Diagonals of rational functions, pullbacked $_2F_1$ hypergeometric functions and modular forms

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Abstract.

We recall that diagonals of rational functions naturally occur in lattice statistical mechanics and enumerative combinatorics. We find that the diagonal of a seven parameter rational function of three variables with a numerator equal to one and a denominator which is a polynomial of degree at most two, can be expressed as a pullbacked $_2F_1$ hypergeometric function. This result can be seen as the simplest non-trivial family of diagonals of rational functions. We focus on some subcases such that the diagonals of the corresponding rational functions can be written as a pullbacked $_2F_1$ hypergeometric function with two possible rational functions pullbacks algebraically related by modular equations, thus showing explicitely that the diagonal is a modular form. We then generalize this result to nine and ten parameter families adding some selected cubic terms at the denominator of the rational function defining the diagonal. We show that each of these rational functions yields an infinite number of rational functions whose diagonals are also pullbacked $_2F_1$ hypergeometric functions and modular forms.

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1. Introduction

It was shown in [1, 2] that different physical related quantities, like the n-fold integrals $\chi^{(n)}$, corresponding to the n-particle contributions of the magnetic susceptibility of the Ising model [3, 4, 5, 6], or the *lattice Green functions* [7, 8, 9, 10, 11], are *diagonals of rational functions* [12, 13, 14, 15, 16, 17].

While showing that the n-fold integrals $\chi^{(n)}$ of the susceptibility of the Ising model are diagonals of rational functions requires some effort, seeing that the lattice

Green functions are diagonals of rational functions nearly follows from their definition. For example, the lattice Green functions (LGF) of the d-dimensional face-centred cubic (fcc) lattice are given [10, 11] by:

$$\frac{1}{\pi^d} \int_0^\pi \cdots \int_0^\pi \frac{\mathrm{d}k_1 \, \cdots \, \mathrm{d}k_d}{1 \, - x \cdot \, \lambda_d}, \qquad \text{with:} \qquad \lambda_d = \binom{d}{2}^{-1} \sum_{i=1}^{d} \sum_{j=i+1}^{d} \cos(k_i) \, \cos(k_j). \quad (1)$$

The LGF can easily be seen to be a diagonal of a rational function: introducing the complex variables $z_j = e^{i\, k_j}$, $j = 1, \cdots, d$, the LGF (1) can be seen as a d-fold generalization of Cauchy's contour integral [1]:

$$\mathsf{Diag}(\mathcal{F}) \;\; = \;\; \frac{1}{2\pi i} \oint_\gamma \mathcal{F}(z_1, z/z_1) \, \frac{dz_1}{z_1}. \qquad (2)$$

Furthermore, the linear differential operators annihilating the physical quantities mentioned earlier $\chi^{(n)}$, are reducible operators. Being reducible they are "breakable" into smaller factors [4, 5] that happen to be elliptic functions, or generalizations thereof: *modular forms*, Calabi-Yau operators [18, 19]... Yet there exists a class of diagonals of rational functions in *three* variables†† whose diagonals are pullbacked $_2F_1$ hypergeometric functions, and in fact *modular forms* [21]. These sets of diagonals of rational functions in *three* variables in [21] were obtained by imposing the coefficients of the polynomial $P(x, y, z)$ appearing in the rational function $1/P(x, y, z)$ defining the diagonal to be 0 or 1¶.

While these constraints made room for exhaustivity, they were quite arbitrary, which raises the question of randomness of the sample : is the emergence of modular forms [20], with the constraints imposed in [21], an artefact of the sample?

Our aim in this paper is to show that *modular forms* emerge for a much larger set of rational functions of three variables, than the one previously introduced in [21], firstly because we obtain a whole family of rational functions whose diagonals give modular forms by adjoining parameters, and secondly through considerations of symmetry.

In particular, we will find that the *seven-parameter* rational function of three variables, with a numerator equal to one and a denominator being a polynomial of degree two at most, given by:

$$R(x, \, y, \, z) \;\; = \;\; \frac{1}{a \; + b_1 \, x \; + b_2 \, y \; + b_3 \, z \; + c_1 \, y \, z \; + c_2 \, x \, z \; + c_3 \, x \, y}, \qquad (3)$$

can be expressed as a particular pullbacked $_2F_1$ hypergeometric function†

$$\frac{1}{P_2(x)^{1/4}} \cdot {}_2F_1\Big([\frac{1}{12}, \, \frac{5}{12}], \, [1], \, 1 \, - \, \frac{P_4(x)^2}{P_2(x)^3}\Big), \qquad (4)$$

where $P_2(x)$ and $P_4(x)$ are two polynomials of degree two and four respectively. We then focus on subcases where the diagonals of the corresponding rational functions can

†† Diagonals of rational functions of two variables are just algebraic functions, so one must consider *at least three* variables to obtain special functions.

¶ Or 0 or ± 1 in the four variable case also examined in [21].

† The selected $_2F_1([1/12, 5/12], [1], \mathcal{P})$ hypergeometric function is closely related to modular forms [22, 23]. This can be seen as a consequence of the identity with the Eisenstein series E_4 and E_6 and this very $_2F_1([1/12, 5/12], [1], \mathcal{P})$ hypergeometric function (see Theorem 3 page 226 in [24] and page 216 of [25]): $E_4(\tau) = {}_2F_1([1/12, 5/12], [1], 1728/j(\tau))^4$ (see also equation (88) in [22] for E_6).

be written as a pullbacked ${}_2F_1$ hypergeometric function, with two rational function pullbacks related algebraically by *modular equations*†.

This seven-parameter family will then be generalized into nine, then ten-parameter familiesx of rational functions that are reciprocal of a polynomial in three variables of degree at most three. We will finally show that each of the previous results yields an *infinite number of new exact pullbacked* ${}_2F_1$ *hypergeometric function results*, through symmetry considerations on monomial transformations and some function-dependent rescaling transformations.

2. Diagonals of rational functions of three variables depending on seven parameters

2.1. Recalls on diagonals of rational functions

Let us recall the definition of the diagonal of a rational function in n variables $\mathcal{R}(x_1, \ldots, x_n) = \mathcal{P}(x_1, \ldots, x_n)/\mathcal{Q}(x_1, \ldots, x_n)$, where $\mathcal{P}$ and $\mathcal{Q}$ are polynomials of $x_1, \cdots, x_n$ with *integer coefficients* such that $\mathcal{Q}(0, \ldots, 0) \neq 0$. The diagonal of $\mathcal{R}$ is defined through its multi-Taylor expansion (for small x_i's)

$$\mathcal{R}\Big(x_1,\, x_2,\, \ldots,\, x_n\Big) \;=\; \sum_{m_1\,=0}^{\infty} \cdots \sum_{m_n\,=0}^{\infty} R_{m_1,\,\ldots,\,m_n} \cdot x_1^{m_1} \,\cdots\, x_n^{m_n}, \qquad (5)$$

as the series in *one variable* x:

$$Diag\Big(\mathcal{R}\Big(x_1,\, x_2,\, \ldots,\, x_n\Big)\Big) \;=\; \sum_{m\,=0}^{\infty} R_{m,\,m,\,\ldots,\,m} \cdot x^m. \qquad (6)$$

Diagonals of rational functions of two variables are algebraic functions [26, 27]. Interesting cases of diagonals of rational functions thus require considering rational functions of at least *three* variables.

2.2. A seven-parameter family of rational functions of three variables

We obtained the diagonal of the rational function in *three* variables depending on *seven* parameters:

$$R(x,\, y,\, z) \;=\; \frac{1}{a \,+\, b_1\, x \,+\, b_2\, y \,+\, b_3\, z \,+\, c_1\, y\, z \,+\, c_2\, x\, z \,+\, c_3\, x\, y}. \qquad (7)$$

This result was obtained by:

- Running the *HolonomicFunctions* [28] package in mathematica for arbitrary parameters $a, b_1, \cdots, c_1, \cdots$ and obtaining a large-sized second order linear differential operator L_2.
- Running the maple command "*hypergeometricsols*" [29] for different sets of values of the parameters on the operator L_2, and guessing¶ the Gauss hypergeometric function ${}_2F_1$ with general parameters solution of L_2.

† Thus providing a nice illustration of the fact that the diagonal is a modular form [23].
¶ The program "*hypergeometricsols*" [29] does not run for arbitrary parameters, hence our recourse to guessing.

2.3. The diagonal of the seven-parameter family of rational functions: the general form

We find the following experimental results: all these diagonals are expressed in terms of *only one pullbacked hypergeometric function*. This is worth pointing out that for an order-two linear differential operator with pullbacked ${}_2F_1$ hypergeometric function solutions, the "*hypergeometricsols*" command in nearly all cases gives the solutions as *a sum of two* ${}_2F_1$ *hypergeometric functions*. Here, quite remarkably, the result is "encapsulated" *in just one* pullbacked hypergeometric function. We find that these diagonals are expressed as pullbacked hypergeometric functions of the form

$$\frac{1}{P_4(x)^{1/6}} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{7}{12}], [1], \frac{1728 \cdot x^3 \cdot P_5(x)}{P_4(x)^2}\Big), \tag{8}$$

where the two polynomials $P_4(x)$ and $P_5(x)$, in the $1728\,x^3\,P_5(x)/P_4(x)^2$ pullback, are polynomials of degree four and five in x respectively. The pullback in (8), given by $1728\,x^3\,P_5(x)/P_4(x)^2$, has the form $1 - \tilde{Q}$ where $\tilde{Q}$ is given by the simpler expression

$$\tilde{Q} \,\,=\,\, \frac{P_2(x)^3}{P_4(x)^2}, \tag{9}$$

where $P_2(x)$ is a polynomial of degree two in x. Recalling the identity

$${}_2F_1\Big([\frac{1}{12}, \frac{7}{12}], [1], x\Big) \,\,=\,\, (1-x)^{-1/12} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \frac{-x}{1-x}\Big), \tag{10}$$

the previous pullbacked hypergeometric function (8) can be rewritten as

$$\frac{1}{P_2(x)^{1/4}} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], -\frac{1728 \cdot x^3 \cdot P_5(x)}{P_2(x)^3}\Big), \tag{11}$$

where $P_5(x)$ is the same polynomial of degree five as the one in the pullback in expression (8). This new pullback also has the form $1 - Q$ with Q given by‡:

$$-\frac{1728 \cdot x^3 \cdot P_5(x)}{P_2(x)^3} \,=\, 1 - Q \qquad \text{where:} \qquad Q \,=\, \frac{P_4(x)^2}{P_2(x)^3}. \tag{12}$$

Finding the exact result for arbitrary values of the seven parameters now boils down to a guessing problem.

2.4. Exact expression of the diagonal for arbitrary parameters a, b_1, ..., c_1, ...

Now that the structure of the result is understood "experimentally" we obtain the result *for arbitrary parameters* a, b_1, b_2, b_3, c_1, c_2, c_3.

Assuming that the diagonal of the rational function (7) has the form explicited in the previous subsection

$$\frac{1}{P_2(x)^{1/4}} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], 1 - \frac{P_4(x)^2}{P_2(x)^3}\Big), \tag{13}$$

where $P_2(x)$ and $P_4(x)$ are two polynomials of degree two and four respectively:

$$P_4(x) \,=\, A_4\,x^4 + A_3\,x^3 + A_2\,x^2 + A_1\,x + A_0, \tag{14}$$

$$P_2(x) \,=\, B_2\,x^2 + B_1\,x + B_0, \tag{15}$$

one can write the order-two linear differential operator having this eight-parameter solution (13), and identify this second order operator depending on eight arbitrary

‡ Note that Q given in (12), is the reciprocal of $\tilde{Q}$ given in (9): $Q \,=\, 1/\tilde{Q}$.

parameters A_i, B_i in (14), with the second order linear differential operator obtained using the *HolonomicFunctions* [28] program for arbitrary parameters. Using the results obtained for specific values of the parameters, one easily guesses that $A_0 = a^6$ and $B_0 = a^4$. One finally gets:

$$P_2(x) =$$
$$8 \cdot \Big(3\, a\, c_1\, c_2\, c_3 \, + 2 \cdot \, (b_1^2\, c_1^2 + \, b_2^2\, c_2^2 \, + \, b_3^2\, c_3^2 - \, b_1\, b_2\, c_1\, c_2 - \, b_1\, b_3\, c_1\, c_3 - \, b_2\, b_3\, c_2\, c_3)\Big) \cdot x^2$$
$$- 8 \cdot a \cdot \Big(a \cdot (b_1\, c_1 \, + \, b_2\, c_2 \, + \, b_3\, c_3) - 3\, b_1\, b_2\, b_3\Big) \cdot x \quad + a^4, \qquad (16)$$

and

$$P_4(x) \, = \, 216 \cdot c_1^2\, c_2^2\, c_3^2 \cdot x^4 \quad - 16 \cdot \Big(9 \cdot a\, c_1\, c_2\, c_3 \cdot (b_1\, c_1 + b_2\, c_2 \, + b_3\, c_3)$$
$$- 6 \cdot (b_1^2\, b_2\, c_1^2\, c_2 \, + b_1\, b_2^2\, c_1\, c_2^2 \, + b_1^2\, b_3\, c_1^2\, c_3 \, + b_1\, b_3^2\, c_1\, c_3^2 \, + b_2^2\, b_3\, c_2^2\, c_3 + \, b_2\, b_3^2\, c_2\, c_3^2)$$
$$+ 4 \cdot (b_1^3\, c_1^3 + \, b_2^3\, c_2^3 \, + \, b_3^3\, c_3^3) \, - 3\, b_1\, b_2\, b_3\, c_1\, c_2\, c_3\Big) \cdot x^3$$
$$+ 12 \cdot \Big(3\, a^3\, c_1\, c_2\, c_3 \, + 4 \cdot a^2 \cdot (b_1^2\, c_1^2 \, + b_2^2\, c_2^2 \, + b_3^2\, c_3^2)$$
$$+ 2 \cdot a^2 \cdot (b_1\, b_2\, c_1\, c_2 \, + b_1\, b_3\, c_1\, c_3 \, + b_2\, b_3\, c_2\, c_3)$$
$$- 12 \cdot a \cdot b_1\, b_2\, b_3 \cdot (b_1\, c_1 + b_2\, c_2 + b_3\, c_3) \, + 18 \cdot b_1^2\, b_2^2\, b_3^2\Big) \cdot x^2$$
$$- 12 \cdot a^3 \cdot \Big(a \cdot (b_1\, c_1 \, + \, b_2\, c_2 \, + \, b_3\, c_3) - 3\, b_1\, b_2\, b_3\Big) \cdot x \quad + a^6. \qquad (17)$$

