Johann Radon Institute for Computational and Applied Mathematics
Austrian Academy of Sciences

Proving and Conjecturing Bounds for Some Floor Function Sums

Elaine Wong
(joint work with Thotsaporn Thanatipanonda)

July 25, 2018

ÖAW RICAM

aec

Outline of the Talk

The Sum in Question

The Sum in Question

- In 1957, Jacobsthal wrote a paper in German introducing a curious sum (sans abstract and references).

The Sum in Question

- In 1957, Jacobsthal wrote a paper in German introducing a curious sum (sans abstract and references).

DET KONGELIGE NORSKE VIDENSKABERS SELSKABS
FORHANDLINGER Bind 30 1957 Nr 6

511.29

Über eine zahlentheoretische Summe

VON

ERNST JACOBSTHAL

(Innsendt til Generalsekretæren 6te juli 1957)

Ist $[x]$ die grösste ganze Zahl $\leqq x$, und sind a, b, m gegebene ganze Zahlen, $m \geqq 1$, und setzt man für ganzes h den Ausdruck

$$(1) \quad \left[\frac{a+b+h}{m}\right]+\left[\frac{h}{m}\right]-\left[\frac{a+h}{m}\right]-\left[\frac{b+h}{m}\right]=\mathrm{D}(a, b, m ; h)=\mathrm{D}(h),$$

so folgt leicht aus den für $[x]$ geltenden Ungleichheiten, dass

$$(1_1) \quad -2<\mathrm{D}(h)<+2$$

ist. $\mathrm{D}(h)$ nimmt also nur die Werte $0, +1, -1$ an. Bildet man für irgend ein natürliches r die Summe

$$(2) \quad \sum_{h=0}^{r-1} \mathrm{D}(h)=\mathrm{S}(a, b, m ; r),$$

so gilt die Ungleichheit

$$(2_1) \quad \mathrm{S}(a, b, m ; r) \geqq 0 .$$

The Sum in Question

- In 1957, Jacobsthal wrote a paper in German introducing a curious sum (sans abstract and references).

$N_1 - N_2 \geqq 0$ richtig. Ist aber $a + r > m$ und N_1 gleich einer der Zahlen r, a, s, so ist

$$w = a + r - m\,, \quad s = a + b + r - m = w + b\,.$$

Ferner wird $r = w + m - a > w$, $a = w + m - r > w$, $s = w + b > w$, also $N_1 > w \geqq N_2$, und damit ist (6_1) gezeigt. Es sei noch bemerkt, dass (14) auch richtig ist unter der Voraussetzung

$$(4_3) \qquad 0 \leqq a \leqq b \leqq m;\ 1 \leqq r \leqq m\,.$$

The Sum in Question

- The terms of Jacobsthal's sum consist of 'alternating' sign floor functions of certain fractions,

$$\left\lfloor \frac{a_1 + a_2 + k}{m} \right\rfloor - \left\lfloor \frac{a_1 + k}{m} \right\rfloor - \left\lfloor \frac{a_2 + k}{m} \right\rfloor + \left\lfloor \frac{k}{m} \right\rfloor ,$$

for fixed $m \in \mathbb{Z}^+$ with $a_1, a_2, k \in \mathbb{Z}^+ \cup \{0\}$.

The Sum in Question

- The terms of Jacobsthal's sum consist of 'alternating' sign floor functions of certain fractions,

$$\left\lfloor \frac{a_1 + a_2 + k}{m} \right\rfloor - \left\lfloor \frac{a_1 + k}{m} \right\rfloor - \left\lfloor \frac{a_2 + k}{m} \right\rfloor + \left\lfloor \frac{k}{m} \right\rfloor,$$

  for fixed $m \in \mathbb{Z}^+$ with $a_1, a_2, k \in \mathbb{Z}^+ \cup \{0\}$.

