Heun functions and diagonals of rational functions‡

Y. Abdelaziz[†], S. Boukraa[£], C. Koutschan[δ], J-M. Maillard[†]

[†] LPTMC, UMR 7600 CNRS, Université Pierre et Marie Curie, Sorbonne Université, Tour 23, 5ème étage, case 121, 4 Place Jussieu, 75252 Paris Cedex 05, France

[£] LPTHIRM and IAESB, Université de Blida, Algeria

[δ] Johann Radon Institute for Computational and Applied Mathematics (RICAM), Altenberger Strasse 69, A-4040 Linz, Austria

Abstract. We provide a set of diagonals of simple rational functions of three and four variables that are squares of Heun functions. These Heun functions obtained through creative telescoping, turn out to be pullbacked $_2F_1$ hypergeometric functions that correspond to classical modular forms. We also obtain Heun functions that are associated with Shimura curves as solutions of telescopers of rational functions.

PACS: 05.50.+q, 05.10.-a, 02.30.Hq, 02.30.Gp, 02.40.Xx
AMS Classification scheme numbers: 34M55, 47E05, 81Qxx, 32G34, 34Lxx, 34Mxx, 14Kxx

Key-words: Diagonals of rational functions, Heun functions, creative telescoping, extremal rational elliptic surfaces, pullbacked hypergeometric functions, Schwarz map, Schwarz function, modular forms, modular equations, Hauptmoduls, Belyi coverings, Shimura curves, automorphic forms.

1. Introduction

Diagonals of rational functions naturally emerge in lattice statistical mechanics, enumerative combinatorics, and more generally, in the context of n-fold integrals of theoretical physics [1, 2]. In previous papers [3, 4, 5] we have seen† that many diagonals of rational functions were pullbacked $_2F_1$ hypergeometric functions‖ that turn out to be related to classical modular forms¶. Sticking with diagonals of rational functions that are solutions of linear differential operators of *order two*, it is natural to study diagonals of rational functions that are Heun functions.

Heun functions emerge in different areas of physics [1, 8, 9, 10] (see also page 60 of [2]) and enumerative combinatorics: the simple cubic lattice Green function [11]

‡ This is the Accepted Manuscript version of an article accepted for publication in JPA. IOP Publishing Ltd is not responsible for any errors or omissions in this version of the manuscript or any version derived from it. The Version of Record is available online at DOI 10.1088/1751-8121/ab67e5.

† These calculations were performed using the creative telescoping program of C. Koutschan [6].

‖ See [7] equation 6 for a definition of a pullback of a hypergeometric function.

¶ In a sense that we define in Appendix A and Appendix B.

can be written as a Heun function, the eigenvalue equation of the Laplace-Beltrami operator on the Eguchi-Hanson space is given by a Heun equation [12, 13]. Heun functions emerging in physics often§ correspond to *globally bounded series* [1, 2], i.e. series that can be recast into series with *integer coefficients*. Most of the time they turn out to be pullbacked ${}_2F_1$ hypergeometric functions [14] and in fact *classical modular forms*. In [3] we found diagonals of "simple" rational functions corresponding to classical modular forms when the operator annihilating the diagonal of the rational function had order two. This leads us here to study the class of Heun functions related to classical modular forms that are diagonals of rational functions††. We will discard the case where the Heun functions are almost trivial, their linear differential operators of order two factorising into two linear differential operators of order one [15, 16]. In this paper we examine Heun functions, which happen to be either *diagonals of simple rational functions* [2] in three or four variables, or solutions of "telescopers". A telescoper is an operator annihilating an n-fold integral *over all possible integration cycles*, including *evanescent integration cycles* which correspond to diagonals of rational functions. More specifically, the "telescoper" of a rational function, say $R(x, y, z)$, we refer to here, is the output of the creative telescoping program [6], applied to the *transformed* rational function $\tilde{R} = R(x/y, y/z, z)/(yz)$. Such a telescoper is a differential operator T in x, D_x such that $T + D_y \cdot U + D_z \cdot V$ annihilates $\tilde{R}$, where U, V are rational functions in x, y, z. In other words, the telescoper T represents a linear ODE that is satisfied by $Diag(R)$. Now the Heun functions examined in this paper fall into one of three categories:

(1) Heun functions that are diagonals of rational functions, having *globally bounded* series expansions, and can be rewritten as pullbacked hypergeometric functions that are *classical modular forms*.