The polynomial $P_5(x)$ in (12), given by $P_5(x) = (P_4(x)^2 - P_2(x)^3)/1728/x^3$, is a slightly larger polynomial of the form

$$P_5(x) = \, 27 \cdot c_1^4\, c_2^4\, c_3^4 \cdot x^5 \, + \, \cdots \, + q_1 \cdot x \, + q_0, \qquad \text{where:}$$
$$q_0 \, = \, - b_1\, b_2\, b_3\, a^3 \cdot (a\, c_1 \, - b_2\, b_3) \cdot (a\, c_2 \, - b_1\, b_3) \cdot (a\, c_3 \, - b_1\, b_2). \qquad (18)$$

The coefficient q_1 in x reads for instance:

$$q_1 \, = \, c_1\, c_2\, c_3\, (b_1\, b_2\, c_1\, c_2 + b_1\, b_3\, c_1\, c_3 + b_2\, b_3\, c_2\, c_3) \cdot a^5$$
$$- \Big(b_1^2\, b_2^2\, c_1^2\, c_2^2 + b_1^2\, b_3^2\, c_1^2\, c_3^2 + b_2^2\, b_3^2\, c_2^2\, c_3^2 \, - 8\, b_1\, b_2\, b_3\, c_1\, c_2\, c_3 \cdot (b_1\, c_1 + \, b_2\, c_2 + b_3\, c_3)\Big) \cdot a^4$$
$$- b_1\, b_2\, b_3 \cdot \Big(57\, b_1\, b_2\, b_3\, c_1\, c_2\, c_3$$
$$+ \, 8 \cdot (b_1^2\, b_2\, c_1^2\, c_2 + \, b_1^2\, b_3\, c_1^2\, c_3 + \, b_1\, b_2^2\, c_1\, c_2^2 \, + \, b_1\, b_3^2\, c_1\, c_3^2 + \, b_2^2\, b_3\, c_2^2\, c_3 + \, b_2\, b_3^2\, c_2\, c_3^2)\Big) \cdot a^3$$
$$+ 8\, b_1^2\, b_2^2\, b_3^2 \cdot (b_1^2\, c_1^2 + \, b_2^2\, c_2^2 + \, b_3^2\, c_3^2) \cdot a^2$$
$$+ 46 \cdot b_1^2\, b_2^2\, b_3^2 \cdot (b_1\, b_2\, c_1\, c_2 + \, b_1\, b_3\, c_1\, c_3 + \, b_2\, b_3\, c_2\, c_3) \cdot a^2$$
$$- 36 \cdot b_1^3\, b_2^3\, b_3^3 \cdot (b_1\, c_1 + b_2\, c_2 + b_3\, c_3) \cdot a \, + 27\, b_1^4\, b_2^4\, b_3^4. \qquad (19)$$

Having "guessed" the exact result, one can easily verify directly that this exact pullbacked hypergeometric result is truly the solution of the large second order linear differential operator obtained using the "*HolonomicFunctions*" program [28].

2.5. Simple symmetries of this seven-parameter result

The different pullbacks

$$\mathcal{P}_1 \, = \, - \, \frac{1728 \cdot x^3 \cdot P_5(x)}{P_2(x)^3}, \qquad \frac{1728 \cdot x^3 \cdot P_5(x)}{P_4(x)^2}, \qquad 1 \, - \, \frac{P_4(x)^2}{P_2(x)^3}, \qquad (20)$$

turn out to be compatible with the symmetries

$$\mathcal{P}_1(\lambda \cdot a,\ \lambda \cdot b_1,\ \lambda \cdot b_2,\ \lambda \cdot b_3,\ \lambda \cdot c_1,\ \lambda \cdot c_2,\ \lambda \cdot c_3,\ x) \\ \quad = \ \mathcal{P}_1(a,\ b_1,\ b_2,\ b_3,\ c_1,\ c_2,\ c_3,\ x). \qquad (21)$$

and

$$\mathcal{P}_1\Big(a,\ \ \lambda_1 \cdot b_1,\ \lambda_2 \cdot b_2,\ \lambda_3 \cdot b_3,\ \ \lambda_2\,\lambda_3 \cdot c_1,\ \lambda_1\,\lambda_3 \cdot c_2,\ \lambda_1\,\lambda_2 \cdot c_3,\ \ \frac{x}{\lambda_1\,\lambda_2\,\lambda_3}\Big) \\ \quad = \ \mathcal{P}_1(a,\ b_1,\ b_2,\ b_3,\ c_1, c_2, c_3,\ x), \qquad (22)$$

where $\lambda,\ \lambda_1,\ \lambda_2$ and λ_3 are arbitrary complex numbers. A demonstration of these symmetry-invariance relations (21) and (22) is sketched in Appendix A.

2.6. A symmetric subcase $\tau \, \rightarrow \, 3\,\tau$*:* $\ {}_2F_1([1/3, 2/3], [1], \mathcal{P})$

2.6.1. A few recalls on Maier's paper

We know from Maier [23] that the *modular equation* associated with† $\tau \, \rightarrow \, 3\,\tau$ corresponds to the elimination of the z variable between the two rational pullbacks:

$$\mathcal{P}_1(z) \ = \ \frac{12^3 \cdot z^3}{(z\,+27) \cdot (z\,+243)^3}, \qquad \mathcal{P}_2(z) \ = \ \frac{12^3 \cdot z}{(z\,+27) \cdot (z\,+3)^3}. \qquad (23)$$

Following Maier [23] one can also write the identities:

$$\Big(9 \cdot \Big(\frac{z\,+27}{z\,+243}\Big)\Big)^{1/4} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], \frac{1728\,z^3}{(z\,+27) \cdot (z\,+243)^3}\Big) \\ = \ \Big(\frac{1}{9} \cdot \Big(\frac{z\,+27}{z\,+3}\Big)\Big)^{1/4} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], \frac{1728\,z}{(z\,+27) \cdot (z\,+3)^3}\Big) \qquad (24)$$

$$= \ {}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], \frac{z}{z\,+27}\Big). \qquad (25)$$

Having a hypergeometric function identity (24) with *two* rational pullbacks (23) related by a *modular equation* provides a good heuristic way to see that we have a *modular form* [22, 23]‡.

2.6.2. The symmetric subcase

Taking the symmetric limit $b_1 \, = \, b_2 \, = \, b_3 \, = \, b$ and $c_1 \, = \, c_2 \, = \, c_3 \, = \, c$ in expression (13), we obtain the solution of the order-two linear differential operator annihilating the diagonal†† in the form

$$\frac{1}{P_2(x)^{1/4}} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1],\ 1\, - \, \frac{P_4(x)^2}{P_2(x)^3}\Big) \\ \quad = \ \frac{1}{P_2(x)^{1/4}} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \, - \, \frac{1728 \cdot x^3 \cdot P_5(x)}{P_2(x)^3}\Big), \qquad (26)$$

† τ denotes the ratio of the two periods of the elliptic functions that naturally emerge in the problem [22].

‡ Something that is obvious here since we are dealing with a ${}_2F_1([1/12, 5/12], [1], x)$ hypergeometric function which is known to be related to modular functions [22, 23] due to its relation with the Eisenstein series E_4, but is less clear for other hypergeometric functions.

†† Called the "telescoper" [30, 31].

with

$$P_2(x) \;=\; a \cdot (24 \cdot c^3 \cdot x^2 \, - 24 \cdot b \cdot (a c - b^2) \cdot x \, + a^3), \tag{27}$$

$$\begin{aligned} P_4(x) \;=\;& 216 \cdot c^6 \cdot x^4 \; - 432 \cdot b c^3 \cdot (a c \, - b^2) \cdot x^3 \\ &+ 36 \cdot (a^3 c^3 \, + 6 \cdot a^2 b^2 c^2 \, - 12 \cdot a b^4 c + 6 \cdot b^6) \cdot x^2 \\ &- 36 \cdot a^3 b \cdot (a c \, - b^2) \cdot x \, + a^6. \end{aligned} \tag{28}$$

and:

$$P_5(x) \;=\; (27 c^3 x^2 \, - 27 b \cdot (a c - b^2) \cdot x \, + a^3) \cdot (c^3 x \, - b \cdot (a c - b^2))^3. \tag{29}$$

In this symmetric case, one can write the pullback in (26) as follows:

$$- \, \frac{1728 \cdot x^3 \cdot P_5(x)}{P_2(x)^3} \;=\; \frac{12^3 \cdot z^3}{(z + 27) \cdot (z + 243)^3}, \tag{30}$$

where z reads:

$$z \;=\; - \, \frac{9^3 \cdot x \cdot (c^3 \cdot x \, - b \cdot (a c - b^2))}{27 \cdot c^3 \cdot x^2 \, - 27 \cdot b \cdot (a c - b^2) \cdot x \, + a^3}. \tag{31}$$

Injecting the expression (31) for z in $\mathcal{P}_2(z)$ given by (23), one gets another pullback

$$\mathcal{P}_2(z) \;=\; - 1728 \cdot x \cdot \frac{\tilde{P}_5}{\tilde{P}_2(x)^3}, \tag{32}$$

with

$$\tilde{P}_5(x) \;=\; (27 c^3 x^2 \, - 27 b \cdot (a c - b^2) \cdot x \, + a^3)^3 \cdot (c^3 x \, - b \cdot (a c - b^2)). \tag{33}$$

and:

$$\tilde{P}_2(x) \;=\; a \cdot (-216 \cdot c^3 \cdot x^2 \, + 216 \cdot b \cdot (a c - b^2) \cdot x \, + a^3), \tag{34}$$

In this case the diagonal of the rational function can be written as a single hypergeometric function with two different pullbacks

$$\begin{aligned} &\frac{1}{P_2(x)^{1/4}} \cdot {}_2F_1\Bigl([\frac{1}{12}, \frac{5}{12}], [1], - \frac{1728 \cdot x^3 \cdot P_5(x)}{P_2(x)^3}\Bigr) \\ &\quad = \; \frac{1}{\tilde{P}_2(x)^{1/4}} \cdot {}_2F_1\Bigl([\frac{1}{12}, \frac{5}{12}], [1], - \frac{1728 \cdot x \cdot \tilde{P}_5(x)}{\tilde{P}_2(x)^3}\Bigr), \end{aligned} \tag{35}$$

with the relation between the two pullbacks given by the *modular equation* associated [22, 23] with $\tau \rightarrow 3\tau$:

$$\begin{aligned} &2^{27} \cdot 5^9 \cdot Y^3 Z^3 \cdot (Y + Z) \; + 2^{18} \cdot 5^6 \cdot Y^2 Z^2 \cdot (27 Y^2 - 45946 Y Z + 27 Z^2) \\ &\quad + 2^9 \cdot 5^3 \cdot 3^5 \cdot \; Y Z \cdot (Y + Z) \cdot (Y^2 + 241433 Y Z + Z^2) \\ &\quad + 729 \cdot (Y^4 \, + \, Z^4) \; - 779997924 \cdot (Y Z^3 \, + Y^3 Z) \; + 31949606 \cdot 3^{10} \cdot Y^2 Z^2 \\ &\quad + 2^9 \cdot 3^{11} \cdot 31 \cdot Y Z \cdot (Y + Z) \; - 2^{12} \cdot 3^{12} \cdot Y Z \;\; = \;\; 0. \end{aligned}$$

2.6.3. Alternative expression for the symmetric subcase

Alternatively, we can obtain the exact expression of the diagonal using directly the "*HolonomicFunctions*" program [28] for arbitrary parameters a, b and c to get an order-two linear differential operator annihilating that diagonal. Then, using "*hypergeometricsols*"†† we obtain that the solution of this second order linear differential operator is given by

$$\frac{1}{a} \cdot {}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], \, -\frac{27}{a^3} \cdot x \cdot (c^3 x \, - b \cdot (a c \, - b^2))\Big), \qquad (36)$$

which looks, at first sight, different from (26) with (27) and (28). Yet this last expression (36) is compatible with the form (26) as a consequence of the identity:

$$\Big(\frac{9 - 8x}{9}\Big)^{1/4} \cdot {}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], x\Big) \, = \, {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], \frac{64 x^3 \cdot (1 - x)}{(9 - 8x)^3}\Big). \qquad (37)$$

The reduction of the (generic) ${}_2F_1([1/12, 5/12], [1], \mathcal{P})$ hypergeometric function to a ${}_2F_1([1/3, 2/3], [1], \mathcal{P})$ form corresponds to a selected $\tau \, \rightarrow \, 3\tau$ modular equation situation (23) well described in [23].