- We denote this expression to be

$$f_m(\{a_1, a_2\}, k).$$

The Sum in Question

- The terms of Jacobsthal's sum consist of 'alternating' sign floor functions of certain fractions,

$$\left\lfloor \frac{a_1 + a_2 + k}{m} \right\rfloor - \left\lfloor \frac{a_1 + k}{m} \right\rfloor - \left\lfloor \frac{a_2 + k}{m} \right\rfloor + \left\lfloor \frac{k}{m} \right\rfloor,$$

for fixed $m \in \mathbb{Z}^+$ with $a_1, a_2, k \in \mathbb{Z}^+ \cup \{0\}$.

- We denote this expression to be

$$f_m(\{a_1, a_2\}, k).$$

- And, we consider its sum over k:

$$\sum_k f_m(\{a_1, a_2\}, k).$$

The Sum in Question

What makes these sums interesting?

The Sum in Question

What makes these sums interesting?

1. The numerators all contain a fixed k but are added to sums of subsets of the multiset $\{a_1, a_2\}$.

The Sum in Question

What makes these sums interesting?

1. The numerators all contain a fixed k but are added to sums of subsets of the multiset $\{a_1, a_2\}$.

2. The signs alternate according to the size of these subsets.

The Sum in Question

What makes these sums interesting?

1. The numerators all contain a fixed k but are added to sums of subsets of the multiset $\{a_1, a_2\}$.
2. The signs alternate according to the size of these subsets.
3. The sums are periodic in nature according to m, and so we can restrict the values of a_1, a_2, k to the interval $[0, m-1]$.

Outline of the Talk

The First Lower Bound

- Jacobsthal 'hand-proved' a lower bound for the sum

$$\sum_{k=0}^{K} f_m(\{a_1, a_2\}, k) \geq 0$$

over all choices of $0 \leq a_1, a_2, K \leq m - 1$ for $m \in \mathbb{Z}^+$.

The First Lower Bound

- Jacobsthal 'hand-proved' a lower bound for the sum

$$\sum_{k=0}^{K} f_m(\{a_1, a_2\}, k) \geq 0$$

over all choices of $0 \leq a_1, a_2, K \leq m - 1$ for $m \in \mathbb{Z}^+$.

- Carlitz (1959) and Grimson (1974) gave different proofs of the same result.

Generalizing the Sum

Tverberg (2012) noticed that this sum can be generalized in a very natural way.

Generalizing the Sum

Tverberg (2012) noticed that this sum can be generalized in a very natural way.

Formula

For $m, n > 0$ and a multiset $\{a_1, \ldots, a_n\}$ such that $0 \leq a_i \leq m - 1$, we define the sum

$$f_m(\{a_1, \ldots, a_n\}, k) := \sum_{T \subseteq \{1, \cdots, n\}} (-1)^{n-|T|} \left\lfloor \frac{k + \sum_{i \in T} a_i}{m} \right\rfloor .$$

Generalizing the Sum

Tverberg (2012) noticed that this sum can be generalized in a very natural way.

Formula

For $m, n > 0$ and a multiset $\{a_1, \ldots, a_n\}$ such that $0 \leq a_i \leq m-1$, we define the sum

$$f_m(\{a_1, \ldots, a_n\}, k) := \sum_{T \subseteq \{1, \cdots, n\}} (-1)^{n-|T|} \left\lfloor \frac{k + \sum_{i \in T} a_i}{m} \right\rfloor .$$

The sign change will play a role when we consider bounds on the sum.

Generalizing the Sum

Following Jacobsthal's lead, Tverberg considered the sum of these sums, taking into account the periodicity,

Generalizing the Sum

Following Jacobsthal's lead, Tverberg considered the sum of these sums, taking into account the periodicity,

$$S_m(\{a_1,\dots,a_n\},K) = \sum_{k=0}^{K} f_m(\{a_1,\dots,a_n\},k)$$

with $0 \le K \le m-1$.

Generalizing the Sum

Following Jacobsthal's lead, Tverberg considered the sum of these sums, taking into account the periodicity,

$$S_m(\{a_1,\dots,a_n\},K)=\sum_{k=0}^{K} f_m(\{a_1,\dots,a_n\},k)$$

with $0\le K\le m-1$. Like the others before him, he also proved that

$$S_m(\{a_1,a_2\},K)\ge 0,$$

over all choices of m, a_i, K.