(2) Heun functions that are diagonals of rational functions, having globally bounded series expansions, and can be rewritten as pullbacked hypergeometric functions that are *derivatives* of classical modular forms.

(3) Heun functions that are solutions of *telescopers* of rational functions that have series expansions that are *not* globally bounded and hence cannot be diagonals of rational functions, but are instead solutions of the telescoper¶. We show that in this case the Heun functions correspond to *Shimura automorphic forms*.

The Heun function $Heun(a, q, \alpha, \beta, \gamma, \delta, x)$ is solution of the order-two Heun linear differential operator with four singularities (D_x denotes d/dx)

$$H_2 \; = \; D_x^2 \; + \; \Big(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-a} \Big) \cdot D_x \; + \; \frac{\alpha \, \beta \, x \, - q}{x \cdot (x-1) \cdot (x-a)}, \qquad (1)$$

where one has the Fuchsian constraint $\epsilon \; = \; \alpha \, + \, \beta \, - \gamma \, - \, \delta \, + \, 1$, where $\alpha, \beta, \gamma, \delta$ need to be rational numbers, and a is an algebraic number. The parameter q is called the *accessory parameter* and the ratio $q/(\alpha\beta)$ is called the *normalised accessory parameter*.

In the first two sections, we examine the Heun functions *emerging from diagonals of simple rational functions* that fall into the first and second category above, and show how they happen to be related to *classical modular forms*, or *derivatives of classical*

§ This is not the case for the Heun functions in [10] which do not correspond to globally bounded series.

†† Diagonals of rational functions are necessarily globally bounded [1, 2].

¶ In this case the diagonal is equal to zero.

modular forms, corresponding to pullbacked ${}_2F_1$ hypergeometric functions. These Heun functions turn out to be globally bounded. This leads us to define a criterion in Appendix A, that allows us to draw a list of parameters of the Gauss hypergeometric function ${}_2F_1([a,b],[c],x)$ that correspond to a *classical modular form* in Appendix B. Furthermore, we show in section 2.2 that some of these Heun functions are periods of *extremal rational surfaces*.

In the third section, we examine the solutions of the telescoper of a rational function, corresponding to a Heun function with a series expansion that is *not globally bounded*, and we show that this Heun function is related to a specific *Shimura curve* [17, 18, 19, 20, 21, 22, 23].

1.1. Lattice Green functions as diagonals of rational functions

The diagonal of a rational function in n variables $\mathcal{R}(x_1,\dots,x_n) = \mathcal{P}(x_1,\dots,x_n)/\mathcal{Q}(x_1,\dots,x_n)$, where $\mathcal{P},\mathcal{Q} \in \mathbb{Q}[x_1,\dots,x_n]$ such that $\mathcal{Q}(0,\dots,0) \neq 0$, is defined through its multi-Taylor expansion around $(0,\dots,0)$:

$$\mathcal{R}\Big(x_1,\,\dots,\,x_n\Big) \;=\; \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} R_{m_1,\,\dots,\,m_n} \cdot x_1^{m_1} \,\dots\, x_n^{m_n}, \tag{2}$$

as the series in *one variable* x:

$$Diag\Big(\mathcal{R}\Big(x_1,\,\dots,\,x_n\Big)\Big) \;=\; \sum_{m=0}^{\infty} R_{m,\,m,\,\dots,\,m} \cdot x^m. \tag{3}$$

With this definition in mind, one can see the simple cubic lattice Green function [24]

$$\frac{1}{(2\pi)^3} \cdot \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{d\theta_1\, d\theta_2\, d\theta_3}{1 \,-\, x \cdot (\cos(\theta_1) \,+\cos(\theta_2) \,+\cos(\theta_3))}, \tag{4}$$

as the diagonal of the rational function in four variables $x,\, z_1,\, z_2,\, z_3$:

$$\frac{1}{1 \,-\, x \cdot z_1\, z_2\, z_3 \cdot ((1+z_1^2)/z_1/2 \,+(1+z_2^2)/z_2/2 \,+(1+z_3^2)/z_3/2)}$$
$$= \;\frac{2}{2 \,-\, x \cdot z_1\, z_2\, z_3 \cdot (z_1 + 1/z_1 \,+\, z_2 + 1/z_2 \,+\, z_3 + 1/z_3)}, \tag{5}$$

where the simple lattice Green function is obtained as the diagonal of a four variable rational function through the following substitution: $\cos(\theta_i) \,=\, (1+z_i^2)/2/z_i$, i.e. $z_i \,=\, \exp(i\,\theta_i)$, and $x \,\to\, x \cdot z_1\, z_2\, z_3$.

