

Bernoulli Symbol and Sum of Powers

Lin Jiu

Dalhousie University
ICMS @ University of Notre Dame

July 25 2018

Joint Work

Christophe Vignat



Bernoulli Numbers & Bernoulli Polynomials

Bernoulli Numbers & Bernoulli Polynomials

Bernoulli numbers B_n and Bernoulli polynomials $B_n(x)$

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \quad \text{and} \quad \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$

Bernoulli Numbers & Bernoulli Polynomials

Bernoulli numbers B_n and Bernoulli polynomials $B_n(x)$

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \quad \text{and} \quad \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$

Examples

$$1^n + 2^n + \cdots + N^n =$$

Bernoulli Numbers & Bernoulli Polynomials

Bernoulli numbers B_n and Bernoulli polynomials $B_n(x)$

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \quad \text{and} \quad \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$

Examples

$$1^n + 2^n + \cdots + N^n = \frac{1}{n+1} \sum_{i=1}^{n+1} \binom{n+1}{i} B_{n+1-i} N^i = \frac{B_{n+1}(N+1) - B_{n+1}}{n+1}.$$

Bernoulli Numbers & Bernoulli Polynomials

Bernoulli numbers B_n and Bernoulli polynomials $B_n(x)$

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \quad \text{and} \quad \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$

Examples

$$1^n + 2^n + \dots + N^n = \frac{1}{n+1} \sum_{i=1}^{n+1} \binom{n+1}{i} B_{n+1-i} N^i = \frac{B_{n+1}(N+1) - B_{n+1}}{n+1}.$$

Riemann-zeta: for $n \in \mathbb{Z}_+$

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n} B_{2n}}{2(2n)!}$$

Bernoulli Numbers & Bernoulli Polynomials

Bernoulli numbers B_n and Bernoulli polynomials $B_n(x)$

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \quad \text{and} \quad \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$

Examples

$$1^n + 2^n + \dots + N^n = \frac{1}{n+1} \sum_{i=1}^{n+1} \binom{n+1}{i} B_{n+1-i} N^i = \frac{B_{n+1}(N+1) - B_{n+1}}{n+1}.$$

Riemann-zeta: for $n \in \mathbb{Z}_+$

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n} B_{2n}}{2(2n)!}, \quad \zeta(-n) = -\frac{B_{n+1}}{n+1}$$

Bernoulli Numbers & Bernoulli Polynomials

Bernoulli numbers B_n and Bernoulli polynomials $B_n(x)$

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \quad \text{and} \quad \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$

Examples

$$1^n + 2^n + \dots + N^n = \frac{1}{n+1} \sum_{i=1}^{n+1} \binom{n+1}{i} B_{n+1-i} N^i = \frac{B_{n+1}(N+1) - B_{n+1}}{n+1}.$$

Riemann-zeta: for $n \in \mathbb{Z}_+$

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n} B_{2n}}{2(2n)!}, \quad \zeta(-n) = -\frac{B_{n+1}}{n+1} = (-1)^n \frac{B_{n+1}}{n+1}.$$

Bernoulli Numbers & Bernoulli Polynomials

Bernoulli numbers B_n and Bernoulli polynomials $B_n(x)$

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \quad \text{and} \quad \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$

Examples

$$1^n + 2^n + \dots + N^n = \frac{1}{n+1} \sum_{i=1}^{n+1} \binom{n+1}{i} B_{n+1-i} N^i = \frac{B_{n+1}(N+1) - B_{n+1}}{n+1}.$$

Riemann-zeta: for $n \in \mathbb{Z}_+$

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n} B_{2n}}{2(2n)!}, \quad \zeta(-n) = -\frac{B_{n+1}}{n+1} = (-1)^n \frac{B_{n+1}}{n+1}.$$

$$B_{2n+1} = 0$$

Bernoulli Symbol (Umbral)

Bernoulli Symbol (Umbral)

Key Idea:

$\mathcal{B}^n \mapsto B_n$: i.e., super index \leftrightarrow lower index.

Bernoulli Symbol (Umbral)

Key Idea:

$\mathcal{B}^n \mapsto B_n$: i.e., super index \leftrightarrow lower index.

Why?

Bernoulli Symbol (Umbral)

Key Idea:

$\mathcal{B}^n \mapsto B_n$: i.e., super index \leftrightarrow lower index.

Why?

$$B_n(x) = \sum_{k=0}^n \binom{n}{k} B_{n-k} x^k = (\mathcal{B} + x)^n.$$

Bernoulli Symbol (Umbral)

Key Idea:

$\mathcal{B}^n \mapsto B_n$: i.e., super index \leftrightarrow lower index.

Why?

$$B_n(x) = \sum_{k=0}^n \binom{n}{k} B_{n-k} x^k = (\mathcal{B} + x)^n.$$

$$B'_n(x) = nB_{n-1}(x)$$

Bernoulli Symbol (Umbral)

Key Idea:

$\mathcal{B}^n \mapsto B_n$: i.e., super index \leftrightarrow lower index.

Why?

$$B_n(x) = \sum_{k=0}^n \binom{n}{k} B_{n-k} x^k = (\mathcal{B} + x)^n.$$

$$B'_n(x) = nB_{n-1}(x) \Leftrightarrow [(\mathcal{B} + x)^n]' = n(\mathcal{B} + x)^{n-1}.$$

Bernoulli Symbol (Umbral)

Key Idea:

$\mathcal{B}^n \mapsto B_n$: i.e., super index \leftrightarrow lower index.

Why?

$$B_n(x) = \sum_{k=0}^n \binom{n}{k} B_{n-k} x^k = (\mathcal{B} + x)^n.$$

$$B'_n(x) = nB_{n-1}(x) \Leftrightarrow [(\mathcal{B} + x)^n]' = n(\mathcal{B} + x)^{n-1}.$$

Probabilitistic Interpretation

Let $L_B \sim \frac{\pi}{2} \operatorname{sech}^2(\pi t)$

Bernoulli Symbol (Umbral)

Key Idea:

$\mathcal{B}^n \mapsto B_n$: i.e., super index \leftrightarrow lower index.