New Bounds

Furthermore, he claimed (without proof) the other bounds:

4. Theorems

THEOREM 4.1. *If* $l = 3$, *then* $F > -2\lfloor m/2 \rfloor$.

THEOREM 4.2. *If* $l = 2$, *then* $F \leq \lfloor m/2 \rfloor$.

THEOREM 4.3. *If* $l = 3$, *then* $F \leq \lfloor m/3 \rfloor$.

As the proofs are elementary and relatively simple we omit them. It would be interesting to see corresponding results for higher values of l, and whether the work by Grimson and Carlitz can be generalized to our general sums.

Simplifying the Sum

Even in the simplest case, the sum can be a bit bulky.

$$S_m(\{a_1\}, K) := \sum_{k=0}^{K} \left(\left\lfloor \frac{a_1 + k}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor \right).$$

Simplifying the Sum

Even in the simplest case, the sum can be a bit bulky.

$$S_m(\{a_1\}, K) := \sum_{k=0}^{K} \left(\left\lfloor \frac{a_1 + k}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor \right).$$

By observing that the fractional parts can be separated out,

$$\left\lfloor \frac{a_1 + k}{m} \right\rfloor = \left\lfloor \frac{a_1}{m} \right\rfloor + \left\lfloor \frac{k}{m} \right\rfloor + \left\lfloor \left\{ \frac{a_1}{m} \right\} + \left\{ \frac{k}{m} \right\} \right\rfloor,$$

Simplifying the Sum

Even in the simplest case, the sum can be a bit bulky.

$$S_m(\{a_1\}, K) := \sum_{k=0}^{K} \left(\left\lfloor \frac{a_1 + k}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor \right).$$

By observing that the fractional parts can be separated out,

$$\left\lfloor \frac{a_1 + k}{m} \right\rfloor = \left\lfloor \frac{a_1}{m} \right\rfloor + \left\lfloor \frac{k}{m} \right\rfloor + \left\lfloor \left\{ \frac{a_1}{m} \right\} + \left\{ \frac{k}{m} \right\} \right\rfloor,$$

a 'simplification' can therefore be made, i.e.,

$$S_m(\{a_1\}, K) = \left\lfloor \tfrac{a_1}{m} \right\rfloor (K+1) + \max\Big(0, (a_1 \bmod m) + K - m + 1\Big).$$

Simplifying the Sum

$$
\begin{aligned}
S_m(\{a_1, a_2\}, K) = &\left(\left\lfloor \frac{a_1 + a_2}{m} \right\rfloor - \left\lfloor \frac{a_1}{m} \right\rfloor - \left\lfloor \frac{a_2}{m} \right\rfloor \right) (K + 1) \\
&+ \max\left(0, ((a_1 + a_2) \bmod m) + K - m + 1\right) \\
&- \max\left(0, (a_1 \bmod m) + K - m + 1\right) \\
&- \max\left(0, (a_2 \bmod m) + K - m + 1\right)
\end{aligned}
$$

Simplifying the Sum

$$
\begin{aligned}
S_m(\{a_1, a_2, a_3\}, K) = \Bigg(& \left\lfloor \frac{a_1 + a_2 + a_3}{m} \right\rfloor - \left\lfloor \frac{a_1 + a_2}{m} \right\rfloor - \left\lfloor \frac{a_2 + a_3}{m} \right\rfloor \\
& - \left\lfloor \frac{a_1 + a_3}{m} \right\rfloor + \left\lfloor \frac{a_1}{m} \right\rfloor + \left\lfloor \frac{a_2}{m} \right\rfloor + \left\lfloor \frac{a_3}{m} \right\rfloor \Bigg) (K + 1) \\
& + \max\left(0, ((a_1 + a_2 + a_3) \bmod m) + K - m + 1\right) \\
& - \max\left(0, ((a_1 + a_2) \bmod m) + K - m + 1\right) \\
& - \max\left(0, ((a_2 + a_3) \bmod m) + K - m + 1\right) \\
& - \max\left(0, ((a_1 + a_3) \bmod m) + K - m + 1\right) \\
& + \max\left(0, (a_1 \bmod m) + K - m + 1\right) \\
& + \max\left(0, (a_2 \bmod m) + K - m + 1\right) \\
& + \max\left(0, (a_3 \bmod m) + K - m + 1\right)
\end{aligned}
$$

The Story of the Bounds

Using the new formulas, we can now prove sharp bounds for multisets of sizes $n = 1, 2, 3$.