The linear differential operator annihilating the diagonal (5), has order three. This operator is the *symmetric square*‖ of a linear differential operator of order two where θ is the homogeneous derivative $x \cdot d/dx$:

$$9\,x^4 \cdot (2\,\theta + 3) \cdot (2\,\theta + 1) \;\; - 4\,x^2 \cdot (10\,\theta^2 + 10\,\theta + 3) \;\; + 4\,\theta^2, \tag{6}$$

whose solution is given by a Heun function. Hence, we see that the diagonal of (5) reads:

$$Heun\Big(\frac{1}{9},\,\frac{1}{12},\,\frac{1}{4},\,\frac{3}{4},\,1,\,\frac{1}{2},\; x^2\Big)^2 \quad \text{or} \quad Heun\Big(9,\,\frac{3}{4},\,\frac{1}{4},\,\frac{3}{4},\,1,\,\frac{1}{2},\; 9\,x^2\Big)^2. \tag{7}$$

‖ The symmetric power of an operator L is the minimal operator M such that for every set of m solutions $y_1,\,\dots,\,y_m$ of L, the product $y_1 y_2 \dots y_m$ is a solution of M. In particular, the symmetric square of an operator L is the symmetric power of the operator L with $m = 2$.

The Heun function on the right in (7) happens to be a period of an *extremal rational curve* as can be seen in the work of Doran and Malmendier [25]. These Heun functions§ in (7) can be rewritten as pullbacked ${}_2F_1$ hypergeometric functions that correspond to *classical modular forms* as can be seen in Example 1 in section 2.1 below†.

2. Diagonals of rational functions in three and four variables, corresponding to Heun functions related to classical modular forms

In the previous section we have mentioned that the rational function whose diagonal is given by the simple cubic lattice is related to modular forms. We will begin by showing this link explicitly in Example 1. In the five other examples we give different rational functions in four variables, some of whom can be found in [27], whose diagonal is given by Heun functions that can be rewritten in terms of Gauss hypergeometric functions related to modular forms. As the reader might guess, the problem of finding rational functions in four variables, whose diagonal is given by Heun functions that can be rewritten in terms of modular forms, is not an easy task!

2.1. Diagonals of rational functions corresponding to Heun functions

• Example 1. Let us consider the following rational function in four variables $x,\, y,\, z$ and w

$$R(x,\, y,\, z,\, w) \;\; = \;\; \frac{1}{1\; -(y+z\; +w\,y+x\,z\; +w\,x\,y+w\,x\,z)}, \qquad (8)$$

or the rational function:

$$R(x,\, y,\, z,\, w) \;\; = \;\; \frac{1}{1\; +x\,y+y\,z+z\,w+w\,x+y\,w+x\,z}. \qquad (9)$$

The diagonals of these two rational functions (8), (9) have the *same* series expansion with integer coefficients:

$$Diag\Big(R(x,\, y,\, z,\, w)\Big) \;\; = \;\; 1\; +6\,x\; +90\,x^2\; +1860\,x^3\; +44730\,x^4\; +1172556\,x^5 \\ +32496156\,x^6\; +936369720\,x^7\; + \;\cdots \qquad (10)$$

The linear differential operator of order three annihilating the series (10) is the *symmetric square* of a linear differential operator of order two. The diagonal (10), solution of this order-three operator, can be written as:

$$Heun\Big(\frac{1}{9},\, \frac{1}{12},\, \frac{1}{4},\, \frac{3}{4},\, 1,\, \frac{1}{2},\; 4\,x\Big)^2 \; = \; (1\,-4\,x)\cdot\, Heun\Big(\frac{1}{9},\, \frac{5}{36},\, \frac{3}{4},\, \frac{5}{4},\, 1,\, \frac{3}{2},\; 4\,x\Big)^2. \quad (11)$$