Why?

$$B_n(x) = \sum_{k=0}^n \binom{n}{k} B_{n-k} x^k = (\mathcal{B} + x)^n.$$

$$B'_n(x) = nB_{n-1}(x) \Leftrightarrow [(\mathcal{B} + x)^n]' = n(\mathcal{B} + x)^{n-1}.$$

Probabilitistic Interpretation

Let $L_B \sim \frac{\pi}{2} \operatorname{sech}^2(\pi t)$, then $\mathcal{B} \sim iL_B - \frac{1}{2}$

Bernoulli Symbol (Umbral)

Key Idea:

$\mathcal{B}^n \mapsto B_n$: i.e., super index \leftrightarrow lower index.

Why?

$$B_n(x) = \sum_{k=0}^n \binom{n}{k} B_{n-k} x^k = (\mathcal{B} + x)^n.$$

$$B'_n(x) = nB_{n-1}(x) \Leftrightarrow [(\mathcal{B} + x)^n]' = n(\mathcal{B} + x)^{n-1}.$$

Probabilistic Interpretation

Let $L_B \sim \frac{\pi}{2} \operatorname{sech}^2(\pi t)$, then $\mathcal{B} \sim iL_B - \frac{1}{2}$

$$B_n = \mathcal{B}^n = \mathbb{E}[B^n] = \frac{\pi}{2} \int_{\mathbb{R}} \left(it - \frac{1}{2}\right)^n \operatorname{sech}^2(\pi t) dt.$$

Bernoulli Symbol (Umbral)

Key Idea:

$\mathcal{B}^n \mapsto B_n$: i.e., super index \leftrightarrow lower index.

Why?

$$B_n(x) = \sum_{k=0}^n \binom{n}{k} B_{n-k} x^k = (\mathcal{B} + x)^n.$$

$$B'_n(x) = nB_{n-1}(x) \Leftrightarrow [(\mathcal{B} + x)^n]' = n(\mathcal{B} + x)^{n-1}.$$

Probabilitistic Interpretation

Let $L_B \sim \frac{\pi}{2} \operatorname{sech}^2(\pi t)$, then $\mathcal{B} \sim iL_B - \frac{1}{2}$

$$B_n = \mathcal{B}^n = \mathbb{E}[\mathcal{B}^n] = \frac{\pi}{2} \int_{\mathbb{R}} \left(it - \frac{1}{2}\right)^n \operatorname{sech}^2(\pi t) dt.$$

$$B_n(x) = (\mathcal{B} + x)^n = \frac{\pi}{2} \int_{\mathbb{R}} \left(x + it - \frac{1}{2}\right)^n \operatorname{sech}^2(\pi t) dt.$$

Probabilitistic Interpretation



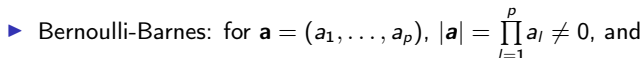
$$B_n = \mathcal{B}^n = \mathbb{E}[\mathcal{B}^n] = \frac{\pi}{2} \int_{\mathbb{R}} \left(it - \frac{1}{2}\right)^n \operatorname{sech}^2(\pi t) dt.$$



$$B_n(x) = (\mathcal{B} + x)^n = \frac{\pi}{2} \int_{\mathbb{R}} \left(x + it - \frac{1}{2}\right)^n \operatorname{sech}^2(\pi t) dt. \left(\frac{t}{e^t - 1} \mid e^{tx}\right)$$



$$\left(\frac{t}{e^t - 1}\right)^p e^{tx} = \sum_{n=0}^{\infty} B_n^{(p)}(x) \frac{t^n}{n!} \Leftrightarrow B_n^{(p)} = \left(\underbrace{\mathcal{B}_1 + \dots + \mathcal{B}_p}_{\text{i. i. d.}} + x\right)^n$$



$$\vec{\mathcal{B}} = (\mathcal{B}_1, \dots, \mathcal{B}_p),$$

$$e^{tx} \prod_{i=1}^p \frac{t}{e^{a_i t} - 1} = \sum_{n=0}^{\infty} B_n(\mathbf{a}; x) \frac{t^n}{n!} \Leftrightarrow B_n(\mathbf{a}; x) = \frac{1}{|\mathbf{a}|} \left(x + \mathbf{a} \cdot \vec{\mathcal{B}}\right)^n$$

Multiple Zeta Values

Riemann-zeta: for $n \in \mathbb{Z}_+$, the AC $\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}$.

Multiple Zeta Values

Riemann-zeta: for $n \in \mathbb{Z}_+$, the AC $\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}$.

DEF.

$$\zeta_r(n_1, \dots, n_r) = \sum_{0 < k_1 < \dots < k_r} \frac{1}{k_1^{n_1} \dots k_r^{n_r}} = \sum_{k_1, \dots, k_r=0}^{\infty} \frac{1}{k_1^{n_1} (k_1 + k_2)^{n_2} \dots (k_1 + \dots + k_r)^{n_r}}$$

Multiple Zeta Values

Riemann-zeta: for $n \in \mathbb{Z}_+$, the AC $\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}$.

DEF.

$$\zeta_r(n_1, \dots, n_r) = \sum_{0 < k_1 < \dots < k_r} \frac{1}{k_1^{n_1} \dots k_r^{n_r}} = \sum_{k_1, \dots, k_r=0}^{\infty} \frac{1}{k_1^{n_1} (k_1 + k_2)^{n_2} \dots (k_1 + \dots + k_r)^{n_r}}$$

Theorem(L. Jiu, V. H. Moll and C. Vignat)

Multiple Zeta Values

Riemann-zeta: for $n \in \mathbb{Z}_+$, the AC $\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}$.