These results can also be expressed in terms of ${}_2F_1([1/3, 1/3], [1], \mathcal{P})$ pullbacked hypergeometric functions [23] using the identities

$$ {}_2F_1\Big([\frac{1}{3}, \frac{1}{3}], [1], x\Big) \, = \, (1 - x)^{-1/3} \cdot {}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], \, -\frac{x}{1 - x}\Big) \qquad (38)$$

$$ = \, \Big((1 - 9x)^3 \cdot (1 - x)\Big)^{-1/12} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \, -\frac{64\, x}{(1 - 9x)^3 \cdot (1 - x)}\Big),$$

or:

$$ {}_2F_1\Big([\frac{1}{3}, \frac{1}{3}], [1], -\frac{x}{27}\Big) \, = \, \Big(1 + \frac{x}{27}\Big)^{-1/3} \cdot {}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], \, \frac{x}{x + 27}\Big) \qquad (39)$$

$$ = \, \Big(\frac{(x+3)^3 \cdot (x + 27)}{729}\Big)^{-1/12} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \, \frac{1728\, x}{(x+3)^3 \cdot (x + 27)}\Big).$$

2.7. A non-symmetric subcase $\tau \, \rightarrow \, 4\tau$: ${}_2F_1([1/2, 1/2], [1], \mathcal{P})$.

Taking the non-symmetric limit $b_1 \, = \, b_2 \, = \, b_3 \, = \, b$ and $c_1 \, = \, c_2 \, = \, 0, \; c_3 \, = \, b^2/a$ in (13), the pullback in (26) reads:

$$\mathcal{P}_1 \, = \, -\frac{1728 \cdot x^3 \cdot P_5(x)}{P_2(x)^3} \, = \, \frac{1728 \cdot a^3 b^{12} \cdot x^4 \cdot (16 b^3 x \, + \, a^3)}{(16 b^6 x^2 \, + \, 16 a^3 b^3 x \, + a^6)^3}. \qquad (40)$$

This pullback can be seen as the first of the two Hauptmoduls

$$\mathcal{P}_1 \, = \, \frac{1728 \cdot z^4 \cdot (z \, + 16)}{(z^2 \, + \, 256 z \, + 4096)^3}, \qquad \mathcal{P}_2 \, = \, \frac{1728 \cdot z \cdot (z \, + 16)}{(z^2 \, + \, 16 z \, + 16)^3}, \qquad (41)$$

provided z is given by‡:

$$z \, = \, \frac{256\, b^3 x}{a^3} \qquad \text{or:} \qquad z \, = \, \frac{-256\, b^3 \cdot x}{a^3 \, + \, 16\, b^3 x}. \qquad (42)$$

These exact expressions (42) of z in terms of x give exact rational expressions of the second Hauptmodul $\mathcal{P}_2$ in terms of x:

$$\mathcal{P}_2^{(1)} \, = \, \frac{1728 \cdot a^{12} b^3 \cdot x \cdot (a^3 \, + 16 b^3 x)^4}{(4096 b^6 x^2 \, + 256 a^3 b^3 x \, + a^6)^3} \qquad \text{or:} \qquad (43)$$

$$\mathcal{P}_2^{(2)} \, = \, \frac{-1728 \cdot a^3 b^3 \cdot x \cdot (a^3 \, + 16 b^3 x)^4}{(256 b^6 x^2 \, - \, 224 a^3 b^3 x \, + a^6)^3}. \qquad (44)$$

†† We use M. van Hoeij "hypergeometricsols" program [29] for many values of a, b and c, and then perform some guessing.

‡ These two expressions are related by the involution $z \, \leftrightarrow \, -16\, z/(z \, + 16)$.

These two pullbacks (40), (43) and (44) (or $\mathcal{P}_1$ and $\mathcal{P}_2$ in (41)) are related by a *modular equation* corresponding† to $\tau \, \rightarrow \, 4\,\tau$.

This subcase thus corresponds to the diagonal of the rational function being expressed in terms of a *modular form* associated to an identity on a hypergeometric function:

$$
\begin{aligned}
&(16\, b^6\, x^2 \, + \, 16\, a^3\, b^3\, x \, + a^6)^{-1/4} \cdot {}_2F_1\Bigl([\frac{1}{12}, \, \frac{5}{12}], \, [1], \, \mathcal{P}_1\Bigr) \\
&\quad = \quad (4096\, b^6\, x^2 \, + 256\, a^3\, b^3\, x \, + a^6)^{-1/4} \cdot {}_2F_1\Bigl([\frac{1}{12}, \, \frac{5}{12}], \, [1], \, \mathcal{P}_2^{(1)}\Bigr) \\
&\quad = \quad (256\, b^6\, x^2 \, - \, 224\, a^3\, x \, + a^6)^{-1/4} \cdot {}_2F_1\Bigl([\frac{1}{12}, \, \frac{5}{12}], \, [1], \, \mathcal{P}_2^{(2)}\Bigr) \\
&\quad = \quad {}_2F_1\Bigl([\frac{1}{2}, \, \frac{1}{2}], \, [1], \, -\, \frac{16 \cdot b^3}{a^3} \cdot x\Bigr).
\end{aligned}
\tag{45}
$$

The last equality is a consequence of the identity:

$$
\begin{aligned}
&{}_2F_1\Bigl([\frac{1}{2}, \, \frac{1}{2}], \, [1], \, -\, \frac{x}{16}\Bigr) \\
&\quad = \quad 2 \cdot \, (x^2 \, + 16\, x \, + 16)^{-1/4} \cdot \, {}_2F_1\Bigl([\frac{1}{12}, \, \frac{5}{12}], \, [1], \, \frac{1728 \cdot \, x \cdot \, (x+16)}{(x^2 \, + 16\, x \, + 16)^3}\Bigr).
\end{aligned}
\tag{46}
$$

Similarly, the elimination of x between the pullback $X \, = \, \mathcal{P}_1$ (given by (40)) and $Y \, = \, \mathcal{P}_2^{(1)}$ gives the *same modular equation* (representing $\tau \, \rightarrow \, 4\,\tau$) as the elimination of x between the pullback $X \, = \, \mathcal{P}_1$ (given by (40)) and $Y \, = \, \mathcal{P}_2^{(2)}$, given in Appendix B by equation (B.1). The elimination of x between the pullback $X \, = \, \mathcal{P}_2^{(1)}$ (given by (40)) and the pullback $Y \, = \, \mathcal{P}_2^{(2)}$ also gives the *same modular equation* (B.1).

2.8. ${}_2F_1([1/4, 3/4], [1], \mathcal{P})$ *subcases: walks in the quarter plane*

The diagonal of the rational function

$$
\frac{4}{4 \;\, + 2 \cdot \, (x+y+z) \, + \, 2 \cdot \, x\,z + \, x\,y}, \tag{47}
$$

is given by the pullbacked hypergeometric function:

$$
\begin{aligned}
&\Bigl(1 \, + \, \frac{3}{4} \cdot \, x^2\Bigr)^{-1/4} \cdot \, {}_2F_1\Bigl([\frac{1}{12}, \, \frac{5}{12}], \, [1], \, \frac{27\, x^4 \cdot \, (x^2+1)}{(3\, x^2 \, + 4)^3}\Bigr) \\
&\qquad = \quad {}_2F_1\Bigl([\frac{1}{4}, \, \frac{3}{4}], \, [1], \, -x^2\Bigr),
\end{aligned}
\tag{48}
$$

which is reminiscent of the hypergeometric series number 5 and 15 in Figure 10 of [32]. Such pullbacked hypergeometric function (48) corresponds to the rook walk problems [33, 34, 35].

Thus the diagonal of the rational function corresponding to the simple rescaling $(x, \, y, \, z) \;\; \longrightarrow \;\; (\pm\sqrt{-1}\, x, \; \pm\sqrt{-1}\, y, \; \pm\sqrt{-1}\, z)$ of (47) given by

$$
R_{\pm} \;\; = \;\; \frac{4}{4 \;\; \pm 2\,\sqrt{-1} \cdot \, (x+y+z) \, - \, 2 \cdot \, x\,z \, - \, x\,y} \tag{49}
$$

or the diagonal of the rational function $(R_+ \, + \, R_-)/2$ reading

$$
\frac{4 \cdot \, (4 \, - xy \, - 2\,xz)}{y^2x^2 \, + 4\,x^2yz + 4\,x^2z^2 + 4\,x^2 \, - 8\,xz + 4\,y^2 + 8\,yz + 4\,z^2 \, + 16}, \tag{50}
$$

† See page 20 in [22].

becomes (as a consequence of identity (48)):

$$\Big(1 - \frac{3}{4} \cdot x^2\Big)^{-1/4} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \frac{27\,x^4 \cdot (1 - x^2)}{(4 - 3\,x^2)^3}\Big) = {}_2F_1\Big([\frac{1}{4}, \frac{3}{4}], [1], x^2\Big). \qquad (51)$$

Though it is not explicitely mentioned in [23] it is worth pointing out that the ${}_2F_1([1/4, 3/4], [1], \mathcal{P})$ hypergeometric functions can be seen as modular forms corresponding to identities with *two* pullbacks related by a modular equation. For example the following identity:

$${}_2F_1\Big([\frac{1}{4}, \frac{3}{4}], [1], \frac{x^2}{(2 - x)^2}\Big) = \Big(\frac{2 - x}{2 \cdot (1 + x)}\Big)^{1/2} \cdot {}_2F_1\Big([\frac{1}{4}, \frac{3}{4}], [1], \frac{4\,x}{(1 + x)^2}\Big), \qquad (52)$$

where the two rational pullbacks

$$A = \frac{4\,x}{(1 + x)^2}, \qquad B = \frac{x^2}{(2 - x)^2}, \qquad (53)$$

are related by the *asymmetric*¶ modular equation:

$$81 \cdot A^2 B^2 - 18\,A\,B \cdot (8\,B + A) + (A^2 + 80 \cdot A\,B + 64\,B^2) - 64\,B = 0. \qquad (54)$$

The modular equation (54) gives an expansion for B that can be seen as an *algebraic series*§ in A:

$$B = \frac{1}{64}A^2 + \frac{5}{256}A^3 + \frac{83}{4096}A^4 + \frac{163}{8192}A^5 + \frac{5013}{262144}A^6 + \cdots \qquad (55)$$

More details are given in Appendix C.

2.9. The generic case: modular forms, pullbacked hypergeometric functions with just one rational pullback

The pullbacks of the ${}_2F_1$ hypergeometric functions in the previous sections can be seen as *Hauptmoduls* [23]. It is only in certain cases like in sections (2.6) or (2.7) that we encounter the situation underlined by Maier [23] of a representation of a modular form as a pullbacked hypergeometric function with *two rational pullbacks*, related by a modular equation of *genus zero*.

Examples of modular equations of genus zero with *rational pullbacks* include for example reductions of the generic ${}_2F_1([1/12, 5/12], [1], \mathcal{P})$ hypergeometric function to particular hypergeometric functions like ${}_2F_1([1/2, 1/2], [1], \mathcal{P})$, ${}_2F_1([1/3, 2/3], [1], \mathcal{P})$, ${}_2F_1([1/4, 3/4], [1], \mathcal{P})$, and also [25] ${}_2F_1([1/6, 5/6], [1], \mathcal{P})$ (see for instance [36]).

In the generic situation corresponding to (13) however, we have a single hypergeometric function with two pullbacks A and B

$${}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], A\Big) = G \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], B\Big), \qquad (56)$$

¶ At first sight one expects the two pullbacks (53) in a relation like (54) to be on the same footing, the *modular equation* between these two pullbacks being *symmetric*: see for instance [22]. This paradox is explained in detail in Appendix C

§ We discard the other root expansion $B = 1 + A + \frac{5}{4}A^2 + \frac{25}{16}A^3 + \frac{31}{16}A^4 + \cdots$ since $B(0) \neq 0$.

with G an algebraic function of x, and where A and B *are related by an algebraic modular equation*, with one of the pullbacks a rational function given by (12) where $P_2(x)$ and $P_4(x)$ are given respectively by (16) and (17). The two pullbacks A and B are also related by a *Schwarzian equation* [22, 37, 38] that can be written in a symmetric way in A and B:

$$\frac{1}{72}\,\frac{32\,B^2\,-41\,B\,+36}{B^2\cdot(B\,-1)^2}\cdot\Big(\frac{dB}{dx}\Big)^2\;+\;\{B,\,x\}$$
$$=\;\frac{1}{72}\,\frac{32\,A^2\,-41\,A\,+36}{A^2\cdot(A\,-1)^2}\cdot\Big(\frac{dA}{dx}\Big)^2\;+\;\{A,\,x\}. \tag{57}$$

One can rewrite the exact expression (13) in the form

$$\frac{1}{P_2(x)^{1/4}}\cdot{}_2F_1\Big([\frac{1}{12},\,\frac{5}{12}],\,[1],\;1\,-\,\frac{P_4(x)^2}{P_2(x)^3}\Big)$$
$$=\;\mathcal{B}\cdot{}_2F_1\Big([\frac{1}{12},\,\frac{5}{12}],\,[1],\;B\Big), \tag{58}$$

where $\mathcal{B}$ is an algebraic function of x, and B is an algebraic pullback related to the *rational* pullback $A\;=\;1\,-\,P_4(x)^2/P_2(x)^3$ by a modular equation. In the generic case, *only one of the two pullbacks* (58) *can be expressed as a rational function* of x.

3. Nine and ten-parameter generalizations

Adding randomly terms in the denominator of (7) yields diagonals annihilated by minimal linear differential operators of order higher than two: this is what happens when quadratic terms like x^2, y^2 or z^2 are added for example. This leads to irreducible telescopers [30, 31] (i.e. linear differential operators annihilating the diagonals) of orders higher two, or to reducible telescopers [30] that factor into several irreducible factors, one of them being of order larger than two.

With the idea of keeping the linear differential operators annihilating the diagonal of order two, we were able to generalize the seven-parameter family (7) by carefully choosing the terms added to the quadratic terms in (7) and still keep the linear differential operator annihilating the diagonal of order two.

