac{\mathrm{Li}(x)'}{\mathrm{Li}(x)} = (\cdots)' + (\log \circ \mathrm{Li}(x))'$$

Application II: creative telescoping

$(F, \{D_x, D_y\})$: a differential field with $D_x D_y = D_y D_x$.

Problem. Given $f \in F$, find nonzero $L := \sum_{i=0}^{d} \ell_i D_x^i$ with $D_y(\ell_i) = 0$ and g in an elementary extension E of F s.t.

$$\underbrace{L(x, D_x)}_{\text{Telescoper}}(f) = D_y(\underbrace{g}_{\text{Certificate}})$$

Application II: creative telescoping

$(F, \{D_x, D_y\})$: a differential field with $D_x D_y = D_y D_x$.

Problem. Given $f \in F$, find nonzero $L := \sum_{i=0}^{d} \ell_i D_x^i$ with $D_y(\ell_i) = 0$ and g in an elementary extension E of F s.t.

$$L(x, D_x)(f) = D_y(g)$$

$\swarrow$ Telescoper $\qquad$ $\searrow$ Certificate

Example. $t := \log(x^2 + y^2)$.

$$f = \underbrace{t + 1 - \frac{2y}{(x^2+y^2)t^2}}_{\text{NOT } D\text{-finite}}$$

Application II: creative telescoping

$(F, \{D_x, D_y\})$: a differential field with $D_x D_y = D_y D_x$.

Problem. Given $f \in F$, find nonzero $L := \sum_{i=0}^{d} \ell_i D_x^i$ with $D_y(\ell_i) = 0$ and g in an elementary extension E of F s.t.

$$L(x, D_x)(f) = D_y(g)$$

Telescoper　　　　Certificate

Example. $t := \log(x^2 + y^2)$.

$$f = \underbrace{t + 1 - \frac{2y}{(x^2+y^2)t^2}}_{\text{NOT } D\text{-finite}} = D_y\left(\frac{1}{t} + yt - y\right) + \frac{2x^2}{x^2+y^2}$$

Application II: creative telescoping

$(F, \{D_x, D_y\})$: a differential field with $D_x D_y = D_y D_x$.

Problem. Given $f \in F$, find nonzero $L := \sum_{i=0}^{d} \ell_i D_x^i$ with $D_y(\ell_i) = 0$ and g in an elementary extension E of F s.t.

$$L(x, D_x)(f) = D_y(g)$$

Telescoper   Certificate

Example. $t := \log(x^2 + y^2)$.

$$f = \underbrace{t + 1 - \frac{2y}{(x^2+y^2)t^2}}_{\text{NOT } D\text{-finite}} = D_y\left(\frac{1}{t} + yt - y\right) + \frac{2x^2}{x^2+y^2}$$

$$\Downarrow$$

$$L = xD_x - 1 \quad \text{and} \quad g = \frac{-2x^2}{t^2(x^2+y^2)} - \frac{1}{t} - yt + y.$$

Summary

Result. Additive decompositions in straight or flat towers.

Summary

Result. Additive decompositions in straight or flat towers.

Plan.

- The general primitive case
- The hyperexponential case
- Creative telescoping for elementary functions

Summary

Result. Additive decompositions in straight or flat towers.

Plan.

- The general primitive case
- The hyperexponential case
- Creative telescoping for elementary functions

Thank you!