DEF.

$$\zeta_r(n_1, \dots, n_r) = \sum_{0 < k_1 < \dots < k_r} \frac{1}{k_1^{n_1} \dots k_r^{n_r}} = \sum_{k_1, \dots, k_r=0}^{\infty} \frac{1}{k_1^{n_1} (k_1 + k_2)^{n_2} \dots (k_1 + \dots + k_r)^{n_r}}$$

Theorem(L. Jiu, V. H. Moll and C. Vignat)

$$\zeta_r(-n_1, \dots, -n_r) = \prod_{k=1}^r (-1)^{n_k} C_{1, \dots, k}^{n_k+1},$$

Multiple Zeta Values

Riemann-zeta: for $n \in \mathbb{Z}_+$, the AC $\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}$.

DEF.

$$\zeta_r(n_1, \dots, n_r) = \sum_{0 < k_1 < \dots < k_r} \frac{1}{k_1^{n_1} \dots k_r^{n_r}} = \sum_{k_1, \dots, k_r=0}^{\infty} \frac{1}{k_1^{n_1} (k_1 + k_2)^{n_2} \dots (k_1 + \dots + k_r)^{n_r}}$$

Theorem(L. Jiu, V. H. Moll and C. Vignat)

$$\zeta_r(-n_1, \dots, -n_r) = \prod_{k=1}^r (-1)^{n_k} C_{1, \dots, k}^{n_k+1},$$

where

$$C_1^n = \frac{B_1^n}{n}, C_{1,2}^n = \frac{(C_1 + B_2)^n}{n}, \dots, C_{1, \dots, k+1}^n = \frac{(C_{1, \dots, k} + B_{k+1})^n}{n}$$

Multiple Zeta Values

Riemann-zeta: for $n \in \mathbb{Z}_+$, the AC $\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}$.

DEF.

$$\zeta_r(n_1, \dots, n_r) = \sum_{0 < k_1 < \dots < k_r} \frac{1}{k_1^{n_1} \dots k_r^{n_r}} = \sum_{k_1, \dots, k_r=0}^{\infty} \frac{1}{k_1^{n_1} (k_1 + k_2)^{n_2} \dots (k_1 + \dots + k_r)^{n_r}}$$

Theorem(L. Jiu, V. H. Moll and C. Vignat)

$$\zeta_r(-n_1, \dots, -n_r) = \prod_{k=1}^r (-1)^{n_k} C_{1, \dots, k}^{n_k+1},$$

where

$$C_1^n = \frac{B_1^n}{n}, C_{1,2}^n = \frac{(C_1 + B_2)^n}{n}, \dots, C_{1, \dots, k+1}^n = \frac{(C_{1, \dots, k} + B_{k+1})^n}{n}$$

Example

$$\begin{aligned} \zeta_2(-n, 0) &= (-1)^n C_1^{n+1} \cdot (-1)^0 C_{1,2}^{0+1} & \zeta(-n) &= (-1)^n C^{n+1} = (-1)^n \frac{B_{n+1}}{n+1}. \\ &= (-1)^n \frac{C_1 + B_2}{1} \cdot C_1^{n+1} \\ &= (-1)^n (C_1^{n+2} + B_2 C_1^{n+1}) \\ &= (-1)^n \left[\frac{B_{n+2}}{n+2} - \frac{1}{2} \frac{B_{n+1}}{n+1} \right]. \end{aligned}$$

Multiple Zeta Values

Riemann-zeta: for $n \in \mathbb{Z}_+$, the AC $\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}$.

DEF.

$$\zeta_r(n_1, \dots, n_r) = \sum_{0 < k_1 < \dots < k_r} \frac{1}{k_1^{n_1} \dots k_r^{n_r}} = \sum_{k_1, \dots, k_r=0}^{\infty} \frac{1}{k_1^{n_1} (k_1 + k_2)^{n_2} \dots (k_1 + \dots + k_r)^{n_r}}$$

Theorem(L. Jiu, V. H. Moll and C. Vignat)

$$\zeta_r(-n_1, \dots, -n_r) = \prod_{k=1}^r (-1)^{n_k} C_{1, \dots, k}^{n_k+1},$$

where

$$C_1^n = \frac{B_1^n}{n}, C_{1,2}^n = \frac{(C_1 + B_2)^n}{n}, \dots, C_{1, \dots, k+1}^n = \frac{(C_{1, \dots, k} + B_{k+1})^n}{n}$$

Example

$$\begin{aligned} \zeta_2(-n, 0) &= (-1)^n C_1^{n+1} \cdot (-1)^0 C_{1,2}^{0+1} & \zeta(-n) &= (-1)^n C^{n+1} = (-1)^n \frac{B_{n+1}}{n+1}. \\ &= (-1)^n \frac{C_1 + B_2}{1} \cdot C_1^{n+1} \\ &= (-1)^n (C_1^{n+2} + B_2 C_1^{n+1}) & \text{Two different ACs.} \\ &= (-1)^n \left[\frac{B_{n+2}}{n+2} - \frac{1}{2} \frac{B_{n+1}}{n+1} \right]. \end{aligned}$$

Analytic Continuation: for n_1, \dots, n_r positive integers

Theorem(Sadaoui)

$$\zeta_r(-n_1, \dots, -n_r) = (-1)^r \sum_{k_2, \dots, k_r} \frac{1}{\bar{n} + r - \bar{k}} \prod_{j=2}^r \frac{\binom{\sum_{i=j}^r n_i - \sum_{i=j+1}^r k_i + r - j + 1}{k_j}}{\sum_{i=j}^r n_i - \sum_{i=j}^r k_i + r - j + 1}$$

$$\times \sum_{l_1, \dots, l_r} \binom{\bar{n} + r - \bar{k}}{l_1} \binom{k_2}{l_2} \dots \binom{k_r}{l_r} B_{l_1} \dots B_{l_r},$$

$$\bar{n} = \sum_{j=1}^n n_j, \quad \bar{k} = \sum_{j=2}^r k_j, \quad k_2, \dots, k_r \geq 0, \quad l_j \leq k_j \text{ for } 2 \leq j \leq r \text{ and } l_1 \leq \bar{n} + r + \bar{k}.$$