3.1. Nine-parameter rational functions giving pullbacked $_2F_1$ hypergeometric functions for their diagonals

Adding the two cubic terms $x^2\,y$ and $y\,z^2$ to the denominator of (7)

$$\frac{1}{a\;+b_1\,x\,+\,b_2\,y\,+\,b_3\,z\;\;+c_1\,y\,z\,+\,c_2\,x\,z\,+\,c_3\,x\,y\;\;+\;d\,x^2\,y\;\;+\;e\,y\,z^2}, \tag{59}$$

gives a linear differential operator annihilating the diagonal of (59) of order two†. After computing the second order linear differential operator annihilating the diagonal of (59) for several values of the parameters with the "*HolonomicFunctions*" program [28], then obtaining their pullbacked hypergeometric solutions using the maple command

† The nine-parameter family (59) singles out x and y, but of course, similar families that single out x and z, or single out y and z exist, with similar results (that can be obtained permuting the three variables x, y and z).

"*hypergeometricsols*" [29], we find that the diagonal of the rational function (59) has the form

$$\frac{1}{P_4(x)^{1/4}} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], 1 - \frac{P_6(x)^2}{P_4(x)^3}\Big), \qquad (60)$$

where $P_4(x)$ and $P_6(x)$ are two polynomials of degree four and six respectively:

$$\begin{aligned} P_4(x) \, = \, & p_2 \, + \, 16 \cdot d^2 \cdot e^2 \cdot x^4 \\ & - 16 \cdot \Big(3 \cdot c_2 \cdot (c_1^2 \cdot d + c_3^2 \cdot e) + (b_1 c_1 + b_3 c_3 - 14 b_2 c_2) \cdot d e\Big) \cdot x^3 \\ & + 8 \cdot (3 a b_3 c_1 d + 3 a b_1 c_3 e - a^2 d e - 6 b_2 b_3^2 d - 6 b_2 b_1^2 e) \cdot x^2, \end{aligned} \qquad (61)$$

and

$$\begin{aligned} P_6(x) \, = \, & p_4 \, - 12 \cdot a^4 d e \cdot x^2 \\ & + 36 \cdot a^2 \Big(b_3 \cdot (a c_1 - 2 b_2 b_3) \cdot d + b_1 \cdot (a c_3 - 2 b_1 b_2) \cdot e\Big) \cdot x^2 \\ & - 72 \cdot a c_1 \cdot (a c_1 c_2 - 10 b_2 b_3 c_2 + 2 b_3^2 c_3) \cdot d \cdot x^3 \\ & - 72 \cdot a c_3 \cdot (a c_2 c_3 - 10 b_1 b_2 c_2 + 2 b_1^2 c_1) \cdot e \cdot x^3 \\ & - 144 \cdot b_2 b_3^2 \cdot (b_1 c_1 + 4 b_2 c_2 - 2 b_3 c_3) \cdot d \cdot x^3 \\ & - 144 \cdot b_2 b_1^2 \cdot (b_3 c_3 + 4 b_2 c_2 - 2 b_1 c_1) \cdot e \cdot x^3 \\ & - 144 \cdot a b_1 b_3 \cdot (c_1^2 \cdot d + c_3^2 \cdot e) \cdot x^3 \\ & + 24 \cdot a \, (a b_3 c_3 + a b_1 c_1 - 20 a b_2 c_2 + 30 b_1 b_2 b_3) \cdot d \cdot e \cdot x^3 \\ & + 216 \cdot (b_3^2 c_1^2 \cdot d^2 + b_1^2 c_3^2 \cdot e^2) \cdot x^4 \\ & - 144 \cdot c_1^2 \, c_2 \cdot (b_3 c_3 + 4 b_2 c_2 - 2 b_1 c_1) \cdot d \cdot x^4 \\ & - 144 \cdot c_3^2 \, c_2 \cdot (b_1 c_1 + 4 b_2 c_2 - 2 b_3 c_3) \cdot e \cdot x^4 \\ & + 48 \cdot a^2 d^2 \cdot e^2 \cdot x^4 \, + 96 \cdot (b_1^2 c_1^2 + b_3^2 c_3^2 + 22 b_2^2 c_2^2) \cdot d \cdot e \cdot x^4 \\ & - 144 \cdot \Big((a b_3 c_1 + 4 b_2 b_3^2) \cdot d + (a b_1 c_3 + 4 b_2 b_1^2) \cdot e\Big) \cdot d \cdot e \cdot x^4 \\ & + 48 \cdot (b_1 b_3 c_1 c_3 + 15 a c_1 c_2 c_3 - 20 b_1 b_2 c_1 c_2 - 20 b_2 b_3 c_2 c_3) \cdot d \cdot e \cdot x^4 \\ & + 96 \cdot (b_1 c_1 + 22 b_2 c_2 + b_3 c_3) \cdot d^2 \cdot e^2 \cdot x^5 \\ & - 576 c_2 \cdot (c_3^2 \cdot e + c_1^2 \cdot d) \cdot d e \cdot x^5 \\ & - 64 \cdot d^3 \cdot e^3 \cdot x^6, \end{aligned} \qquad (62)$$

where the polynomials p_2 and p_4 are the polynomials $P_2(x)$ and $P_4(x)$ of degree two and four in x given by (16) and (17) in section (2): p_2 and p_4 correspond to the $d = e = 0$ limit

It is worth pointing out two facts, firstly that the $d \leftrightarrow e$ symmetry corresponds to keeping c_2 fixed, but changing $c_1 \leftrightarrow c_3$ (or equivalently y fixed, $x \leftrightarrow z$), secondly that the simple symmetry arguments displayed in section (2.5) for the seven-parameter family straightforwardly generalize for this nine-parameter family (see relations (A.6) and (A.7) in Appendix A.3).

3.2. Ten-parameter rational functions giving pullbacked ${}_2F_1$ hypergeometric functions for their diagonals

Adding the *three cubic terms*‡ $x^2\, y$, $y^2\, z$ and $z^2\, x$ to the denominator of (7) we get the rational function:

$$R(x,\, y,\, z) \,\,\, = \qquad\qquad (63)$$

$$\frac{1}{a \,\, + b_1\, x \, + \, b_2\, y \, + \, b_3\, z \,\,\, + c_1\, y\, z \, + \, c_2\, x\, z \, + \, c_3\, x\, y \,\,\, + \,\, d_1\, x^2\, y \,\, + \,\, d_2\, y^2\, z \, + \,\, d_3\, z^2\, x}.$$

Note that (63) is *not* a generalization of (59).

After computing the second order linear differential operator annihilating the diagonal of (63) for several values of the parameters with the "*HolonomicFunctions*" program [28], then their pullbacked hypergeometric solutions using "*hypergeometricsols*" [29], we find that the diagonal of the rational function (63) has the experimentally observed form:

$$\frac{1}{P_3(x)^{1/4}} \cdot {}_2F_1\Big([\frac{1}{12}, \, \frac{5}{12}], \, [1], \,\, 1 \, - \, \frac{P_6(x)^2}{P_3(x)^3}\Big). \qquad (64)$$

Furthermore, the pullback in (64) is seen to be of the form:

$$1 \, - \, \frac{P_6(x)^2}{P_3(x)^3} \quad = \quad \frac{1728\, x^3 \cdot \, P_9}{P_3(x)^3}. \qquad (65)$$

The polynomial $P_3(x)$ reads

$$P_3(x) \, = \, p_2 \,\,\, - 24 \cdot \Big(9 \cdot \, a \cdot \, d_1\, d_2\, d_3 \,\,\, - 6 \cdot \, (b_1\, c_3 \cdot \, d_2\, d_3 \, + \, b_2\, c_1 \cdot \, d_1\, d_3 \, + \, b_3\, c_2 \cdot \, d_1\, d_2)$$
$$+ 2 \cdot \, (c_1^2\, c_2\, d_1 \, + \, c_1\, c_3^2\, d_3 \, + \, c_2^2\, c_3\, d_2)\Big) \cdot \, x^3 \qquad (66)$$
$$+ 24 \cdot \, \Big(a \cdot \, (b_1\, c_2\, d_2 \, + \, b_2\, c_3\, d_3 \, + \, b_3\, c_1\, d_1) \,\,\, - 2 \, \cdot \, (b_1^2\, b_3\, d_2 \, + \, b_1\, b_2^2\, d_3 \, + \, b_2\, b_3^2\, d_1)\Big) \cdot \, x^2,$$

where p_2 is the polynomial $P_2(x)$ of degree two in x given by (16) in section (2): p_2 corresponds to the $d_1 \, = \, d_2 \, = \, d_3 \, = \, 0$ limit. The expression of the polynomial $P_6(x)$ is more involved. It reads:

$$P_6(x) \,\, = \,\,\, p_4 \,\,\, + \,\, \Delta_6(x), \qquad (67)$$

where p_4 is the polynomial $P_4(x)$ of degree four in x given by (17) in section (2). The expression of polynomial $\Delta_6(x)$ of degree six in x is quite large and is given in Appendix D.

A set of results and subcases (sections (3.2.2) and (3.2.3)), were used to "guess" the general exact expressions of the polynomials $P_3(x)$ and $P_6(x)$ in (64) for the ten-parameters family (63). From the subcase $d_3 \, = \, 0$ of section (3.2.1) below, it is easy to see that one can deduce similar exact results for $d_1 \, = 0$ or $d_2 \, = 0$ by performing the cyclic transformation $x \, \rightarrow \, y \, \rightarrow \, z \, \rightarrow \, x$ corresponding to the transformation $b_1 \, \rightarrow \, b_2 \, \rightarrow \, b_3 \, \rightarrow \, b_1$, $c_1 \, \rightarrow \, c_2 \, \rightarrow \, c_3 \, \rightarrow \, c_1$, $d_1 \, \rightarrow \, d_2 \, \rightarrow \, d_3 \, \rightarrow \, d_1$. So one can see P_3 and $P_6(x)$ as the polynomials p_2 and p_4 given by (16) and (17) with corrections terms given, in Appendix E, by (E.1) and (E.2) for $d_3 \, = \, 0$. Similar corrections† for

‡ An equivalent family of ten-parameter rational functions amounts to adding $x\, y^2$, $y\, z^2$ and $z\, x^2$.
† Taking care of the double counting !

$d_1 = 0$ and $d_2 = 0$, as well as correction terms having the form $d_1 d_2 d_3 \times (\cdots)$, and so on and so forth, these terms being the most difficult to obtain¶.

Similarly to the previous section the symmetry arguments displayed in section (2.5) for the seven-parameter family also apply to this ten-parameter family (see (A.8) and (A.9) in Appendix A.3).

Remark : Do note that adding arbitrary sets of cubic terms yields telescopers [30, 31] of order larger than two: *the corresponding diagonals are no longer pullbacked* $_2F_1$ *hypergeometric functions.*

Let us just now focus on simpler subcases whose results are easier to obtain than in the general case (63).

3.2.1. Subcase of (63): a nine-parameter rational function

Instead of adding three cubic terms, let us add two cubic terms. This amounts to restricting the rational function (63) to the $d_3 = 0$ subcase

$$\frac{1}{a + b_1 x + b_2 y + b_3 z + c_1 y z + c_2 x z + c_3 x y + d_1 x^2 y + d_2 y^2 z}, \qquad (68)$$

which *cannot be reduced to the nine parameter family* (59) even if it looks similar. The diagonal of the rational function (68) has the *experimentally observed* form

$$\frac{1}{P_3(x)^{1/4}} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \, 1 - \frac{P_5(x)^2}{P_3(x)^3}\Big), \qquad (69)$$

where $P_3(x)$ and $P_5(x)$ are two polynomials of degree respectively three and five in x. Furthermore the pullback in (69) has the form:

$$1 - \frac{P_5(x)^2}{P_3(x)^3} = \frac{1728 \, x^3 \cdot P_7}{P_3(x)^3}. \qquad (70)$$

The two polynomials $P_3(x)$ and $P_5(x)$ are given in Appendix E.

3.2.2. Cubic terms subcase of (63)

Taking the limit $b_1 = b_2 = b_3 = c_1 = c_2 = c_3 = 0$ in (63) we obtain:

$$R(x, y, z) = \frac{1}{a + d_1 \cdot x^2 y + d_2 \cdot y^2 z + d_3 \cdot z^2 x},$$

whose diagonal reads

$$\begin{aligned} &{}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], \, -27 \cdot \frac{d_1 d_2 d_3}{a^3} \cdot x^3\Big) \qquad (71) \\ &= \Big(1 - 216 \cdot \frac{d_1 d_2 d_3}{a^3} \cdot x^3\Big)^{-1/4} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \, 1 - \frac{P_6(x)^2}{P_3(x)^3}\Big), \end{aligned}$$

with:

$$P_3(x) = -216 \cdot a \, d_1 \, d_2 \, d_3 \cdot x^3 + a^4, \qquad (72)$$

$$P_6(x) = -5832 \cdot d_1^2 \, d_2^2 \, d_3^2 \cdot x^6 + 540 \cdot a^3 \, d_1 d_2 d_3 \cdot x^3 + a^6. \qquad (73)$$

¶ We already know some of these terms from (72) and (73) in section (3.2.2) below. Furthermore, the symmetry constraints (A.9) and (A.8) in Appendix A.3, as well as other constraints corresponding to the symmetric subcase of section (3.2.3), give additional constraints on the kind of allowed final correction terms.