The Heun function (11) can be written as a pullbacked ${}_2F_1$ hypergeometric function

$$Heun\Big(\frac{1}{9},\, \frac{1}{12},\, \frac{1}{4},\, \frac{3}{4},\, 1,\, \frac{1}{2},\; 4\,x\Big) \qquad (12) \\ = \;\; \mathcal{A}_\pm^{(1)}\cdot\, {}_2F_1\Big([\frac{1}{6},\, \frac{2}{3}],\, [1],\, \mathcal{H}_\pm^{(1)}\Big) \;\; = \;\; \mathcal{A}_\pm^{(2)}\cdot\, {}_2F_1\Big([\frac{1}{8},\, \frac{5}{8}],\, [1],\, \mathcal{H}_\pm^{(2)}\Big),$$

§ These Heun functions can be alternatively written as $Heun\Big(\frac{1}{9},\, \frac{1}{3},\, 1,\, 1,\, 1,\, 1,\; x\Big)$. See Appendix A equation (A.12) in [2] for more details.

† An example of emergence of modular functions in the context of K3 surfaces through Dedekind's η-functions can be found in section 4 of [26].

where the two pullbacks $\mathcal{H}_\pm^{(1)}$, $\mathcal{H}_\pm^{(2)}$ are square root algebraic functions

$$\mathcal{H}_\pm^{(1)} \; = \; -54 \cdot x \cdot \frac{1 \, -27\, x \, -108\, x^2}{(1\, -54\, x)^2} \\ \pm \, 54 \cdot x \cdot (1\, -9\, x) \cdot \frac{(1\, -4\, x)^{1/2} \cdot (1\, -36\, x)^{1/2}}{(1\, -54\, x)^2}, \tag{13}$$

$$\mathcal{H}_\pm^{(2)} \; = \; -128 \cdot x \cdot \frac{1 \, -38\, x \, +200\, x^2}{(1\, -100\, x)^2 \cdot (1\, -4\, x)} \\ \pm \, 128 \cdot x \cdot (1\, -20\, x) \cdot \frac{(1\, -36\, x)^{1/2}}{(1\, -100\, x)^2 \cdot (1\, -4\, x)}, \tag{14}$$

where $Y_\pm \, = \, \left(\mathcal{A}_\pm^{(1)}\right)^{12}$ are simple algebraic functions, respectively solutions of

$$64 \, + p_3(x) \cdot Y_+ \, + \, (1 \, - \, 54\, x)^4 \cdot Y_+^2 \; = \; 0, \tag{15}$$

$$1 \, + p_3(x) \cdot Y_- \, + 64 \cdot (1 \, - \, 54\, x)^4 \cdot Y_-^2 \; = \; 0, \tag{16}$$

where

$$p_3(x) \; = \; 186624\, x^3 \, - 15552\, x^2 \, + 2484\, x \, - 65, \tag{17}$$

and where $Y_\pm \, = \, \left(\mathcal{A}_\pm^{(2)}\right)^{8}$ are simple algebraic functions, respectively solutions of

$$81 \, - 2 \cdot (41 \, - 900\, x) \cdot (1\, -4\, x) \cdot Y_+ + (1\, -100\, x)^2 \cdot (1\, -4\, x)^2 \cdot Y_+^2 \; = \; 0, \tag{18}$$

$$1 \, - 2 \cdot (41 \, - 900\, x) \cdot (1\, -4\, x) \cdot Y_- + 81 \cdot (1\, -100\, x)^2 \cdot (1\, -4\, x)^2 \cdot Y_-^2 \; = \; 0. \tag{19}$$

The two Hauptmoduls $\mathcal{H}_\pm^{(1)}$ have the following series expansions

$$\mathcal{H}_-^{(1)} \; = \; -108\, x \, - 8640\, x^2 \, - 615168\, x^3 \, - 41167872\, x^4 \, - 2650337280\, x^5 \\ - 166137937920\, x^6 - 10213026103296\, x^7 - 618505440067584\, x^8 \, + \, \cdots \tag{20}$$

and

$$\mathcal{H}_+^{(1)} \; = \; -108\, x^2 \, - 3024\, x^3 \, - 87696\, x^4 \, - 2616192\, x^5 \, - 79800768\, x^6 \\ - 2477350656\, x^7 \, - 78006945024\, x^8 \, - 2485113716736\, x^9 \, + \, \cdots \tag{21}$$

and are related by the *genus zero* modular equation:

$$625\, A^3\, B^3 \, - 525\, A^2\, B^2 \cdot (A+B) \, - 96\, AB \cdot (A^2+B^2) \, - 3\, A^2\, B^2 \\ - 4 \cdot (A^3+B^3) \, + \, 528 \cdot A\, B \cdot (A+B) \, - 432 \cdot A\, B \; = \; 0. \tag{22}$$