Theorem(Akiyama and Tanigawa)

$$\zeta_r(-n_1, \dots, -n_r) = -\frac{\zeta_{r-1}(-n_1, \dots, -n_{r-2}, -n_{r-1} - n_r - 1)}{1 + n_r}$$

$$- \frac{\zeta_{r-1}(-n_1, \dots, -n_{r-2}, -n_{r-1} - n_r)}{2}$$

$$+ \sum_{q=1}^{n_r} (-n_r)_q a_q \zeta_{r-1}(-n_1, \dots, -n_{r-2}, -n_{r-1} - n_r + q),$$

Analytic Continuation: for n_1, \dots, n_r positive integers

Theorem(Sadaoui)

$$\zeta_r(-n_1, \dots, -n_r) = (-1)^r \sum_{k_2, \dots, k_r} \frac{1}{\bar{n} + r - \bar{k}} \prod_{j=2}^r \frac{\binom{\sum_{i=j}^r n_i - \sum_{i=j+1}^r k_i + r - j + 1}{k_j}}{\sum_{i=j}^r n_i - \sum_{i=j}^r k_i + r - j + 1}$$

$$\times \sum_{l_1, \dots, l_r} \binom{\bar{n} + r - \bar{k}}{l_1} \binom{k_2}{l_2} \dots \binom{k_r}{l_r} B_{l_1} \dots B_{l_r},$$

$$\bar{n} = \sum_{j=1}^n n_j, \quad \bar{k} = \sum_{j=2}^r k_j, \quad k_2, \dots, k_r \geq 0, \quad l_j \leq k_j \text{ for } 2 \leq j \leq r \text{ and } l_1 \leq \bar{n} + r + \bar{k}.$$

Theorem(Akiyama and Tanigawa)

$$\zeta_r(-n_1, \dots, -n_r) = -\frac{\zeta_{r-1}(-n_1, \dots, -n_{r-2}, -n_{r-1} - n_r - 1)}{1 + n_r}$$

$$- \frac{\zeta_{r-1}(-n_1, \dots, -n_{r-2}, -n_{r-1} - n_r)}{2}$$

$$+ \sum_{q=1}^{n_r} (-n_r)_q a_q \zeta_{r-1}(-n_1, \dots, -n_{r-2}, -n_{r-1} - n_r + q),$$

Sum of Powers

$$\sum_{N > i_1 > \dots > i_r > 0} \frac{1}{i_1^{n_1} \dots i_r^{n_r}}$$

Sum of Powers

$$\sum_{N > i_1 > \dots > i_r > 0} \frac{1}{i_1^{n_1} \dots i_r^{n_r}} \left(\sum_{N > i_1 > \dots > i_r > 0} i_1^{n_1} \dots i_r^{n_r} \right)$$

Sum of Powers

$$\sum_{N > i_1 > \dots > i_r > 0} \frac{1}{i_1^{n_1} \dots i_r^{n_r}} \left(\sum_{N > i_1 > \dots > i_r > 0} i_1^{n_1} \dots i_r^{n_r} \right)$$

Faulhaber's formula

$$\sum_{k=1}^N k^n = \frac{B_{n+1}(N+1) - B_{n+1}}{n+1}, \quad n \geq 1, \quad N \geq 1$$

Sum of Powers

$$\sum_{N > i_1 > \dots > i_r > 0} \frac{1}{i_1^{n_1} \dots i_r^{n_r}} \left(\sum_{N > i_1 > \dots > i_r > 0} i_1^{n_1} \dots i_r^{n_r} \right)$$

Faulhaber's formula

$$\sum_{k=1}^N k^n = \frac{B_{n+1}(N+1) - B_{n+1}}{n+1}, \quad n \geq 1, \quad N \geq 1$$

G. H. E. Duchamp, V. H. N. Minh and N. Q. Hoan, Harmonic sums and polylogarithms at non-positive multi-indices, *J. Symbolic Comput.* **83** (2017), 166–186.

\mathcal{H} Symbol

$$(\mathcal{H}(N))^n = H_{-n}(N) = 1^n + 2^n + \cdots + (N-1)^n$$

\mathcal{H} Symbol

$$(\mathcal{H}(N))^n = H_{-n}(N) = 1^n + 2^n + \cdots + (N-1)^n = \sum_{N>k>0} k^n.$$

\mathcal{H} Symbol

$$(\mathcal{H}(N))^n = H_{-n}(N) = 1^n + 2^n + \cdots + (N-1)^n = \sum_{N>k>0} k^n.$$

Faulhaber's formula

$$(\mathcal{H}(N))^n = \sum_{k=1}^{N-1} k^n = \frac{B_{n+1}(N) - B_{n+1}}{n+1}$$

\mathcal{H} Symbol

$$(\mathcal{H}(N))^n = H_{-n}(N) = 1^n + 2^n + \cdots + (N-1)^n = \sum_{N>k>0} k^n.$$

Faulhaber's formula

$$(\mathcal{H}(N))^n = \sum_{k=1}^{N-1} k^n = \frac{B_{n+1}(N) - B_{n+1}}{n+1} = \int_0^N (\mathcal{B} + x)^n dx$$

\mathcal{H} Symbol

$$(\mathcal{H}(N))^n = H_{-n}(N) = 1^n + 2^n + \cdots + (N-1)^n = \sum_{N>k>0} k^n.$$

Faulhaber's formula

$$(\mathcal{H}(N))^n = \sum_{k=1}^{N-1} k^n = \frac{B_{n+1}(N) - B_{n+1}}{n+1} = \int_0^N (\mathcal{B} + x)^n dx$$