The Story of the Bounds

Using the new formulas, we can now prove sharp bounds for multisets of sizes $n = 1, 2, 3$.

n	Lower Bound	Lower Bound Credit	Upper Bound	Upper Bound Credit
1	0	Trivial	$m - 1$	Trivial
2	0	Jacobsthal; Carlitz; Grimson; Tverberg; **TT, EW**	$\left\lfloor \frac{m}{2} \right\rfloor$	Tverberg*; **TT, EW**
3	$-2\left\lfloor \frac{m}{2} \right\rfloor$	Tverberg*; Onphaeng, Pongsriiam	$\left\lfloor \frac{m}{3} \right\rfloor$	Tverberg*; **TT, EW**

Table: Bounds for $S_m(A, K)$ where $n = |A|$; *Conjectured

The Story of the Bounds

Using the new formulas, we can now prove sharp bounds for multisets of sizes $n = 1, 2, 3$.

n	Lower Bound	Lower Bound Credit	Upper Bound	Upper Bound Credit
1	0	Trivial	$m-1$	Trivial
2	0	Jacobsthal; Carlitz; Grimson; Tverberg; **TT, EW**	$\left\lfloor \frac{m}{2} \right\rfloor$	Tverberg*; **TT, EW**
3	$-2\left\lfloor \frac{m}{2} \right\rfloor$	Tverberg*; Onphaeng, Pongsriiam	$\left\lfloor \frac{m}{3} \right\rfloor$	Tverberg*; **TT, EW**

Table: Bounds for $S_m(A, K)$ where $n = |A|$; *Conjectured

In particular, we prove the upper bounds that Tverberg conjectured but did not prove.

The Story of the Bounds

Unfortunately, an extension to higher cases resulted in extremely complicated case analysis and we chose not to pursue that route.

The Story of the Bounds

Unfortunately, an extension to higher cases resulted in extremely complicated case analysis and we chose not to pursue that route.

We were, however, able to prove a "weakened version" of the lower bound for $n = 4$ using similar techniques of Onphaeng and Pongsriiam, namely:

The Story of the Bounds

Unfortunately, an extension to higher cases resulted in extremely complicated case analysis and we chose not to pursue that route.

We were, however, able to prove a "weakened version" of the lower bound for $n = 4$ using similar techniques of Onphaeng and Pongsriiam, namely:

n	Lower Bound	Lower Bound Credit	Upper Bound	Upper Bound Credit
4	$-2\left\lfloor \frac{m}{2} \right\rfloor - \left\lfloor \frac{m}{3} \right\rfloor$ (Not So Sharp)	**TT, EW**	$4\left\lfloor \frac{m}{2} \right\rfloor$	Onphaeng, Pongsriiam

Table: Bounds for $S_m(A, K)$ where $n = |A|$

Proof.

We can combine the following bounds from $n = 2, 3$,

$$\begin{aligned} 0 &\leq S_m(\{a_1 + a_2 + a_3, a_4\}, K), \\ -\left\lfloor \frac{m}{2} \right\rfloor &\leq -S_m(\{a_1 + a_2, a_4\}, K), \\ -\left\lfloor \frac{m}{2} \right\rfloor &\leq -S_m(\{a_1 + a_3, a_4\}, K), \\ -\left\lfloor \frac{m}{3} \right\rfloor &\leq -S_m(\{a_2, a_3, a_4\}, K), \\ 0 &\leq S_m(\{a_1, a_4\}, K), \end{aligned}$$

along with the identity,

$$\begin{aligned} S_m(\{a_1, a_2, a_3, a_4\}, K) = {} & S_m(\{a_1 + a_2 + a_3, a_4\}, K) \\ & - S_m(\{a_1 + a_2, a_4\}, K) - S_m(\{a_1 + a_3, a_4\}, K) \\ & - S_m(\{a_2, a_3, a_4\}, K) + S_m(\{a_1, a_4\}, K), \end{aligned}$$