• Example 2. The diagonal of the rational function

$$R(x,\, y,\, z,\, w) \; = \; \frac{1}{1 \, - \, (w\, x \, + y\, z \, + w\, x\, y \, + w\, x\, z \, + w\, y\, z \, + x\, y\, z)}, \tag{23}$$

reads:

$$Diag\Big(R(x,\, y,\, z,\, w)\Big) \; = \; 1 \, + 2\, x \, + 18\, x^2 \, + 164\, x^3 \, + 1810\, x^4 \, + \, \cdots \tag{24}$$

The linear differential operator annihilating the diagonal (24) of the rational function (23) has order three:

$$L_3 \; = \; 2 \, + 60\, x \, - (1\, - 40x \, - 444\, x^2) \cdot D_x \, - 3\, x \cdot (1\, -18\, x \, - 128\, x^2) \cdot D_x^2 \\ - x^2 \cdot (1\, +4\, x) \cdot (1 \, - \, 16\, x) \cdot D_x^3 . \tag{25}$$

The operator (25) is the *symmetric square* of a linear differential operator of order two. Hence the solution corresponding to the diagonal of (23) is given by the square of a Heun function:

$$Heun\Bigl(-\frac{1}{4},\, \frac{1}{16},\, \frac{3}{8},\, \frac{5}{8},\, 1,\, \frac{1}{2},\, -4\,x\Bigr)^2$$
$$= \quad 1 \;+2\,x \;+18\,x^2 \;+164\,x^3 \;+1810\,x^4 \;+21252\,x^5 \;+263844\,x^6 \; +3395016\,x^7 \;+44916498\,x^8 \;+ \;\cdots \qquad (26)$$

This Heun function can be written as a pullbacked ${}_2F_1$ hypergeometric function:

$$Heun\Bigl(-\frac{1}{4},\, \frac{1}{16},\, \frac{3}{8},\, \frac{5}{8},\, 1,\, \frac{1}{2},\, -4\,x\Bigr) \;=\; \mathcal{A}_\pm \cdot {}_2F_1\Bigl([\frac{1}{8},\, \frac{3}{8}],\, [1],\, \mathcal{H}_\pm\Bigr), \qquad (27)$$

where $\mathcal{A}_\pm$ and the Hauptmodul $\mathcal{H}_\pm$ are algebraic functions expressed with square roots:

$$\mathcal{H}_\pm \;=\; -\,128\,x \cdot \frac{1\,-20\,x\,+50\,x^2\,+400\,x^3\,-224\,x^4\,-512\,x^5}{(1\,-88\,x\,-\,112\,x^2\,-256\,x^3)^2} \qquad (28)$$
$$\pm\; 128\,x \cdot \frac{(1\,+2\,x)\,(1\,-12\,x)\,(1\,-4\,x)\cdot(1\,+4\,x)^{1/2}\cdot(1\,-16\,x)^{1/2}}{(1\,-88\,x\,-\,112\,x^2\,-256\,x^3)^2}.$$

These Hauptmoduls (28) are also given by the quadratic relation having *genus zero*:

$$(256\,x^3\,+112\,x^2+88\,x\,-1)^2\cdot\mathcal{H}_\pm^2$$
$$-\;256\cdot x\cdot(512\,x^5+224\,x^4-400\,x^3-50\,x^2+20\,x\,-1)\cdot\mathcal{H}_\pm$$
$$+\,65536\,x^6 \;=\; 0, \qquad (29)$$

and have the series expansions:

$$\mathcal{H}_- \;=\; -256\,x \;-39936\,x^2 \;-5116416\,x^3 \;-595357696\,x^4 \;-65525931776\,x^5 \;-6954923846656\,x^6 \;-719583708750336\,x^7 \;+\;\cdots$$
$$\mathcal{H}_+ \;=\; -256\,x^5 \;-5120\,x^6 \;-89600\,x^7 \;-1433600\,x^8 \;-22201600\,x^9 \;-337755136\,x^{10} \;-5094679040\,x^{11} \;+\;\cdots \qquad (30)$$

The relation between these two Hauptmoduls corresponds to a genus zero $q \leftrightarrow q^5$ *modular equation* (q denotes the nome of the operator of order two).