$y \in \mathbb{C}$

$$(\mathcal{H}(y))^n = \frac{1}{n+1} \sum_{k=1}^{n+1} \binom{n+1}{k} B_{n+1-k} y^k$$

\mathcal{H} Symbol

$$(\mathcal{H}(N))^n = H_{-n}(N) = 1^n + 2^n + \cdots + (N-1)^n = \sum_{N>k>0} k^n.$$

Faulhaber's formula

$$(\mathcal{H}(N))^n = \sum_{k=1}^{N-1} k^n = \frac{B_{n+1}(N) - B_{n+1}}{n+1} = \int_0^N (\mathcal{B} + x)^n dx$$

$y \in \mathbb{C}$

$$(\mathcal{H}(y))^n = \frac{1}{n+1} \sum_{k=1}^{n+1} \binom{n+1}{k} B_{n+1-k} y^k$$

$$H_{-n_1, \dots, -n_r}(N) = \sum_{N>i_1>\dots>i_r>0} i_1^{n_1} \cdots i_r^{n_r}.$$

Sum of Powers

Theorem(L. J and C. Vignat)

$$H_{-n_1, \dots, -n_r}(N) = \prod_{k=1}^r \mathcal{H}_{1, \dots, k}^{n_k},$$

where $\mathcal{H}_1 = \mathcal{H}(N)$ and recursively $\mathcal{H}_{1, \dots, k} = \mathcal{H}(\mathcal{H}_{1, \dots, k-1})$.

Sum of Powers

Theorem(L. J and C. Vignat)

$$H_{-n_1, \dots, -n_r}(N) = \prod_{k=1}^r \mathcal{H}_{1, \dots, k}^{n_k},$$

where $\mathcal{H}_1 = \mathcal{H}(N)$ and recursively $\mathcal{H}_{1, \dots, k} = \mathcal{H}(\mathcal{H}_{1, \dots, k-1})$. $r = 2$:

Sum of Powers

Theorem(L. J and C. Vignat)

$$H_{-n_1, \dots, -n_r}(N) = \prod_{k=1}^r \mathcal{H}_{1, \dots, k}^{n_k},$$

where $\mathcal{H}_1 = \mathcal{H}(N)$ and recursively $\mathcal{H}_{1, \dots, k} = \mathcal{H}(\mathcal{H}_{1, \dots, k-1})$. $r = 2$:

$$H_{-n, -m}(N) = \sum_{N > i > j > 0} i^n j^m$$

Sum of Powers

Theorem(L. J and C. Vignat)

$$H_{-n_1, \dots, -n_r}(N) = \prod_{k=1}^r \mathcal{H}_{1, \dots, k}^{n_k},$$

where $\mathcal{H}_1 = \mathcal{H}(N)$ and recursively $\mathcal{H}_{1, \dots, k} = \mathcal{H}(\mathcal{H}_{1, \dots, k-1})$. $r = 2$:

$$H_{-n, -m}(N) = \sum_{N > i > j > 0} i^n j^m = \mathcal{H}_1^n \cdot \mathcal{H}_{1,2}^m$$

Sum of Powers

Theorem(L. J and C. Vignat)

$$H_{-n_1, \dots, -n_r}(N) = \prod_{k=1}^r \mathcal{H}_{1, \dots, k}^{n_k},$$

where $\mathcal{H}_1 = \mathcal{H}(N)$ and recursively $\mathcal{H}_{1, \dots, k} = \mathcal{H}(\mathcal{H}_{1, \dots, k-1})$. $r = 2$:

$$H_{-n, -m}(N) = \sum_{N > i > j > 0} i^n j^m = \mathcal{H}_1^n \cdot \mathcal{H}_{1,2}^m = \mathcal{H}_1^n [\mathcal{H}(\mathcal{H}_1)]^m$$

Sum of Powers

Theorem(L. J and C. Vignat)

$$H_{-n_1, \dots, -n_r}(N) = \prod_{k=1}^r \mathcal{H}_{1, \dots, k}^{n_k},$$

where $\mathcal{H}_1 = \mathcal{H}(N)$ and recursively $\mathcal{H}_{1, \dots, k} = \mathcal{H}(\mathcal{H}_{1, \dots, k-1})$. $r = 2$:

$$\begin{aligned} H_{-n, -m}(N) &= \sum_{N > i > j > 0} i^n j^m = \mathcal{H}_1^n \cdot \mathcal{H}_{1,2}^m = \mathcal{H}_1^n [\mathcal{H}(\mathcal{H}_1)]^m \\ &= \frac{\mathcal{H}_1^n}{m+1} \sum_{k=1}^{m+1} \binom{m+1}{k} B_{m+1-k} \mathcal{H}_1^k \end{aligned}$$

Sum of Powers

Theorem(L. J and C. Vignat)

$$H_{-n_1, \dots, -n_r}(N) = \prod_{k=1}^r \mathcal{H}_{1, \dots, k}^{n_k},$$

where $\mathcal{H}_1 = \mathcal{H}(N)$ and recursively $\mathcal{H}_{1, \dots, k} = \mathcal{H}(\mathcal{H}_{1, \dots, k-1})$. $r = 2$:

$$\begin{aligned} H_{-n, -m}(N) &= \sum_{N > i > j > 0} i^n j^m = \mathcal{H}_1^n \cdot \mathcal{H}_{1,2}^m = \mathcal{H}_1^n [\mathcal{H}(\mathcal{H}_1)]^m \\ &= \frac{\mathcal{H}_1^n}{m+1} \sum_{k=1}^{m+1} \binom{m+1}{k} B_{m+1-k} \mathcal{H}_1^k = \frac{1}{m+1} \sum_{k=1}^{m+1} \binom{m+1}{k} B_{m+1-k} \mathcal{H}_1^{n+k} \end{aligned}$$