3.2.3. A symmetric subcase of (63)

Taking the limit symmetric limit $b_1 = b_2 = b_3 = b$, $c_1 = c_2 = c_3 = c$, $d_1 = d_2 = d_3 = d$ in (63), the diagonal reads‡

$$\frac{1}{a - 6\, d \cdot x} \cdot {}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], \mathcal{P}\Big), \tag{74}$$

where the pullback $\mathcal{P}$ reads:

$$\mathcal{P} \, = \, - \, \frac{27\, x \cdot \Big(a^2\, d \, - a\, b\, c \, + b^3 \, + (c^3 \, - 3\, b\, c\, d \, - 3\, a\, d^2) \cdot x \, + 9\, d^3 \cdot x^2\Big)}{(a \, - 6\, d \cdot x)^3}. \tag{75}$$

At first sight the hypergeometric result (74) with the pullback (75) does not seem to be in agreement with the hypergeometric result (71) of section (3.2.2). In fact these two results are in agreement as a consequence of the hypergeometric identity:

$$\frac{1}{1 - 6\,X} \cdot {}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], \, -\frac{27 \cdot X \cdot (1 \, - 3\,X \, + 9\,X^2)}{(1 \, - 6\,X)^3}\Big)$$
$$= \, {}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], \, -27 \cdot X^3\Big) \qquad \text{with:} \qquad X \, = \, \frac{d \cdot x}{a}. \tag{76}$$

This hypergeometric result (71) can also be rewritten in the form (64) where the two polynomials $P_3(x)$ and $P_6(x)$ read respectively:

$$P_3(x) \, = \, -72 \cdot d \cdot (3\,ad^2 \, - 6\,bcd \, + 2\,c^3) \cdot x^3 \, + 24 \cdot (3\,abc\,d \, + ac^3 \, - 6\,b^3\,d) \cdot x^2$$
$$- 24 \cdot a\,b \cdot (ac - b^2) \cdot x \quad + a^4, \tag{77}$$

$$P_6(x) \, = \, -5832 \cdot d^6 \cdot x^6 \, + 3888 \cdot c\,d^3 \cdot (3\,b\,d \, - c^2) \cdot x^5$$
$$- 216 \cdot (18\,abc\,d^3 \, + 18\,b^3\,d^3 \, - 12\,ac^3d^2 \, - 9\,b^2c^2\,d^2 \, + 6\,bc^4\,d \, - c^6) \cdot x^4$$
$$+ 108 \cdot (5\,a^3\,d^3 \, - 18\,a^2bc\,d^2 \, - 2\,a^2c^3\,d + 12\,ab^2c^2\,d \, + 24\,ab^3\,d^2 \, - 4\,a\,bc^4$$
$$- 12\,b^4c\,d + 4\,b^3c^3) \cdot x^3$$
$$+ 36 \cdot (3\,a^3bc\,d \, - 6\,a^2b^3\,d \, + a^3c^3 + 6\,a^2b^2c^2 \, - 12\,ab^4c + 6\,b^6) \cdot x^2$$
$$- 36 \cdot a^3\,b \cdot (ac - b^2) \cdot x \quad + a^6. \tag{78}$$

4. Transformation symmetries of the diagonals of rational functions

The previous results can be expanded through symmetry considerations: performing monomial transformations on each of the previous (seven, eight, nine or ten-parameter) rational functions yields an *infinite number* of rational functions whose diagonals are pullbacked ${}_2F_1$ hypergeometric functions.

4.1. $(x, y, z) \, \rightarrow \, (x^n, y^n, z^n)$ symmetries

We have a first remark: once we have an exact result for a diagonal, we immediately get another diagonal by changing (x, y, z) into (x^n, y^n, z^n) for any positive integer n in the rational function. As a result we obtain a new expression for the diagonal changing x into x^n.

A simple example amounts to revisiting the fact that the diagonal of (49) given above is the hypergeometric function (51). Changing (x, y, z) into $(8\,x^2, 8\,y^2, 8\,z^2)$

‡ Trying to mix the two previous subcases by imposing $b_1 = b_2 = b_3 = b$, $c_1 = c_2 = c_3 = c$ with d_1, d_2 , d_3 not being equal, does not yield a ${}_2F_1([1/3, 2/3], [1], \mathcal{P})$ hypergeometric function.

in (49), one obtains the pullbacked ${}_2F_1$ hypergeometric function number 5 or 15 in Figure 10 of [32] (see also [33, 34, 35])

$$ {}_2F_1\Bigl([\frac{1}{4}, \frac{3}{4}], [1], 64\, x^4\Bigr), \qquad (79)$$

can be seen as the diagonal of

$$ \frac{2}{2 \;+ 8\sqrt{-1}\cdot\, (x^2+y^2+z^2) \;-\; 64\, x^2\, z^2 \;-\; 32\cdot x^2\, y^2}, \qquad (80)$$

which is tantamount to saying that the transformation $(x, \, y, \, z) \, \rightarrow \, (x^n, \, y^n, \, z^n)$ is a symmetry.

4.2. Monomial transformations on rational functions

More generally, let us consider the monomial transformation

$$ (x, \, y, \, z) \quad \longrightarrow \quad M(x, \, y, \, z) \; = \; (x_M, \, y_M, \, z_M) $$
$$ = \; \Bigl(x^{A_1} \cdot y^{A_2} \cdot z^{A_3}, \; x^{B_1} \cdot y^{B_2} \cdot z^{B_3}, \; x^{C_1} \cdot y^{C_2} \cdot z^{C_3}\Bigr), \qquad (81)$$

where the A_i's, B_i's and C_i's are positive integers such that $A_1 \, = \, A_2 \, = \, A_3$ is excluded (as well as $B_1 \, = \, B_2 \, = \, B_3$ as well as $C_1 \, = \, C_2 \, = \, C_3$), where the determinant of the $3 \, \times \, 3$ matrix

$$ \left[\begin{array}{ccc} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{array} \right], \qquad (82)$$

is not equal to zero††, and where:

$$ A_1 \, + B_1 \, + C_1 \; = \; A_2 \, + B_2 \, + C_2 \; = \; A_3 \, + B_3 \, + C_3. \qquad (83)$$

We will denote by n the integer in these three equal† sums (83): $n \, = \, A_i \, + B_i \, + C_i$. The condition (83) is introduced in order to force the product¶ of $x_M \, y_M \, z_M$ to be an integer power of the product of $x \, y \, z$: $x_M \, y_M \, z_M \; = \; (x \, y \, z)^n$.

If we take a rational function $\mathcal{R}(x, \, y, \, z)$ in three variables and perform a monomial transformation (81) $(x, \, y, \, z) \, \rightarrow \, M(x, \, y, \, z)$, on the rational function $\mathcal{R}(x, \, y, \, z)$, we get another rational function that we denote by $\tilde{\mathcal{R}} \; = \; \mathcal{R}(M(x, \, y, \, z))$. Now the diagonal of $\tilde{\mathcal{R}}$ is the diagonal of $\mathcal{R}(x, \, y, \, z)$ where we have changed x into x^n:

$$ \Phi(x) \; = \; Diag\Bigl(\mathcal{R}\Bigl(x, \, y, \, z\Bigr)\Bigr), \qquad Diag\Bigl(\tilde{\mathcal{R}}\Bigl(x, \, y, \, z\Bigr)\Bigr) \; = \; \Phi(x^n). \qquad (84)$$

A demonstration of this result is sketched in Appendix F.

From the fact that the diagonal of the rational function

$$ \frac{1}{1 \; + x \; + y \; + z \; \; + 3 \cdot \, (x \, y \; + \; y \, z + \; x \, z)}, \qquad (85)$$

is the hypergeometric function

$$ {}_2F_1\Bigl([\frac{1}{3}, \frac{2}{3}], [1], \;\; 27\, x \cdot \, (2 \, - 27\, x)\Bigr), \qquad (86)$$

††We want the rational function $\tilde{\mathcal{R}} \, = \, \mathcal{R}(M(x, \, y, \, z))$ deduced from the monomial transformation (81) to remain a rational function of three variables and not of two, or one, variables.
† For $n \, = \, 1$ the $3{\times}3$ matrix (82) is stochastic and transformation (81) is a *birational transformation*.
¶ Recall that taking the diagonal of a rational function of three variables extracts, in the multi-Taylor expansion (5), only the terms that are n-th power of the *product* $x \, y \, z$.

one deduces for example that the diagonal of the rational function (87) transformed by the monomial transformation $(x, y, z) \longrightarrow (z, x^2 y, y z)$

$$\frac{1}{1 + y z + x^2 y + 3 \cdot (y z^2 + x^2 y z + x^2 y^2 z)}, \tag{87}$$

is the pullbacked hypergeometric function

$${}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], \; 27 x^2 \cdot (2 - 27 x^2)\Big), \tag{88}$$

which is (86) where $x \rightarrow x^2$.

To illustrate the point further, from the fact that the diagonal of the rational function

$$\frac{1}{1 + x + y + z + 3 x y + 5 y z + 7 x z}, \tag{89}$$

is the hypergeometric function

$$\frac{1}{(2712 x^2 - 96 x + 1)^{1/4}} \tag{90}$$
$$\times \; {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \; 1 - \frac{(2381400 x^4 - 181440 x^3 + 7524 x^2 - 144 x + 1)^2}{(2712 x^2 - 96 x + 1)^3}\Big),$$

one deduces immediately that the diagonal of the rational function (89) transformed by the monomial transformation $(x, y, z) \rightarrow (x z, x^2 y, y^2 z^2)$

$$\frac{1}{1 + x z + x^2 y + y^2 z^2 + 3 x^2 y^3 + 5 x y^2 z^3 + 7 x^3 y z}, \tag{91}$$

is the hypergeometric function

$$\frac{1}{(2712 x^6 - 96 x^3 + 1)^{1/4}} \tag{92}$$
$$\times \; {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \; 1 - \frac{(2381400 x^{12} - 181440 x^9 + 7524 x^6 - 144 x^3 + 1)^2}{(2712 x^6 - 96 x^3 + 1)^3}\Big),$$

which is nothing but (90) where x has been changed into x^3.

4.3. More symmetries on diagonals

Other transformation symmetries of the diagonals include the function dependent rescaling transformation

$$(x, y, z) \quad \longrightarrow \quad \Big(F(x y z) \cdot x, \quad F(x y z) \cdot y, \quad F(x y z) \cdot z\Big), \tag{93}$$

where $F(x y z)$ is a rational function† of the product of the three variables x, y and z. Under such a transformation the previous diagonal $\Delta(x)$ becomes $\Delta(x \cdot F(x)^3)$.

To illustrate the point take

$$(x, y, z) \quad \longrightarrow \quad \Big(x \cdot F, \; y \cdot F, \; z \cdot F\Big), \qquad \text{with:} \tag{94}$$

$$F \; = \; \frac{1 + 2 x y z}{1 + 3 x y z + 5 x^2 y^2 z^2} \; = \; \Phi(x y z), \tag{95}$$

$$\text{where:} \qquad \Phi(x) \; = \; \frac{1 + 2 x}{1 + 3 x + 5 x^2}, \tag{96}$$

† More generally one can imagine that $F(x y z)$ is the series expansion of an algebraic function.

the rational function

$$\frac{1}{1\ +x\ +y\ +z\ \ +y\,z+\,x\,z\,+\,x\,y}, \qquad (97)$$

whose diagonal is ${}_2F_1([1/3,\,2/3],\,[1],\,-27\,x^2)$, becomes the rational function $P(x,\,y,\,z)/Q(x,\,y,\,z)$, where the numerator $P(x,\,y,\,z)$ and the denominator $Q(x,\,y,\,z)$, read respectively:

$$P(x,\,y,\,z)\ =\ (1\,+\,3\,x\,y\,z\,+\,5\,x^2\,y^2\,z^2)^2, \qquad (98)$$

$$\begin{aligned}Q(x,\,y,\,z)\ =\ &25\,x^4y^4z^4\ +10\cdot\,(x^4y^3z^3\,+\,x^3y^4z^3\,+\,x^3y^3z^4)\,+30\,x^3y^3z^3\\ &+4\cdot\,(x^3y^3z^2\,+\,x^3y^2z^3\,+\,x^2y^3z^3)\ +11\cdot\,(x^3y^2z^2\,+\,x^2y^3z^2\,+\,x^2y^2z^3)\\ &+19\,x^2y^2z^2\ +4\cdot\,(x^2y^2z\,+\,x^2yz^2\,+\,xy^2z^2)\ +5\cdot\,(x^2yz\,+\,xy^2z\,+\,xyz^2)\\ &+6\,xyz\ \ +xy+xz+yz\ \ +x+y+z\ \ +1. \qquad (99)\end{aligned}$$

The diagonal of this last rational function is equal to:

$$\begin{aligned}&{}_2F_1\Big([\frac{1}{3},\,\frac{2}{3}],\,[1],\ \ -27\cdot\,\Big(x\,\cdot\,\Phi(x)^3\Big)^2\Big)\\ &\quad =\ {}_2F_1\Big([\frac{1}{3},\,\frac{2}{3}],\,[1],\ \ -\,27\,x^2\cdot\,\Big(\frac{1\,+\,2\,x}{1\,+\,3\,x\,+\,5\,x^2}\Big)^6\Big). \qquad (100)\end{aligned}$$

Let us give another example: let us consider again the rational function (89) whose diagonal is (90), and let us consider the same function-rescaling transformation (94) with (95). One finds that the diagonal of the rational function

$$\frac{1}{1\ +F\cdot\,x\,+F\cdot\,y\,+F\cdot\,z\ \ +\,3\cdot\,F^2\cdot\,x\,y\,+\,5\cdot\,F^2\cdot\,y\,z+\,7\cdot\,F^2\cdot\,x\,z}, \qquad (101)$$

is the hypergeometric function

$$\frac{1}{(2712\ x^2\,\Phi(x)^6\ -96\ x\ \Phi(x)^3\ +1)^{1/4}}\ \times\ {}_2F_1\Big([\frac{1}{12},\,\frac{5}{12}],\,[1],1\,-\,\mathcal{H}\Big), \qquad (102)$$

where the pullback $1\,-\,\mathcal{H}$ reads:

$$1\ -\ \frac{(2381400\,x^4\,\Phi(x)^{12}\ -181440\,x^3\,\Phi(x)^9\ +7524\,x^2\,\Phi(x)^6\ -144\,x\,\Phi(x)^3\ +1)^2}{(2712\ x^2\ \Phi(x)^6\ -\,96\ x\,\Phi(x)^3\ \ +1)^3}.$$

The pullbacked hypergeometric function (103) is nothing but (90) where x has been changed into $x\,\Phi(x)^3$. A demonstration of these results is sketched in Appendix G.