• Example 3. The rational function in four variables:

$$R(x,\, y,\, z,\, w) \;=\; \frac{1}{1\;-\;(y\,+z\,+x\,y\,+x\,z\,+w\,x\,y\,+w\,x\,z\,+w\,y\,z)}, \qquad (31)$$

has a diagonal that reads:

$$Diag\Bigl(R(x,\, y,\, z,\, w)\Bigr) \;=\; 1 \;+4\,x \;+48\,x^2 \;+760\,x^3 \;+13840\,x^4 \;+273504\,x^5 \;+5703096\,x^6 \;+123519792\,x^7 \;+\;\cdots \qquad (32)$$

The linear differential operator annihilating the diagonal of this rational function is the following linear differential operator of order three, which is the *symmetric square* of an operator of order two:

$$x^2\cdot(1\,+\,x)\cdot(1\,-27\,x)\cdot D_x^3 \;\;+3\,x\cdot(1\,-39\,x\,-54\,x^2)\cdot D_x^2$$
$$+\,(1\,-86\,x\,-186\,x^2)\cdot D_x \;\;-4\cdot(1\,+6\,x). \qquad (33)$$

The operator (33) admits a Heun function that has series expansion with integer coefficients as a solution:

$$Heun\Big(-\frac{1}{27}, \frac{2}{27}, \frac{1}{3}, \frac{2}{3}, 1, \frac{1}{2}, -x\Big)^2 \;=\; 1 \;+4\,x \;+48\,x^2 \;+760\,x^3 \;+13840\,x^4$$
$$+\,273504\,x^5 \;+5703096\,x^6 \;+\;123519792\,x^7 \;+\;\cdots \qquad (34)$$

We also have the following series expansion with integer coefficients:

$$Heun\Big(-\frac{1}{27}, \frac{2}{27}, \frac{1}{3}, \frac{2}{3}, 1, \frac{1}{2}, -x\Big) \;=\; 1 \;+2\,x \;+22\,x^2 \;+336\,x^3 \;+6006\,x^4$$
$$+\,117348\,x^5 \;+2428272\,x^6 \;+52303680\,x^7 \;+\;\cdots \qquad (35)$$

The Heun function (35) can be written as a pullbacked ${}_2F_1$ hypergeometric function

$$Heun\Big(-\frac{1}{27}, \frac{2}{27}, \frac{1}{3}, \frac{2}{3}, 1, \frac{1}{2}, -x\Big) \;= \qquad (36)$$
$$\Big(25 \,-80\,x \,-24\cdot\,(1\,+\,x)^{1/2}\cdot\,(1\,-27\,x)^{1/2}\Big)^{-1/4}\,\cdot\,{}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \mathcal{H}_+\Big),$$

where the Hauptmodul $\mathcal{H}$ reads:

$$\mathcal{H}_\pm \;=\; 864\cdot\,x\cdot\,\frac{(1\,-21\,x\,+8\,x^2)\,\cdot\,(1\,-42\,x\,+454\,x^2-1008\,x^3\,-1280\,x^4)}{(1\,+224\,x\,+448\,x^2)^3}$$
$$\pm\;864\cdot\,x\cdot\,(1\,-8\,x)\cdot\,(1\,-2\,x)\cdot\,(1\,-24\,x)\cdot\,(1\,-16\,x\,-8\,x^2)$$
$$\times\;\frac{(1\,+\,x)^{1/2}\cdot\,(1\,-27\,x)^{1/2}}{(1\,+224\,x\,+448\,x^2)^3}. \qquad (37)$$

The series expansions of these two Hauptmoduls (37) read respectively

$$\mathcal{H}_+ \;=\; 1728\,x \;-1270080\,x^2 \;+593381376\,x^3 \;-226343666304\,x^4$$
$$+\,76907095308288\,x^5 \;-24246668175851520\,x^6 \;+\;\cdots \qquad (38)$$

and:

$$\mathcal{H}_- \;=\; 1728\,x^7 \,+108864\,x^8 \;+4536000\,x^9 \;+158251968\,x^{10} \qquad (39)$$
$$+\,5017070016\,x^{11} \;+150134378688\,x^{12} \;+4328271255168\,x^{13} \;+\;\cdots$$