The Story of the Bounds

to obtain the following result:

to obtain the following result:

Theorem

For $0 \leq a_1, a_2, a_3, a_4, K \leq m - 1$,

$$-2\left\lfloor \frac{m}{2} \right\rfloor - \left\lfloor \frac{m}{3} \right\rfloor \leq S_m(\{a_1, a_2, a_3, a_4\}, K).$$

The Story of the Bounds

n	Lower Bound	Lower Bound Credit	Upper Bound	Upper Bound Credit
1	0	Trivial	$m-1$	Trivial
2	0	Jacobsthal; Carlitz; Grimson; Tverberg; **TT, EW**	$\left\lfloor \frac{m}{2} \right\rfloor$	Tverberg*; **TT, EW**
3	$-2\left\lfloor \frac{m}{2} \right\rfloor$	Tverberg*; Onphaeng, Pongsriiam	$\left\lfloor \frac{m}{3} \right\rfloor$	Tverberg*; **TT, EW**
4	$-2\left\lfloor \frac{m}{2} \right\rfloor - \left\lfloor \frac{m}{3} \right\rfloor$ (Not So Sharp)	**TT, EW**	$4\left\lfloor \frac{m}{2} \right\rfloor$	Onphaeng, Pongsriiam
odd (≥ 5)	$-2^{n-2}\left\lfloor \frac{m}{2} \right\rfloor$	Onphaeng, Pongsriiam		
even (≥ 5)			$2^{n-2}\left\lfloor \frac{m}{2} \right\rfloor$	Onphaeng, Pongsriiam

Table: Bounds for $S_m(A, K)$ where $n = |A|$; *Conjectured

Enter Computer Algebra

- We can enumerate patterns of the bounds for $S_m(A, K)$ for a range of m, multisets $A = \{a_1, \ldots, a_n\}$, and K with $0 \leq a_i, K \leq m - 1$. In doing so, we can identify exactly which A and K gives the maximum and minimum values of the sums and furthermore conjecture the pattern for these 'locations.'

Enter Computer Algebra

- We can enumerate patterns of the bounds for $S_m(A, K)$ for a range of m, multisets $A = \{a_1, \ldots, a_n\}$, and K with $0 \leq a_i, K \leq m - 1$. In doing so, we can identify exactly which A and K gives the maximum and minimum values of the sums and furthermore conjecture the pattern for these 'locations.'

- We use this information to generate enough data to guess a recurrence for the extreme values of S_m when $n \geq 4$ (by plugging in those specific values to our new formulas).

Enter Computer Algebra

Define

$$M(m,n) := \begin{cases} \max\limits_{A,K} S_m(A,K), & n \text{ odd;} \\ \min\limits_{A,K} S_m(A,K), & n \text{ even.} \end{cases}$$

Enter Computer Algebra

Define

$$M(m,n) := \begin{cases} \max_{A,K} S_m(A,K), & n \text{ odd;} \\ \min_{A,K} S_m(A,K), & n \text{ even.} \end{cases}$$

Using this, we conjecture

$$M(m,n) = m \cdot f(n),$$

where $f(n)$ satisfies a ninth order recurrence with polynomial coefficients of degree at most 2.

Enter Computer Algebra

Recurrence 1

$$
\begin{aligned}
-5(n+3)(n-2)f(n)& \\
&= 10(n^2+n-8)f(n-1) - 4(2n^2-10n+3)f(n-2) \\
&\quad - 24(2n-11)f(n-3) - 32(2n^2-10n-1)f(n-4) \\
&\quad - 192(n-1)(n-5)f(n-5) + 64(2n^2-22n+51)f(n-6) \\
&\quad + 384(2n-13)f(n-7) - 256(n-3)(n-8)f(n-8) \\
&\quad + 512(n-9)(n-8)f(n-9)
\end{aligned}
$$