Sum of Powers

Theorem(L. J and C. Vignat)

$$H_{-n_1, \dots, -n_r}(N) = \prod_{k=1}^r \mathcal{H}_{1, \dots, k}^{n_k},$$

where $\mathcal{H}_1 = \mathcal{H}(N)$ and recursively $\mathcal{H}_{1, \dots, k} = \mathcal{H}(\mathcal{H}_{1, \dots, k-1})$. $r = 2$:

$$\begin{aligned} H_{-n, -m}(N) &= \sum_{N > i > j > 0} i^n j^m = \mathcal{H}_1^n \cdot \mathcal{H}_{1,2}^m = \mathcal{H}_1^n [\mathcal{H}(\mathcal{H}_1)]^m \\ &= \frac{\mathcal{H}_1^n}{m+1} \sum_{k=1}^{m+1} \binom{m+1}{k} B_{m+1-k} \mathcal{H}_1^k = \frac{1}{m+1} \sum_{k=1}^{m+1} \binom{m+1}{k} B_{m+1-k} \mathcal{H}_1^{n+k} \\ &= \frac{1}{m+1} \sum_{k=1}^{m+1} \binom{m+1}{k} B_{m+1-k} \left[\frac{B_{n+k+1}(N) - B_{n+k+1}}{n+k+1} \right] \end{aligned}$$

Sum of Powers

Theorem(L. J and C. Vignat)

$$H_{-n_1, \dots, -n_r}(N) = \prod_{k=1}^r \mathcal{H}_{1, \dots, k}^{n_k},$$

where $\mathcal{H}_1 = \mathcal{H}(N)$ and recursively $\mathcal{H}_{1, \dots, k} = \mathcal{H}(\mathcal{H}_{1, \dots, k-1})$. $r = 2$:

$$\begin{aligned} H_{-n, -m}(N) &= \sum_{N > i > j > 0} i^n j^m = \mathcal{H}_1^n \cdot \mathcal{H}_{1,2}^m = \mathcal{H}_1^n [\mathcal{H}(\mathcal{H}_1)]^m \\ &= \frac{\mathcal{H}_1^n}{m+1} \sum_{k=1}^{m+1} \binom{m+1}{k} B_{m+1-k} \mathcal{H}_1^k = \frac{1}{m+1} \sum_{k=1}^{m+1} \binom{m+1}{k} B_{m+1-k} \mathcal{H}_1^{n+k} \\ &= \frac{1}{m+1} \sum_{k=1}^{m+1} \binom{m+1}{k} B_{m+1-k} \left[\frac{B_{n+k+1}(N) - B_{n+k+1}}{n+k+1} \right] \\ &= \sum_{k=0}^m \sum_{l=0}^{p-1-k} \sum_{q=0}^{p-k-l} \frac{B_k B_l}{(m+1)(p-k)} \binom{m+1}{k} \binom{p-k}{l} \binom{p-k-l}{q} (N-1)^q \end{aligned}$$

Remarks

Remarks

- ▶ The computation rules for $\mathcal{H}_{1,\dots,k}$ is reminiscent of the chain rule for differentiation

$$\frac{d}{dx} (f_r \circ \dots \circ f_1(x)) = f'_r (f_{r-1} \circ \dots \circ f_1) \cdots f'_1(x).$$

Remarks

- ▶ The computation rules for $\mathcal{H}_{1,\dots,k}$ is reminiscent of the chain rule for differentiation

$$\frac{d}{dx} (f_r \circ \dots \circ f_1(x)) = f'_r (f_{r-1} \circ \dots \circ f_1) \cdots f'_1(x).$$

- ▶ for r polynomials P_1, \dots, P_r without constant terms,

$$\sum_{N > i_1 > \dots > i_r > 0} P_1(i_1) \cdots P_r(i_r) = \prod_{k=1}^r P_k(\mathcal{H}_{1,\dots,k})$$

Remarks

- ▶ The computation rules for $\mathcal{H}_{1,\dots,k}$ is reminiscent of the chain rule for differentiation

$$\frac{d}{dx} (f_r \circ \dots \circ f_1(x)) = f'_r (f_{r-1} \circ \dots \circ f_1) \cdots f'_1(x).$$

- ▶ for r polynomials P_1, \dots, P_r without constant terms,

$$\sum_{N > i_1 > \dots > i_r > 0} P_1(i_1) \dots P_r(i_r) = \prod_{k=1}^r P_k(\mathcal{H}_{1,\dots,k})$$

- ▶ Recurrence

$$H_{-n_1, \dots, -n_r}(N) = \frac{1}{n_r + 1} \sum_{k=1}^{n_r+1} \binom{n_r + 1}{k} B_{n_r+1-k} H_{-n_1, \dots, -n_{r-1}-k}(N)$$

Remarks

Remarks

- ▶ Generating function

$$\mathcal{F}_r(w_1, \dots, w_r; N) = \sum_{n_1, \dots, n_r=0}^{\infty} \frac{w_1^{n_1} \dots w_r^{n_r}}{n_1! \dots n_r!} H_{-n_1, \dots, -n_r}(N)$$

Remarks

- ▶ Generating function

$$\mathcal{F}_r(w_1, \dots, w_r; N) = \sum_{n_1, \dots, n_r=0}^{\infty} \frac{w_1^{n_1} \dots w_r^{n_r}}{n_1! \dots n_r!} H_{-n_1, \dots, -n_r}(N)$$

$$\mathcal{F}_r(w_1, \dots, w_r; N) = \frac{\mathcal{F}_{r-1}(w_1, \dots, w_{r-1} + w_r; N) - \mathcal{F}_{r-1}(w_1, \dots, w_{r-1}; N)}{e^{w_r} - 1},$$