These two Hauptmoduls are the two solutions of the *quadratic* genus zero relation:

$$1728^2\cdot\,x^8 \;\;+1728\cdot\,(1\,-21\,x\,+8\,x^2)\,(1280\,x^4+1008\,x^3-454\,x^2+42\,x-1)\cdot\,x\cdot\,\mathcal{H}_\pm$$
$$+\,(1\,+\,224\,x\,+\,448\,x^2)^3\cdot\mathcal{H}_\pm^2 \;=\; 0, \qquad (40)$$

and the two j-invariants ($\mathcal{H}_\pm = 1728/j_\pm$) are solutions of the quadratic relation:

$$x^8\cdot\;j_\pm^2 \;\;+(1\,-21\,x\,+8\,x^2)\,(1280\,x^4+1008\,x^3-454\,x^2+42\,x-1)\cdot\,x\cdot\,j_\pm$$
$$+\,(1\,+\,224\,x\,+\,448\,x^2)^3 \;=\; 0. \qquad (41)$$

Denoting $A = \mathcal{H}_+$ and $B = \mathcal{H}_-$ and considering the two (identical) quadratic relations (40) $Q(x, A) = 0$ and $Q(x, B) = 0$, one easily gets by performing the resultant between $Q(x,A) = 0$ and $Q(x,B) = 0$ in x, and thus eliminating x, the *modular equation* $P(A, B) = 0$. One gets a large *modular equation* of *genus zero* corresponding to $q \leftrightarrow q^7$ in the nome q (see (38) and (39)):

$$81600^9\cdot\,A^6\,B^6\cdot\,(343\,A^2\,+286\,A\,B\,+343\,B^2)\;+\;\cdots\;-2^{36}\cdot 3^{18}\cdot\,A\,B \;=\; 0. \qquad (42)$$

Now the previous Heun function can be written with a different algebraic Hauptmodul H and a different algebraic function $\mathcal{A}$:

$$Heun\Big(-\frac{1}{27}, \frac{2}{27}, \frac{1}{3}, \frac{2}{3}, 1, \frac{1}{2}, -x\Big) \;=\; \mathcal{A}\cdot\,{}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], H\Big), \qquad (43)$$

where this Hauptmodul is solution of the degree six equation:

$$p_6(x)^3 \cdot (1 - 2x)^6 \cdot H^6 + 3 \cdot 1728 \cdot x^4 \cdot p_{20}(x) \cdot (1 - 2x)^3 \cdot H^5$$
$$- 1728^2 \cdot x \cdot p_{23}(x) \cdot H^4 + 1728^3 \cdot x^3 \cdot p_{21}(x) \cdot H^3 + 1728^4 \cdot x^8 \cdot p_{16}(x) \cdot H^2$$
$$- 1728^5 \cdot x^{10} \cdot p_{14}(x) \cdot H + 1728^6 \cdot x^{24} = 0, \qquad (44)$$

where the polynomials $p_n(x)$ are polynomials of degree n. Note that the curve (44) is a *genus one* curve. This degree six polynomial equation (44) in H, gives Hauptmoduls having the following series expansions:

$$1728\,x^2 + 31104\,x^3 - 689472\,x^4 - 34193664\,x^5 - 431329536\,x^6 + \cdots \qquad (45)$$

and

$$1728\,x^{14} + 217728\,x^{15} + 15930432\,x^{16} + 888039936\,x^{17} + \cdots \qquad (46)$$

corresponding to $q \leftrightarrow q^7$ in the nome q. By denoting A and B two Hauptmoduls solutions of degree six of (44), $Q_6(x, A) = 0$ and $Q_6(x, B) = 0$, one gets by elimination of x through a resultant of $Q_6(x, A)$ and $Q_6(x, B)$ in x, the modular equation $P(A, B) = 0$. Now this *modular curve* is also a *genus one* curve.