Enter Computer Algebra

Recurrence 1

$$
\begin{aligned}
&-5(n+3)(n-2)f(n) \\
&\qquad = 10(n^2+n-8)f(n-1) - 4(2n^2-10n+3)f(n-2) \\
&\qquad\quad - 24(2n-11)f(n-3) - 32(2n^2-10n-1)f(n-4) \\
&\qquad\quad - 192(n-1)(n-5)f(n-5) + 64(2n^2-22n+51)f(n-6) \\
&\qquad\quad + 384(2n-13)f(n-7) - 256(n-3)(n-8)f(n-8) \\
&\qquad\quad + 512(n-9)(n-8)f(n-9)
\end{aligned}
$$

for $n \geq 11$, with the initial conditions

$$
\begin{aligned}
&f(2) = 0, f(3) = 1/3, f(4) = -1, f(5) = 2, f(6) = -3, \\
&f(7) = 8, f(8) = -18, f(9) = 36, f(10) = -65.
\end{aligned}
$$

Enter Computer Algebra

Such recurrences can be found by packages such as Guess.m (Kauers). In doing so, we can do better by obtaining a fifth order recurrence with polynomial coefficients of degree at most 5:

Enter Computer Algebra

Such recurrences can be found by packages such as Guess.m (Kauers). In doing so, we can do better by obtaining a fifth order recurrence with polynomial coefficients of degree at most 5:

Recurrence 2

$$\begin{aligned}
n(n+5)(-13+28n+28n^2-36n^3+8n^4)f(n) \\
&= -2(-2+n)(-25+25n+28n^2-116n^3+20n^4+8n^5)f(n-1) \\
&\quad + 4(5-14n+4n^2)(-3-8n+6n^2)f(n-2) \\
&\quad - 8(-115+92n+134n^2-112n^3+16n^4)f(n-3) \\
&\quad + 16(n-2)(-85+13n+168n^2-56n^3-28n^4+8n^5)f(n-4) \\
&\quad + 32(-3+n)(-2+n)(15+8n-32n^2-4n^3+8n^4)f(n-5)
\end{aligned}$$

Enter Computer Algebra

Such recurrences can be found by packages such as Guess.m (Kauers). In doing so, we can do better by obtaining a fifth order recurrence with polynomial coefficients of degree at most 5:

Recurrence 2

$$
\begin{aligned}
&n(n+5)(-13+28n+28n^2-36n^3+8n^4)f(n) \\
&\quad = -2(-2+n)(-25+25n+28n^2-116n^3+20n^4+8n^5)f(n-1) \\
&\qquad + 4(5-14n+4n^2)(-3-8n+6n^2)f(n-2) \\
&\qquad - 8(-115+92n+134n^2-112n^3+16n^4)f(n-3) \\
&\qquad + 16(n-2)(-85+13n+168n^2-56n^3-28n^4+8n^5)f(n-4) \\
&\qquad + 32(-3+n)(-2+n)(15+8n-32n^2-4n^3+8n^4)f(n-5)
\end{aligned}
$$

for $n \geq 7$, with the initial conditions

$$f(2) = 0, f(3) = 1/3, f(4) = -1, f(5) = 2, \text{ and } f(6) = -3.$$

Enter Computer Algebra

n	Lower Bound	Lower Bound Credit	Upper Bound	Upper Bound Credit
1	0	Trivial	$m-1$	Trivial
2	0	Jacobsthal; Carlitz; Grimson; Tverberg; **TT, EW**	$\left\lfloor \frac{m}{2} \right\rfloor$	Tverberg*; **TT, EW**
3	$-2\left\lfloor \frac{m}{2} \right\rfloor$	Tverberg*; Onphaeng, Pongsriiam	$\left\lfloor \frac{m}{3} \right\rfloor$	Tverberg*; **TT, EW**
4	$-3\left\lfloor \frac{m}{3} \right\rfloor$ **(Conjecture)**	**TT, EW***	$4\left\lfloor \frac{m}{2} \right\rfloor$	Onphaeng, Pongsriiam
odd (≥ 5)	$-2^{n-2}\left\lfloor \frac{m}{2} \right\rfloor$	Onphaeng, Pongsriiam	**(Conjectures)**	**TT, EW***
even (≥ 5)	**(Conjectures)**	**TT, EW***	$2^{n-2}\left\lfloor \frac{m}{2} \right\rfloor$	Onphaeng, Pongsriiam