Remarks

- ▶ Generating function

$$\mathcal{F}_r(w_1, \dots, w_r; N) = \sum_{n_1, \dots, n_r=0}^{\infty} \frac{w_1^{n_1} \dots w_r^{n_r}}{n_1! \dots n_r!} H_{-n_1, \dots, -n_r}(N)$$

$$\mathcal{F}_r(w_1, \dots, w_r; N) = \frac{\mathcal{F}_{r-1}(w_1, \dots, w_{r-1} + w_r; N) - \mathcal{F}_{r-1}(w_1, \dots, w_{r-1}; N)}{e^{w_r} - 1},$$

where

$$\mathcal{F}_1(w_1; N) = \frac{e^{Nw_1} - 1}{e^{w_1} - 1}.$$

Remarks

- ▶ Generating function

$$\mathcal{F}_r(w_1, \dots, w_r; N) = \sum_{n_1, \dots, n_r=0}^{\infty} \frac{w_1^{n_1} \dots w_r^{n_r}}{n_1! \dots n_r!} H_{-n_1, \dots, -n_r}(N)$$

$$\mathcal{F}_r(w_1, \dots, w_r; N) = \frac{\mathcal{F}_{r-1}(w_1, \dots, w_{r-1} + w_r; N) - \mathcal{F}_{r-1}(w_1, \dots, w_{r-1}; N)}{e^{w_r} - 1},$$

where

$$\mathcal{F}_1(w_1; N) = \frac{e^{Nw_1} - 1}{e^{w_1} - 1}.$$

- ▶ S-Sum:

$$S_{-n_1, \dots, -n_r}(N) = \sum_{N \geq i_1 \geq \dots \geq i_r \geq 1} i_1^{n_1} \dots i_r^{n_r}$$

Remarks

- ▶ Generating function

$$\mathcal{F}_r(w_1, \dots, w_r; N) = \sum_{n_1, \dots, n_r=0}^{\infty} \frac{w_1^{n_1} \dots w_r^{n_r}}{n_1! \dots n_r!} H_{-n_1, \dots, -n_r}(N)$$

$$\mathcal{F}_r(w_1, \dots, w_r; N) = \frac{\mathcal{F}_{r-1}(w_1, \dots, w_{r-1} + w_r; N) - \mathcal{F}_{r-1}(w_1, \dots, w_{r-1}; N)}{e^{w_r} - 1},$$

where

$$\mathcal{F}_1(w_1; N) = \frac{e^{Nw_1} - 1}{e^{w_1} - 1}.$$

- ▶ S-Sum:

$$S_{-n_1, \dots, -n_r}(N) = \sum_{N \geq i_1 \geq \dots \geq i_r \geq 1} i_1^{n_1} \dots i_r^{n_r}$$

$$S_{-n_1, \dots, -n_r}(N) = \prod_{k=1}^r \bar{\mathcal{H}}_{1, \dots, k}^{n_k},$$

where $\bar{\mathcal{H}}_1 = \mathcal{H}(N+1)$ and $\bar{\mathcal{H}}_{1, \dots, k} = \mathcal{H}(\bar{\mathcal{H}}_{1, \dots, k-1} + 1)$.

Remarks

Remarks

- ▶ Extended Bernoulli polynomials

Remarks

- ▶ Extended Bernoulli polynomials

$$B_{n+1}(x+1) - B_{n+1}(x) = (n+1)x^n$$

Remarks

- ▶ Extended Bernoulli polynomials

$$B_{n+1}(x+1) - B_{n+1}(x) = (n+1)x^n$$

$$B_{n_1, \dots, n_r}(z+1) = B_{n_1, \dots, n_r}(z) + n_1 z^{n_1-1} B_{n_2, \dots, n_r}(z)$$

Remarks

- ▶ Extended Bernoulli polynomials

$$B_{n+1}(x+1) - B_{n+1}(x) = (n+1)x^n$$

$$B_{n_1, \dots, n_r}(z+1) = B_{n_1, \dots, n_r}(z) + n_1 z^{n_1-1} B_{n_2, \dots, n_r}(z)$$

Theorem (G. H. E. Duchamp, V. H. N. Minh and N. Q. Hoan)

$$H_{-n_1, \dots, -n_r}(N) = \frac{B_{n_1+1, \dots, n_r+1}(N+1) - \sum_{k=1}^{r-1} b'_{n_k+1, \dots, n_r+1} B_{n_1+1, \dots, n_k+1}(N+1)}{\prod_{i=1}^r (n_i+1)}.$$

Remarks

- ▶ Extended Bernoulli polynomials

$$B_{n+1}(x+1) - B_{n+1}(x) = (n+1)x^n$$

$$B_{n_1, \dots, n_r}(z+1) = B_{n_1, \dots, n_r}(z) + n_1 z^{n_1-1} B_{n_2, \dots, n_r}(z)$$

Theorem (G. H. E. Duchamp, V. H. N. Minh and N. Q. Hoan)

$$H_{-n_1, \dots, -n_r}(N) = \frac{B_{n_1+1, \dots, n_r+1}(N+1) - \sum_{k=1}^{r-1} b'_{n_k+1, \dots, n_r+1} B_{n_1+1, \dots, n_k+1}(N+1)}{\prod_{i=1}^r (n_i+1)}.$$

- ▶ Theorem (L. J and C. Vignat)

$$\begin{aligned} \beta_{n_1, \dots, n_r}(z) &:= B_{n_1, \dots, n_r}(z) - B_{n_1, \dots, n_r}(0) \\ &= \left(\prod_{k=1}^r \frac{\partial}{\partial \mathcal{B}_k} \right) H_{-n_1, \dots, -n_r}(z) \end{aligned}$$

Remarks

- ▶ Extended Bernoulli polynomials

$$B_{n+1}(x+1) - B_{n+1}(x) = (n+1)x^n$$

$$B_{n_1, \dots, n_r}(z+1) = B_{n_1, \dots, n_r}(z) + n_1 z^{n_1-1} B_{n_2, \dots, n_r}(z)$$