Thus for each rational function belonging to one of the seven, eight, nine or ten parameter families of rational functions yielding a pullbacked ${}_2F_1$ hypergeometric function, one can deduce from the function dependent rescaling transformations (93) and the monomial transformations (81) as well as through the combination of these two transformations an *infinite number* of other rational functions, having denominators with a higher degree than three, yielding pullbacked ${}_2F_1$ hypergeometric functions related to modular forms for their diagonals.

5. Conclusion

We found here that a seven-parameter rational function of three variables with a numerator equal to one and a of polynomial denominator of degree two at most, can be expressed as a pullbacked ${}_2F_1$ hypergeometric function. We then generalized that result to eight, then nine and ten parameters, by adding specific cubic terms. We focused on subcases where the diagonals of the corresponding rational functions are pullbacked ${}_2F_1$ hypergeometric function with two possible rational function pullbacks algebraically related by *modular equations*, thus obtaining the result that the diagonal is a *modular form*†.

We have finally seen that monomial transformations, as well as a function rescaling of the three (resp. N) variables, are symmetries of the diagonals of rational functions of three (resp. N) variables. Consequently, each of our previous families of rational functions, once transformed by these symmetries, yields an *infinite number* of families of rational functions of three variables (of higher degree) whose diagonals are also pullbacked ${}_2F_1$ hypergeometric functions, related to modular forms.

Since diagonals of rational functions emerge naturally in integrable lattice statistical mechanics and enumerative combinatorics, exploring the kind of exact results we obtain for diagonals of rational functions (modular forms, Calabi-Yau operators, pullbacked ${}_nF_{n-1}$ hypergeometric functions, ...) is an important work to be performed to provide results and tools in integrable lattice statistical mechanics and enumerative combinatorics.

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Appendix A. Simple symmetries of the diagonal of the rational function (7)

Let us recall the pullbacks (20) in section (2.5), that we denote $\mathcal{P}_1$.

Appendix A.1. Overall parameter symmetry

The seven parameters are defined up to an overall parameter (they must be seen as homogeneous variables). Changing $(a,\, b_1,\, b_2,\, b_3,\, c_1,\, c_2,\, c_3)$ into $(\lambda \cdot a,\, \lambda \cdot b_1,\, \lambda \cdot b_2,\, \lambda \cdot b_3,\, \lambda \cdot c_1,\, \lambda \cdot c_2,\, \lambda \cdot c_3)$ the rational function R given by (7) and its diagonal $Diag(R)$ are changed into R/λ and $Diag(R)/\lambda$. It is thus clear that the previous pullbacks (20), which totally "encode" the exact expression of the diagonal as a pullbacked hypergeometric function, must be invariant under this transformation. This is actually the case:

$$\begin{aligned}&\mathcal{P}_1(\lambda \cdot a,\ \lambda \cdot b_1,\ \lambda \cdot b_2,\ \lambda \cdot b_3,\ \lambda \cdot c_1,\ \lambda \cdot c_2,\ \lambda \cdot c_3,\ x)\\ &\quad = \ \mathcal{P}_1(a,\ b_1,\ b_2,\ b_3,\ c_1,\ c_2,\ c_3,\ x). \end{aligned} \qquad (A.1)$$

† Differently from the usual definition of modular forms in the τ variables.

This result corresponds to the fact that $P_2(x)$ (resp. $P_4(x)$) is a *homogeneous polynomial* in the seven parameters $a,\, b_1, \cdots, c_1, \cdots$ of degree two (resp. four).

Appendix A.2. Variable rescaling symmetry

On the other hand, the rescaling of the three variables $(x,\, y,\, z)$ in (7), $(x,\, y,\, z) \, \rightarrow \, (\lambda_1 \cdot x, \,\, \lambda_2 \cdot y, \,\, \lambda_3 \cdot z)$ is a change of variables that is compatible with the operation of taking the diagonal of the rational function R.

When taking the diagonal and performing this change of variables, the monomials in the multi-Taylor expansion of (7) transform as:

$$a_{m,\,n,\,p} \cdot x^m \, y^n \, z^p \qquad \longrightarrow \qquad a_{m,\,n,\,p} \cdot \lambda_1^m \cdot \lambda_2^n \cdot \lambda_3^p \cdot x^m \, y^n \, z^p. \tag{A.2}$$

Taking the diagonal yields

$$a_{m,\,m,\,m} \cdot x^m \qquad \longrightarrow \qquad a_{m,\,m,\,m} \cdot (\lambda_1 \, \lambda_2 \, \lambda_3)^m \cdot x^m. \tag{A.3}$$

Therefore it amounts to changing $x \, \rightarrow \, \lambda_1 \, \lambda_2 \, \lambda_3 \cdot x$. With that rescaling $(x,\, y,\, z) \, \rightarrow \, (\lambda_1 \cdot x, \,\, \lambda_2 \cdot y, \,\, \lambda_3 \cdot z)$ the diagonal of the rational function remains invariant if one changes the seven parameters as follows:

$$\begin{aligned} &(a, \,\, b_1, \, b_2, \, b_3, \,\, c_1, \, c_2, \, c_3) \qquad \longrightarrow \\ &\qquad (a, \,\, \lambda_1 \cdot b_1, \,\, \lambda_2 \cdot b_2, \,\, \lambda_3 \cdot b_3, \,\, \lambda_2 \, \lambda_3 \cdot c_1, \,\, \lambda_1 \, \lambda_3 \cdot c_2, \,\, \lambda_1 \, \lambda_2 \cdot c_3). \end{aligned} \tag{A.4}$$

One deduces that the pullbacks (20) verify:

$$\begin{aligned} &\mathcal{P}_1 \Bigl(a, \quad \lambda_1 \cdot b_1, \,\, \lambda_2 \cdot b_2, \,\, \lambda_3 \cdot b_3, \quad \lambda_2 \, \lambda_3 \cdot c_1, \,\, \lambda_1 \, \lambda_3 \cdot c_2, \,\, \lambda_1 \, \lambda_2 \cdot c_3, \quad {{x} \over {\lambda_1 \, \lambda_2 \, \lambda_3}} \Bigr) \\ &\qquad = \quad \mathcal{P}_1(a, \,\, b_1, \,\, b_2, \,\, b_3, \,\, c_1, \, c_2, \, c_3, \,\, x). \end{aligned} \tag{A.5}$$

Appendix A.3. Generalization to nine and ten-parameter families

The previous arguments can also be generalized for the nine and ten-parameter families analysed in sections (3.1) and (3.2).

- The pullback $\mathcal{H}$ in (60) verifies (as it should)

$$\begin{aligned} &\mathcal{H} \Bigl(a, \quad \lambda_1 \cdot b_1, \,\, \lambda_2 \cdot b_2, \,\, \lambda_3 \cdot b_3, \quad \lambda_2 \, \lambda_3 \cdot c_1, \,\, \lambda_1 \, \lambda_3 \cdot c_2, \,\, \lambda_1 \, \lambda_2 \cdot c_3, \\ &\qquad\qquad \lambda_1^2 \, \lambda_2 \cdot d, \quad \lambda_3^2 \, \lambda_2 \cdot e, \quad {{x} \over {\lambda_1 \, \lambda_2 \, \lambda_3}} \Bigr) \\ &\qquad = \quad \mathcal{H}(a, \,\, b_1, \,\, b_2, \,\, b_3, \,\, c_1, \, c_2, \, c_3, \,\, d, \,\, e, \,\, x), \end{aligned} \tag{A.6}$$

and:

$$\begin{aligned} &\mathcal{H} \Bigl(\lambda \cdot \, a, \,\, \lambda \cdot \, b_1, \,\, \lambda \cdot \, b_2, \,\, \lambda \cdot \, b_3, \,\, \lambda \cdot \, c_1, \, \lambda \cdot \, c_2, \, \lambda \cdot \, c_3, \,\, \lambda \cdot \, d, \,\, \lambda \cdot \, e, \,\, x \Bigr) \\ &\qquad = \quad \mathcal{H}(a, \,\, b_1, \,\, b_2, \,\, b_3, \,\, c_1, \, c_2, \, c_3, \,\, d, \,\, e, \,\, x). \end{aligned} \tag{A.7}$$

- The $\mathcal{H}$ pullback (65) in (64) verifies (as it should):

$$\begin{aligned} &\mathcal{H} \Bigl(a, \quad \lambda_1 \cdot b_1, \,\, \lambda_2 \cdot b_2, \,\, \lambda_3 \cdot b_3, \quad \lambda_2 \, \lambda_3 \cdot c_1, \,\, \lambda_1 \, \lambda_3 \cdot c_2, \,\, \lambda_1 \, \lambda_2 \cdot c_3, \\ &\qquad\qquad \lambda_1^2 \, \lambda_2 \cdot d_1, \quad \lambda_2^2 \, \lambda_3 \cdot d_2, \,\, \lambda_3^2 \, \lambda_1 \cdot d_3, \quad {{x} \over {\lambda_1 \, \lambda_2 \, \lambda_3}} \Bigr) \\ &\qquad = \quad \mathcal{H}(a, \,\, b_1, \,\, b_2, \,\, b_3, \,\, c_1, \, c_2, \, c_3, \,\, d_1, \,\, d_2, \,\, d_3, \,\, x), \end{aligned} \tag{A.8}$$

and:

$$\begin{aligned} &\mathcal{H} \Bigl(\lambda \cdot \, a, \,\, \lambda \cdot \, b_1, \,\, \lambda \cdot \, b_2, \,\, \lambda \cdot \, b_3, \,\, \lambda \cdot \, c_1, \, \lambda \cdot \, c_2, \, \lambda \cdot \, c_3, \,\, \lambda \cdot \, d_1, \,\, \lambda \cdot \, d_2, \,\, \lambda \cdot \, d_3, \,\, x \Bigr) \\ &\qquad = \quad \mathcal{H}(a, \,\, b_1, \,\, b_2, \,\, b_3, \,\, c_1, \, c_2, \, c_3, \,\, d_1, \,\, d_2, \,\, d_3, \,\, x). \end{aligned} \tag{A.9}$$

Appendix B. Modular equation for the non-symmetric $\tau \rightarrow 4\tau$ subcase: ${}_2F_1([1/2, 1/2], [1], \mathcal{P})$

The pullback $X = \mathcal{P}_1$ (given by (40)) and the pullback $Y = \mathcal{P}_2^{(2)}$ are related by the *modular equation* (representing $\tau \rightarrow 4\tau$):

$$825^9 \cdot X^6 Y^6 \quad - 389 \cdot 11^6 \cdot 5^{16} \cdot 3^{10} \cdot 2^6 \cdot X^5 Y^5 \cdot (X+Y)$$
$$+ 11^3 \cdot 5^{12} \cdot 3^7 \cdot 2^4 \cdot X^4 Y^4 \cdot \Big(26148290096 \cdot (X^2+Y^2) - 15599685235 \cdot XY\Big)$$
$$- 105955481959 \cdot 5^{10} \cdot 3^7 \cdot 2^{15} \cdot X^3 Y^3 \cdot (X+Y) \cdot (X^2+Y^2)$$
$$+ 503027637092599 \cdot 5^{10} \cdot 3^7 \cdot 2^6 \cdot X^4 Y^4 \cdot (X+Y)$$
$$+ 5^6 \cdot 3^4 \cdot 2^{16} \cdot X^2 Y^2 \cdot \Big(1634268131 \cdot (X^4+Y^4) + 1788502080642816 \cdot X^2 Y^2$$
$$+ 848096080668355 \cdot (X^3 Y + X Y^3)\Big)$$
$$- 5^4 \cdot 3^4 \cdot 2^{22} \cdot XY \cdot (X+Y) \cdot \Big(389 \cdot (X^4+Y^4) + 41863592956503 \cdot X^2 Y^2$$
$$- 54605727143 \cdot (X^3 Y + X Y^3)\Big)$$
$$+ 2^{24} \cdot \Big(X^6 + Y^6 + 561444609 \cdot (X^5 Y + X Y^5)$$
$$+ 1425220456750080 \cdot (X^4 Y^2 + X^2 Y^4) + 2729942049541120 \cdot X^3 Y^3\Big)$$
$$- 5 \cdot 3^7 \cdot 2^{34} \cdot XY \cdot (X+Y) \cdot (391 X^2 - 12495392 XY + 391 Y^2) \qquad \text{(B.1)}$$
$$+ 31 \cdot 3^7 \cdot 2^{40} \cdot XY \cdot (X+2Y) \cdot (2X+Y) \quad - 3^9 \cdot 2^{42} \cdot XY \cdot (X+Y) \quad = \quad 0.$$

Instead of identities on ${}_2F_1([1/12, 5/12], [1], \mathcal{P})$ hypergeometric functions like (45), one can consider directly identities on ${}_2F_1([1/2, 1/2], [1], \mathcal{P})$ hypergeometric functions. One has for instance the following identity:

$${}_2F_1\Big([\frac{1}{2}, \frac{1}{2}], [1], \frac{8x \cdot (1+x^2)}{(1+x)^4}\Big) \quad = \quad (1+x)^2 \cdot {}_2F_1\Big([\frac{1}{2}, \frac{1}{2}], [1], x^4\Big). \qquad \text{(B.2)}$$

Denoting

$$A \;=\; \frac{8x \cdot (1+x^2)}{(1+x)^4}, \qquad\qquad B \;=\; x^4, \qquad \text{(B.3)}$$

the two pullbacks in (B.2), one has the following *asymmetric* modular equation (of the $\tau \rightarrow 4\tau$ type [22]) between these two pullbacks (B.3):

$$A^4 B^4 \; - 4 A^4 B^3 \; + 6 A^3 B^2 \cdot (A + 128 B) \; - 4 A^2 B \cdot (A^2 - 640 AB + 1216 B^2)$$
$$+ A \cdot (A^3 + 768 A^2 B + 5632 AB^2 + 8192 B^3) \; - 256 B \cdot (19 A^2 + 64 AB + 16 B^2)$$
$$+ 8192 B \cdot (A+B) \; - 4096 B \; = \; 0, \qquad \text{(B.4)}$$

Note that changing $B \rightarrow 1 - B$ the previous algebraic equation becomes a *symmetric* modular equation:

$$A^4 B^4 \; - 768 A^3 B^3 \; + 4864 (A^3 B^2 + A^2 B^3) \; - 8960 A^2 B^2 \; - 8192 (A^3 B + A B^3)$$
$$+ 4096 (A^3 + B^3) \; + 8192 (A^2 B + A B^2) \; - 4096 (A^2 + B^2) \; = \; 0. \qquad \text{(B.5)}$$

As far as representations of $\tau \rightarrow 4\tau$ isogenies, the modular equations (B.4) and (B.5) are clearly much simpler than (B.1).