Table: Bounds for $S_m(A, K)$ where $n = |A|$; *Conjectured

Examples

For m being a multiple of 3, we have the following conjectures:

Examples

For m being a multiple of 3, we have the following conjectures:

Case	Bound (L/U)	'Location' of Bound (i.e. $M(m,n)$ occurs at...)
$n=4$	$-m$ (L)	$\{\frac{m}{3}, \frac{m}{3}, \ldots, \frac{m}{3}\}$, $K = \frac{m}{3} - 1$
$n=5$	$2m$ (U)	or
$n=6$	$-3m$ (L)	$\{\frac{2m}{3}, \frac{2m}{3}, \ldots, \frac{2m}{3}\}$, $K = \frac{2m}{3} - 1$

Examples

For m being a multiple of 3, we have the following conjectures:

Case	Bound (L/U)	'Location' of Bound (i.e. $M(m,n)$ occurs at...)
$n = 4$	$-m$ (L)	$\{\frac{m}{3}, \frac{m}{3}, \ldots, \frac{m}{3}\}$, $K = \frac{m}{3} - 1$
$n = 5$	$2m$ (U)	or
$n = 6$	$-3m$ (L)	$\{\frac{2m}{3}, \frac{2m}{3}, \ldots, \frac{2m}{3}\}$, $K = \frac{2m}{3} - 1$

For m being a multiple of 5, we have the following conjectures:

Examples

For m being a multiple of 3, we have the following conjectures:

Case	Bound (L/U)	'Location' of Bound (i.e. $M(m,n)$ occurs at...)
$n=4$ $n=5$ $n=6$	$-m$ (L) $2m$ (U) $-3m$ (L)	$\{\frac{m}{3},\frac{m}{3},\ldots,\frac{m}{3}\}$, $K=\frac{m}{3}-1$ or $\{\frac{2m}{3},\frac{2m}{3},\ldots,\frac{2m}{3}\}$, $K=\frac{2m}{3}-1$

For m being a multiple of 5, we have the following conjectures:

Case	Bound (L/U)	'Location' of Bound (i.e. $M(m,n)$ occurs at...)
$n=6$ $n=7$ $n=8$ $n=9$	$-3m$ (L) $8m$ (U) $-18m$ (L) $36m$ (U)	$\{\frac{2m}{5},\frac{2m}{5},\ldots,\frac{2m}{5}\}$, $K=\frac{2m}{5}-1$ or $\{\frac{3m}{5},\frac{3m}{5},\ldots,\frac{3m}{5}\}$, $K=\frac{3m}{5}-1$

Thank you!

References

[1] **1957**: E. Jacobsthal, Über eine zahlentheoretische Summe, *Norske Vid. Selsk. Forh. Trondheim* **30** (1957), 35–41.

[2] **1959**: L. Carlitz, An arithmetic sum connected with the greatest integer function, *Norske Vidensk. Selsk. Forh. Trondheim* **32** (1959), 24-30.

[3] **1974**: R. C. Grimson, The evaluation of a sum of Jacobsthal, *Norske Vid. Selsk. Skr. Trondheim* (1974), No. 4.

[4] **2012**: H. Tverberg, On some number-theoretic sums introduced by Jacobsthal, *Acta Arith.*, **155** (2012), 349–351.

[5] **2017**: K. Onphaeng and P. Pongsriiam, Jacobsthal and Jacobsthal-Lucas numbers and sums introduced by Jacobsthal and Tverberg, *J. Integer Sequences* **20** (2017), Article 17.3.6.

[6] **2018**: T. Thanatipanonda and E. Wong, Curious Bounds for Floor Function Sums, *J. Integer Sequences* **21** (2018), Article 18.1.8 (arXiv: 1708.03190v2)