Theorem (G. H. E. Duchamp, V. H. N. Minh and N. Q. Hoan)

$$H_{-n_1, \dots, -n_r}(N) = \frac{B_{n_1+1, \dots, n_r+1}(N+1) - \sum_{k=1}^{r-1} b'_{n_k+1, \dots, n_r+1} B_{n_1+1, \dots, n_k+1}(N+1)}{\prod_{i=1}^r (n_i+1)}.$$

- ▶ Theorem (L. J and C. Vignat)

$$\begin{aligned} \beta_{n_1, \dots, n_r}(z) &:= B_{n_1, \dots, n_r}(z) - B_{n_1, \dots, n_r}(0) \\ &= \left(\prod_{k=1}^r \frac{\partial}{\partial \mathcal{B}_k} \right) H_{-n_1, \dots, -n_r}(z) \end{aligned}$$

$$(\mathcal{H}(z))^n = \frac{B_{n+1}(z) - B_{n+1}}{n+1} = \int_0^z (\mathcal{B} + x)^n dx$$

Generating Function

Generating Function

Let

$$\beta_{n_1, \dots, n_r}(z) = B_{n_1, \dots, n_r}(z) - B_{n_1, \dots, n_r}(0),$$

then,

$$\beta_{n_1, \dots, n_r}(z+1) = \beta_{n_1, \dots, n_r}(z) + n_1 z^{n_1-1} \beta_{n_2, \dots, n_r}(z).$$

Generating Function

Let

$$\beta_{n_1, \dots, n_r}(z) = B_{n_1, \dots, n_r}(z) - B_{n_1, \dots, n_r}(0),$$

then,

$$\beta_{n_1, \dots, n_r}(z+1) = \beta_{n_1, \dots, n_r}(z) + n_1 z^{n_1-1} \beta_{n_2, \dots, n_r}(z).$$

Consider the generating function

$$\mathcal{G}_r(w_1, \dots, w_r; z) = \sum_{n_1, \dots, n_r=0}^{\infty} \frac{w_1^{n_1} \dots w_r^{n_r}}{n_1! \dots n_r!} \beta_{n_1, \dots, n_r}(z)$$

Generating Function

Let

$$\beta_{n_1, \dots, n_r}(z) = B_{n_1, \dots, n_r}(z) - B_{n_1, \dots, n_r}(0),$$

then,

$$\beta_{n_1, \dots, n_r}(z+1) = \beta_{n_1, \dots, n_r}(z) + n_1 z^{n_1-1} \beta_{n_2, \dots, n_r}(z).$$

Consider the generating function

$$\mathcal{G}_r(w_1, \dots, w_r; z) = \sum_{n_1, \dots, n_r=0}^{\infty} \frac{w_1^{n_1} \dots w_r^{n_r}}{n_1! \dots n_r!} \beta_{n_1, \dots, n_r}(z)$$

Then,

$$\mathcal{G}_r(w_1, \dots, w_r; z) = \frac{w_r \left(\frac{w_r-1}{w_r-1+w_r} \mathcal{G}_{r-1}(w_1, \dots, w_{r-2}, w_{r-1}+w_r; z) - \mathcal{G}_{r-1}(w_1, \dots, w_{r-2}, w_{r-1}; z) \right)}{e^{w_r} - 1},$$

where,

$$\mathcal{G}_1(w_1; z) = \frac{w(e^{zw} - 1)}{e^w - 1}.$$

Current Work

Current Work

$$\text{Li}_{-n_1, \dots, -n_r}(z) = \sum_{i_1 > \dots > i_r > 0} i_1^{n_1} \dots i_r^{n_r} z^{i_1}$$

Current Work

$$\begin{aligned} \text{Li}_{-n_1, \dots, -n_r}(z) &= \sum_{i_1 > \dots > i_r > 0} i_1^{n_1} \dots i_r^{n_r} z^{i_1} \\ &= \sum_{i_1 > 0} i_1^{n_1} z^{i_1} \left(\sum_{i_1 > i_2 > \dots > i_r > 0} i_2^{n_2} \dots i_r^{n_r} \right) \end{aligned}$$

Current Work

$$\begin{aligned} \text{Li}_{-n_1, \dots, -n_r}(z) &= \sum_{i_1 > \dots > i_r > 0} i_1^{n_1} \dots i_r^{n_r} z^{i_1} \\ &= \sum_{i_1 > 0} i_1^{n_1} z^{i_1} \left(\sum_{i_1 > i_2 > \dots > i_r > 0} i_2^{n_2} \dots i_r^{n_r} \right) \\ &= \sum_{i_1 > 0} i_1^{n_1} z^{i_1} H_{-n_2, \dots, -n_r}(i_1) \end{aligned}$$

Current Work

$$\begin{aligned} \text{Li}_{-n_1, \dots, -n_r}(z) &= \sum_{i_1 > \dots > i_r > 0} i_1^{n_1} \dots i_r^{n_r} z^{i_1} \\ &= \sum_{i_1 > 0} i_1^{n_1} z^{i_1} \left(\sum_{i_1 > i_2 > \dots > i_r > 0} i_2^{n_2} \dots i_r^{n_r} \right) \\ &= \sum_{i_1 > 0} i_1^{n_1} z^{i_1} H_{-n_2, \dots, -n_r}(i_1) \\ &= \sum_{i_1 > 0} i_1^{n_1} z^{i_1} [\mathcal{H}(i_1)]^{n_2} [\mathcal{H}(\mathcal{H}(i_1))]^{n_3} \dots [\mathcal{H}(\dots \mathcal{H}(i_1))]^{n_r}. \end{aligned}$$

Current Work

$$\zeta_r(-n_1, \dots, -n_r) = \prod_{k=1}^r (-1)^{n_k} C_{1, \dots, k}^{n_k+1} H_{-n_1, \dots, -n_r}(N) = \prod_{k=1}^r \mathcal{H}_{1, \dots, k}^{n_k}$$

End

Thank you