Appendix C. ${}_2F_1([1/4, 3/4], [1], \mathcal{P})$ hypergeometric as modular forms

Appendix C.1. ${}_2F_1([1/4, 3/4], [1], \mathcal{P})$ identities

Let us focus on the ${}_2F_1([1/4, 3/4], [1], \mathcal{P})$ hypergeometric function:

$$ {}_2F_1\Bigl([\frac{1}{4}, \frac{3}{4}], [1], x\Bigr) \quad = \quad (1 + 3x)^{-1/4} \cdot {}_2F_1\Bigl([\frac{1}{12}, \frac{5}{12}], \frac{27x \cdot (1-x)^2}{(1+3x)^3}\Bigr). \qquad (C.1) $$

The emergence of ${}_2F_1([1/4, 3/4], [1], \mathcal{P})$ hypergeometric functions in physics, walk problems in the quarter of a plane [33, 34, 35] in enumerative combinatorics, or in interesting subcases of diagonals (see section (2.8)), raises the question if ${}_2F_1([1/4, 3/4], [1], \mathcal{P})$ should be seen as associated to the isogenies [22] $\tau \rightarrow 2\tau$ or $\tau \rightarrow 4\tau$. The identity

$$ {}_2F_1\Bigl([\frac{1}{4}, \frac{3}{4}], [1], 64x^2\Bigr) \quad = \quad (1 + 8x)^{-1/2} \cdot {}_2F_1\Bigl([\frac{1}{2}, \frac{1}{2}], [1], \frac{16x}{1+8x}\Bigr), \qquad (C.2) $$

or equivalently

$$ {}_2F_1\Bigl([\frac{1}{4}, \frac{3}{4}], [1], \Bigl(\frac{x}{2-x}\Bigr)^2\Bigr) \quad = \quad \Bigl(\frac{2-x}{2}\Bigr)^{1/2} \cdot {}_2F_1\Bigl([\frac{1}{2}, \frac{1}{2}], [1], x\Bigr), \qquad (C.3) $$

seems to relate ${}_2F_1([1/4, 3/4], [1], \mathcal{P})$ to ${}_2F_1([1/2, 1/2], [1], x)$, and thus seems to relate ${}_2F_1([1/4, 3/4], [1], \mathcal{P})$ rather $\tau \rightarrow 4\tau$. Yet things are more subtle.

Let us see how ${}_2F_1([1/4, 3/4], [1], \mathcal{P})$ can be described as a modular form corresponding to pullbacked ${}_2F_1([1/4, 3/4], [1], \mathcal{P})$ hypergeometric functions with two different rational pullbacks. For instance, one deduces from (B.2) combined with (C.3), several identities on the hypergeometric function ${}_2F_1([1/4, 3/4], [1], \mathcal{P})$ like

$$ {}_2F_1\Bigl([\frac{1}{4}, \frac{3}{4}], [1], \frac{x^2}{(2-x)^2}\Bigr) $$
$$ = \quad \Bigl(\frac{2-x}{2 \cdot (1-2x)}\Bigr)^{1/2} \cdot {}_2F_1\Bigl([\frac{1}{4}, \frac{3}{4}], [1], -4 \cdot \frac{x \cdot (1-x)}{(1-2x)^2}\Bigr), \qquad (C.4) $$

or

$$ {}_2F_1\Bigl([\frac{1}{4}, \frac{3}{4}], [1], \frac{x^2}{(2-x)^2}\Bigr) $$
$$ = \quad \Bigl(\frac{2-x}{2 \cdot (1+x)}\Bigr)^{1/2} \cdot {}_2F_1\Bigl([\frac{1}{4}, \frac{3}{4}], [1], \frac{4x}{(1+x)^2}\Bigr). \qquad (C.5) $$

and thus:

$$ {}_2F_1\Bigl([\frac{1}{4}, \frac{3}{4}], [1], \frac{4x}{(1+x)^2}\Bigr) $$
$$ = \quad \Bigl(\frac{1+x}{1-2x}\Bigr)^{1/2} \cdot {}_2F_1\Bigl([\frac{1}{4}, \frac{3}{4}], [1], -4 \cdot \frac{x \cdot (1-x)}{(1-2x)^2}\Bigr). \qquad (C.6) $$

Appendix C.2. Schwarzian equations

Recalling the viewpoint developed in our previous paper [22] these identities can be seen to be of the form

$$ {}_2F_1\Bigl([\frac{1}{4}, \frac{3}{4}], [1], B\Bigr) \quad = \quad G \cdot {}_2F_1\Bigl([\frac{1}{4}, \frac{3}{4}], [1], A\Bigr), $$

where G is some algebraic factor.

The important result of [22] is that after elimination of the algebraic factor G one finds that the two pullbacks A and B verify the following *Schwarzian equation*:

$$ - {{1} \over {8}} {{3\, A^2 \, -3\, A \, +4 } \over {A^2 \, (A-1)^2 }} \quad + {{1} \over {8}} {{3\, B^2 \, -3\, B \, +4 } \over {B^2 \, (B-1)^2 }} \cdot \Bigl({{dB} \over {dA}}\Bigr)^2 \quad + \quad \{B, \, A\} \quad = \quad 0, \qquad (C.7) $$

where $\{B, \, A\}$ denotes the Schwarzian derivative.

Do note that ${}_2F_1([1/4, 3/4], [1], \mathcal{P})$ is a selected hypergeometric function since the rational function in the Schwarzian derivative (C.7)

$$ W(A) \,\, = \,\, \, -{{1} \over {8}} \, {{3\, A^2 \, \, -3\, A \, +4 } \over {A^2 \cdot \, (A-1)^2 }}, \qquad (C.8) $$

is invariant under the $A \, \rightarrow \, 1 \, - \, A$ transformation: $W(A) \, = \, W(1 \, - \, A)$.

This Schwarzian equation can be written in a more symmetric way between A and B, namely:

$$ {{1} \over {8}} {{3\, B^2 \, -3\, B \, +4 } \over {B^2 \, (B-1)^2 }} \cdot \Bigl({{dB} \over {dx}}\Bigr)^2 \quad + \quad \{B, \, x\} \quad = \quad {{1} \over {8}} {{3\, A^2 \, -3\, A \, +4 } \over {A^2 \, (A-1)^2 }} \cdot \Bigl({{dA} \over {dx}}\Bigr)^2 \quad + \quad \{A, \, x\}. \qquad (C.9) $$

Let us denote $\rho(x)$ the rational function of the LHS or the RHS of equality (C.9). For the three identities (C.4), (C.5), this rational function is (of course†) the same rational function, namely

$$ \rho(x) \quad = \quad {{1} \over {2}} \cdot \, {{x^2 \, \, -x \, +1 } \over { x \cdot \, (x-1)^2 }}. \qquad (C.10) $$

Let us consider the first two identities (C.4) and (C.5), denoting by A and B the corresponding pullbacks:

$$ A \,\, = \,\, \, -4 \cdot \, {{ x \cdot \, (1\, -\, x) } \over { (1\, -\, 2\, x)^2 }}, \qquad \hbox{or:} \qquad {{ 4\, x } \over { (1\, +\, x)^2 }}, \qquad B \,\, = \,\, {{x^2 } \over { (2\, -x)^2 }} \qquad (C.11) $$

These two pullbacks are related by the *asymmetric* modular equation:

$$ 81 \cdot \, A^2 \, B^2 \, -18\, A\, B \cdot \, (8\, B \, +A) \, + (A^2 \, +80 \cdot \, A\, B \, +64\, B^2) \, \, -64\, B \,\, = \,\, 0. \qquad (C.12) $$

giving the following expansion for A seen as an *algebraic series*‡ in B:

$$ B \,\, = \,\, \, {{1} \over {64}} A^2 \,\, + {{5} \over {256}} A^3 \,\, + {{83} \over {4096}} A^4 \,\, + {{163} \over {8192}} A^5 \,\, + {{5013} \over {262144}} A^6 \,\,\, + \,\, \cdots \qquad (C.13) $$

Such an algebraic series is clearly†† a $\tau \, \rightarrow \, 2\, \tau$ (or $q \, \rightarrow \, q^2$ in the nome q) isogeny [22].

The modular curve (C.12) is unpleasantly asymmetric: the two pullbacks are not on the same footing. Note however, that using the $A \, \leftrightarrow \, 1 \, - \, A$ symmetry (see (C.8)) on the Schwarzian equations (C.9), and changing $A \, \rightarrow \, 1 \, - \, A$ in the asymmetric modular curve (54), one gets the *symmetric* modular curve:

$$ 81 \cdot \, A^2 \, B^2 \,\, -18 \cdot \, (A^2 \, B \, + A\, B^2) \, \, + A^2 \, -44\, A\, B \, + B^2 \, \, -2 \cdot \, (A \, + B) \, \, +1 \,\, = \,\, 0. \qquad (C.14) $$

† Since these identities share one pullback.
‡ We discard the other root expansion $B \, = \, 1 \, + A \, + {{5} \over {4}} A^2 \, + {{25} \over {16}} A^3 \, + {{31} \over {16}} A^4 \,\, + \cdots$
††From (C.13) see [22].

Changing $B \rightarrow 1 - B$ in the asymmetric modular curve (54), one also gets another *symmetric* modular curve:

$$81 \cdot A^2 B^2 - 144 \cdot (A^2 B + A B^2) + 208 A B + 64 \cdot (A^2 + B^2 - A - B) = 0. \qquad (C.15)$$

The two pullbacks for (C.15) read:

$$A = \frac{4x}{(1+x)^2}, \qquad B = \frac{4 \cdot (1-x)}{(2-x)^2}. \qquad (C.16)$$

The price to pay to restore the symmetry between the two pullbacks (C.16) is that the corresponding pullbacks do not yield hypergeometric identities *expandable for x small.*

Finally, the identity (C.6) corresponds to a symmetric relation between these two-pullbacks which reads:

$$81 \cdot C^2 D^2 - 144 \cdot (C^2 D + C D^2) + 16 \cdot (4 C^2 + 13 C D + 4 D^2) - 64 \cdot (C + D) = 0. \qquad (C.17)$$

The corresponding series expansion

$$D = -C - \frac{5}{4} C^2 - \frac{25 C^3}{16} - \frac{31 C^4}{16} - \frac{305 C^5}{128} - \frac{2979 C^6}{1024} - \frac{14457 C^7}{4096} - \frac{17445 C^8}{4096} - \frac{167615 C^9}{32768} - \frac{801941 C^{10}}{131072} - \frac{3822989 C^{11}}{524288} + \cdots \qquad (C.18)$$

is *an involutive series.*

Appendix D. Exact expression of polynomial P_6 for the ten-parameter rational function (63)

The diagonal of the ten-parameters rational function (63) is the pullbacked hypergeometric function

$$\frac{1}{P_3(x)^{1/4}} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], 1 - \frac{P_6(x)^2}{P_3(x)^3}\Big), \qquad (D.1)$$

where $P_3(x)$ is given by (66) and $P_6(x)$ is a polynomial of degree six in x of the form

$$P_6(x) = p_4 + \Delta_6(x), \qquad (D.2)$$

where p_4 is the polynomial $P_4(x)$ given by (17) in section (2), and where $\Delta_6(x)$ is the following polynomial of degree six in x:

$$\begin{aligned}
\Delta_6(x) = & -5832 \cdot d_1^2 d_2^2 d_3^2 \cdot x^6 \\
& + 3888 \cdot d_1 d_2 d_3 \cdot (b_1 c_2 d_2 + b_2 c_3 d_3 + b_3 c_1 d_1) \cdot x^5 \\
& - 864 \cdot (c_1^3 d_1^2 d_3 + c_2^3 d_1 d_2^2 + c_3^3 d_2 d_3^2) \cdot x^5 \\
& - 1296 \cdot c_1 c_2 c_3 d_1 d_2 d_3 \cdot x^5 \\
& - 1296 \cdot b_1 b_2 b_3 d_1 d_2 d_3 \cdot x^4 \\
& - 1296 \cdot a \cdot d_1 d_2 d_3 (b_1 c_1 + b_2 c_2 + b_3 c_3) \cdot x^4 \\
& - 1296 \cdot (b_1 b_2 c_2 c_3 d_2 d_3 + b_1 b_3 c_1 c_2 d_1 d_2 + b_2 b_3 c_1 c_3 d_1 d_3) \cdot x^4 \\
& + 864 \cdot (c_1^2 c_3 d_1 d_3 + c_1 c_2^2 d_1 d_2 + c_2 c_3^2 d_2 d_3) \cdot a \cdot x^4 \\
& - 864 \cdot (b_1^3 d_2^2 d_3 + b_2^3 d_1 d_3^2 + b_3^3 d_1^2 d_2) \cdot x^4
\end{aligned}$$

$$
\begin{aligned}
&+ 864 \cdot \Bigl(b_1^2\, c_1\, c_3\, d_2\, d_3 + b_1\, b_2\, c_1^2\, d_1\, d_3 + b_1\, b_3\, c_3^2\, d_2\, d_3 \\
&\qquad + b_2^2\, c_1\, c_2\, d_1\, d_3 + b_2\, b_3\, c_2^2\, d_1\, d_2 + b_3^2\, c_2\, c_3\, d_1\, d_2\Bigr) \cdot x^4 \\
&+ 216 \cdot (b_1^2\, c_2^2\, d_2^2 + b_2^2\, c_3^2\, d_3^2 + b_3^2\, c_1^2\, d_1^2) \cdot x^4 \\
&+ 288 \cdot (b_1\, c_1^3\, c_2\, d_1 + b_2\, c_2^3\, c_3\, d_2 + b_3\, c_1\, c_3^3\, d_3) \cdot x^4 \\
&- 576 \cdot (b_1\, c_1^2\, c_3^2\, d_3 + b_2\, c_1^2\, c_2^2\, d_1 + b_3\, c_2^2\, c_3^2\, d_2) \cdot x^4 \\
&- 144 \cdot c_1\, c_2\, c_3 \cdot (b_1\, c_2\, d_2 + b_2\, c_3\, d_3 + b_3\, c_1\, d_1) \cdot x^4 \\
&\quad + 540 \cdot d_1\, d_2\, d_3\, a^3 \cdot x^3 \\
&\quad - 648 \cdot (b_1\, c_3\, d_2\, d_3 + b_2\, c_1\, d_1\, d_3 + b_3\, c_2\, d_1\, d_2) \cdot a^2 \cdot x^3 \\
&\qquad - 72 \cdot (c_1^2\, c_2\, d_1 + c_1\, c_3^2\, d_3 + c_2^2\, c_3\, d_2) \cdot a^2 \cdot x^3 \\
&\quad + 288 \cdot (b_1^3\, b_3\, c_1\, d_2 + b_1\, b_2^3\, c_2\, d_3 + b_2\, b_3^3\, c_3\, d_1) \cdot x^3 \\
&\quad - 576 \cdot (b_1^2\, b_2^2\, c_1\, d_3 + b_1^2\, b_3^2\, c_3\, d_2 + b_2^2\, b_3^2\, c_2\, d_1) \cdot x^3 \\
&\quad - 144 \cdot b_1\, b_2\, b_3\, (b_1\, c_2\, d_2 + b_2\, c_3\, d_3 + b_3\, c_1\, d_1) \cdot x^3 \\
&\quad + 864 \cdot (b_1^2\, b_2\, d_2\, d_3 + b_1\, b_3^2\, d_1\, d_2 + b_2^2\, b_3\, d_1\, d_3) \cdot a \cdot x^3 \\
&\quad - 144 \cdot \Bigl(b_1^2\, c_1\, c_2\, d_2 + b_1\, b_2\, c_2^2\, d_2 + b_1\, b_3\, c_1^2\, d_1 \\
&\qquad\quad + b_2^2\, c_2\, c_3\, d_3 + b_2\, b_3\, c_3^2\, d_3 + b_3^2\, c_1\, c_3\, d_1\Bigr) \cdot a \cdot x^3 \\
&\quad + 720 \cdot (b_1\, b_2\, c_1\, c_3\, d_3 + b_1\, b_3\, c_2\, c_3\, d_2 + b_2\, b_3\, c_1\, c_2\, d_1) \cdot a \cdot x^3 \\
&\quad + 36 \cdot a^3 \cdot (b_1\, c_2\, d_2 + b_2\, c_3\, d_3 + b_3\, c_1\, d_1) \cdot x^2 \\
&\qquad - 72 \cdot a^2 \cdot (b_1^2\, b_3\, d_2 + b_1\, b_2^2\, d_3 + b_2\, b_3^2\, d_1) \cdot x^2. \qquad \text{(D.3)}
\end{aligned}
$$

Appendix E. Polynomials $P_3(x)$ and $P_5(x)$ for the nine-parameter rational function (63)

The two polynomials $P_3(x)$ and $P_5(x)$ encoding the pullback of the pullbacked hypergeometric function (69) for the nine-parameter rational function (63) in section (3.2.1), read

$$
\begin{aligned}
P_3(x) \;=\; & p_2 \;+\; 48 \cdot c_2 \cdot (3\, b_3\, d_1\, d_2 - c_1^2\, d_1 - c_2\, c_3\, d_2) \cdot x^3 \\
& + 24 \cdot (a\, b_1\, c_2\, d_2 + a\, b_3\, c_1\, d_1 - 2\, b_1^2\, b_3\, d_2 - 2\, b_2\, b_3^2\, d_1) \cdot x^2, \qquad \text{(E.1)}
\end{aligned}
$$

and

$$
\begin{aligned}
P_5(x) \;=\; & p_4 \;-\; 864 \cdot c_2^3\, d_1\, d_2^2 \cdot x^5 \\
& + 864 \cdot (a\, c_1\, c_2^2\, d_1\, d_2 + b_2\, b_3\, c_2^2\, d_1\, d_2 + b_3^2\, c_2\, c_3\, d_1\, d_2 - b_3^3\, d_1^2\, d_2) \cdot x^4 \\
& - 576 \cdot (b_2\, c_1^2\, c_2^2\, d_1 + b_3\, c_2^2\, c_3^2\, d_2) \cdot x^4 \\
& + 288 \cdot (b_1\, c_1^3\, c_2\, d_1 + b_2\, c_2^3\, c_3\, d_2) \cdot x^4 \\
& - 144 \cdot (b_1\, c_1\, c_2^2\, c_3\, d_2 + b_3\, c_1^2\, c_2\, c_3\, d_1) \cdot x^4 \\
& + 216 \cdot (b_1^2\, c_2^2\, d_2^2 + b_3^2\, c_1^2\, d_1^2 - 6\, b_1\, b_3\, c_1\, c_2\, d_1\, d_2) \cdot x^4
\end{aligned}
$$

$$- 72 \cdot (9\,a^2\,b_3\,c_2\,d_1\,d_2 + a^2\,c_1^2\,c_2\,d_1 + a^2\,c_2^2\,c_3\,d_2) \cdot x^3$$
$$- 144 \cdot a \cdot (b_1^2\,c_1\,c_2\,d_2 + b_1\,b_2\,c_2^2\,d_2 + b_1\,b_3\,c_1^2\,d_1 + b_3^2\,c_1\,c_3\,d_1) \cdot x^3$$
$$- 144 \cdot (b_1^2\,b_2\,b_3\,c_2\,d_2 + b_1\,b_2\,b_3^2\,c_1\,d_1) \cdot x^3$$
$$+ 720 \cdot (a\,b_1\,b_3\,c_2\,c_3\,d_2 + a\,b_2\,b_3\,c_1\,c_2\,d_1) \cdot x^3$$
$$- 576 \cdot (b_1^2\,b_3^2\,c_3\,d_2 + b_2^2\,b_3^2\,c_2\,d_1) \cdot x^3$$
$$+ 288 \cdot (b_1^3\,b_3\,c_1\,d_2 + b_2\,b_3^3\,c_3\,d_1 + 3\,a\,b_1\,b_3^2\,d_1\,d_2) \cdot x^3$$
$$+ 36 \cdot a^2 \cdot (a\,b_1\,c_2\,d_2 + a\,b_3\,c_1\,d_1 - 2\,b_1^2\,b_3\,d_2 - 2\,b_2\,b_3^2\,d_1) \cdot x^2, \qquad \text{(E.2)}$$

where the polynomials p_2 and p_4 are the polynomials $P_2(x)$ and $P_4(x)$ of degree two and four in x given by (16) and (17) in section (2): p_2 and p_4 correspond to the $d_1 = d_2 = 0$ limit.

Appendix F. Monomial symmetries on diagonals

Let us sketch the demonstration of the monomial symmetry results of section (81), with the condition that the determinant of (82) is not zero and the conditions (83) are verified. We will denote by n the integer in the three equal sums (83): $n = A_i + B_i + C_i$. The diagonal of the rational function of three variables $\mathcal{R}$ is defined through its multi-Taylor expansion (for small x, y and z):

$$\mathcal{R}\Big(x,\, y,\, z\Big) \;=\; \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum_{m_3=0}^{\infty} R_{m_1,\, \dots,\, m_n} \cdot x^{m_1} \cdot y^{m_2} \cdot z^{m_3}, \qquad \text{(F.1)}$$

as the series in one variable x:

$$\Phi(x) \;=\; Diag\Big(\mathcal{R}\Big(x,\, y,\, z\Big)\Big) \;=\; \sum_{m=0}^{\infty} R_{m,\,m,\,m} \cdot x^m. \qquad \text{(F.2)}$$

The monomial transformation (81) changes the multi-Taylor expansion (F.1) into

$$\tilde{\mathcal{R}}\Big(x,\, y,\, z\Big) \;=\; \sum_{M_1=0}^{\infty} \sum_{M_2=0}^{\infty} \sum_{M_3=0}^{\infty} \tilde{R}_{M_1,\, M_2\, M_3} \cdot x^{M_1} \cdot y^{M_2} \cdot z^{M_3} \;=$$

$$\sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum_{m_3=0}^{\infty} R_{m_1,\, m_2\, m_3} \cdot \Big(x^{A_1}\, y^{A_2}\, z^{A_3}\Big)^{m_1} \Big(x^{B_1}\, y^{B_2}\, z^{B_3}\Big)^{m_2} \Big(x^{C_1}\, y^{C_2}\, z^{C_3}\Big)^{m_3}$$

$$= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum_{m_3=0}^{\infty} R_{m_1,\, m_2\, m_3} \cdot x^{M_1} \cdot y^{M_2} \cdot z^{M_3}$$

where:

$$M_1 \;=\; A_1 \cdot m_1 + B_1 \cdot m_2 + C_1 \cdot m_3, \qquad \text{(F.3)}$$
$$M_2 \;=\; A_2 \cdot m_1 + B_2 \cdot m_2 + C_2 \cdot m_3, \qquad \text{(F.4)}$$
$$M_3 \;=\; A_3 \cdot m_1 + B_3 \cdot m_2 + C_3 \cdot m_3. \qquad \text{(F.5)}$$

Taking the diagonal amounts to forcing the exponents m_1, m_2 and m_3 to be equal. It is easy to see that when condition (83) is verified, $m_1 = m_2 = m_3$ yields $M_1 = M_2 = M_3$. Conversely if the determinant of (82) is not zero it is straightforward to see that the conditions $M_1 = M_2 = M_3$ yield $m_1 = m_2 = m_3$.

Then if one knows an exact expression for the diagonal of a rational function, the diagonal of this rational function changed by the monomial transformation (81) reads

$$Diag\Big(\tilde{\mathcal{R}}\Big(x,\, y,\, z\Big)\Big) \;=\; \sum_{M\,=\,0}^{\infty} \tilde{R}_{M,\, M,\, M} \cdot x^M \;=\; \sum_{m\,=\,0}^{\infty} R_{m,\, m,\, m} \cdot x^{n \cdot m} \;=\; \Phi(x^n), \quad \text{(F.6)}$$

and is thus equal to the previous exact expression $\Phi(x)$, where we have changed $x \rightarrow x^n$, where n is the integer $n \, = \, A_1 + B_1 + C_1 \, = \, A_2 + B_2 + C_2 \, = \, A_3 + B_3 + C_3$. These monomial symmetries for diagonal of rational functions are not specific of rational functions of three variables: they can be straightforwardly generalized to an arbitrary number of variables.

Appendix G. Rescaling symmetries on diagonals

We sketch the demonstration of the result in section (4.3). One recalls that the diagonal of the rational function of three variables $\mathcal{R}$ is defined through its multi-Taylor expansion (for small x, y and z)

$$\mathcal{R}\Big(x,\, y,\, z\Big) \;=\; \sum_{m_1\,=\,0}^{\infty} \, \sum_{m_2\,=\,0}^{\infty} \, \sum_{m_3\,=\,0}^{\infty} R_{m_1,\, \dots,\, m_n} \cdot x^{m_1} \cdot y^{m_2} \cdot z^{m_3}, \qquad \text{(G.1)}$$

as the series in one variable x:

$$\Phi(x) \;=\; Diag\Big(\mathcal{R}\Big(x,\, y,\, z\Big)\Big) \;=\; \sum_{m\,=\,0}^{\infty} R_{m,\, m,\, m} \cdot x^m. \qquad \text{(G.2)}$$

The (function rescaling) transformation (93) transforms the multi-Taylor expansion (G.1) into:

$$\mathcal{R}\Big(x,\, y,\, z\Big) \;= \qquad \text{(G.3)}$$
$$\sum_{m_1\,=\,0}^{\infty} \, \sum_{m_2\,=\,0}^{\infty} \, \sum_{m_3\,=\,0}^{\infty} R_{m_1,\, \dots,\, m_n} \cdot x^{m_1} \cdot y^{m_2} \cdot z^{m_3} \cdot F(x\,y\,z)^{m_1 + m_2 + m_3}.$$

We assume that the function $F(x)$ has some simple Taylor series expansion. Each time taking the diagonal of (G.3) forces the exponents m_1, m_2 and m_3 to be equal in the term $x^{m_1} \cdot y^{m_2} \cdot z^{m_3}$ of the multi-Taylor expansion (G.3), one gets a factor $F(x\,y\,z)^{m_1 + m_2 + m_3} \, = \, F(x\,y\,z)^{3\,m}$. Consequently, the diagonal of (G.3) becomes:

$$Diag\Big(\tilde{\mathcal{R}}\Big(x,\, y,\, z\Big)\Big) \;=\; \sum_{m\,=\,0}^{\infty} R_{m,\, m,\, m} \cdot x^n \cdot F(x)^{3\,n}$$
$$\;=\; Diag\Big(\mathcal{R}\Big(x,\, y,\, z\Big)\Big)\Big(x \cdot F(x)^3\Big). \qquad \text{(G.4)}$$

Clearly, these function-dependent rescaling symmetries for diagonals of rational functions are not specific of rational functions of three variables: they can be straightforwardly generalized to an *arbitrary number of variables.*

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