• Example 4. The rational function in four variables

$$R(x, y, z, w) = \frac{1}{1 - (y + z + w\,z + x\,y + x\,z + w\,x\,y)}, \qquad (47)$$

has a diagonal whose series expansion reads:

$$Diag\Big(R(x, y, z, w)\Big) = 1 + 4\,x + 60\,x^2 + 1120\,x^3 + 24220\,x^4 + 567504\,x^5$$
$$+ 14030016\,x^6 + 360222720\,x^7 + \cdots \qquad (48)$$

The linear differential operator annihilating the diagonal of this rational function (47) has order three:

$$4 + 96 \cdot x - (1 - 92 \cdot x - 864 \cdot x^2) \cdot D_x - 3\,x \cdot (1 - 42 \cdot x - 256 \cdot x^2) \cdot D_x^2$$
$$- x^2 \cdot (1 + 4\,x) \cdot (1 - 32\,x) \cdot D_x^3. \qquad (49)$$

This order-three linear differential operator is the *symmetric square* of a linear differential operator of order two, admitting as solution the square of a Heun function, which has a series expansion with integer coefficents:

$$Heun\Big(-\frac{1}{8}, \frac{1}{16}, \frac{1}{4}, \frac{3}{4}, 1, \frac{1}{2}, -4\,x\Big)^2 = 1 + 4\,x + 60\,x^2 + 1120\,x^3$$
$$+ 24220\,x^4 + 567504\,x^5 + \cdots, \qquad (50)$$

which is related to the Heun function of example 1 through the following relation:

$$Heun\Big(-\frac{1}{8}, \frac{1}{16}, \frac{1}{4}, \frac{3}{4}, 1, \frac{1}{2}, -4\,x\Big)^2$$
$$= (1 + 4\,x)^{-1/2} \cdot Heun\Big(\frac{1}{9}, \frac{1}{12}, \frac{1}{4}, \frac{3}{4}, 1, \frac{1}{2}, \frac{4\,x}{1 + 4\,x}\Big)^2. \qquad (51)$$

The linear differential operator (49) is the *symmetric square* of a linear differential operator of order two having a pullbacked ${}_2F_1$ hypergeometric function as a solution:

$$Heun\Big(-\frac{1}{8}, \frac{1}{16}, \frac{1}{4}, \frac{3}{4}, 1, \frac{1}{2}, -4\,x\Big) = 1 + 2\,x + 28\,x^2 + 504\,x^3 + 10710\,x^4$$
$$+ 248220\,x^5 + 6091680\,x^6 + 155580000\,x^7 + 4092325500\,x^8 + \cdots$$
$$= \mathcal{A}_\pm^{(1)} \cdot {}_2F_1\Big([\frac{1}{6}, \frac{2}{3}], [1], \mathcal{H}_\pm^{(1)}\Big) = \mathcal{A}_\pm^{(2)} \cdot {}_2F_1\Big([\frac{1}{8}, \frac{5}{8}], [1], \mathcal{H}_\pm^{(2)}\Big), \qquad (52)$$

where $\mathcal{A}_\pm^{(1)}$, $\mathcal{A}_\pm^{(2)}$ and the two Hauptmoduls $\mathcal{H}_\pm^{(1)}$ are square root algebraic functions:

$$\mathcal{H}_\pm^{(1)} \,\, = \,\, -54\,x \cdot \frac{1\,-19\,x\,-200\,x^2}{(1\,+4\,x)\cdot\,(1\,-50\,x)^2} \pm\, 54\cdot\, x\cdot\,(1\,-32\,x)^{1/2}\cdot\, \frac{1\,-5\,x}{(1\,+4\,x)\cdot\,(1\,-50\,x)^2}. \qquad (53)$$

The two Hauptmoduls $\mathcal{H}_\pm^{(1)}$ are solutions of the quadratic relation:

$$(1\,+4\,x)\cdot\,(1\,-50\,x)^2\cdot\,\big(\mathcal{H}_\pm^{(1)}\big)^2 \,\,-108\,x\cdot\,(200\,x^2\,+19\,x\,-1)\cdot\,\mathcal{H}_\pm^{(1)} +11664\,x^3 \,\,=\,\, 0. \qquad (54)$$

The two Hauptmoduls $\mathcal{H}_\pm^{(2)}$ in (52) are also square root algebraic functions:

$$\mathcal{H}_\pm^{(2)} \,\, = \,\, -28\cdot\, x\cdot\, \frac{1\,-30\,x\,+64\,x^2}{(1\,-96\,x)^2} \pm\, 28\cdot\, x\cdot\,(1\,-16\,x)\cdot\, \frac{(1\,+4\,x)^{1/2}\cdot\,(1\,-32\,x)^{1/2}}{(1\,-96\,x)^2}, \qquad (55)$$

which are solutions of the quadratic relation

$$(1\,-96\,x)^2\cdot\,\big(\mathcal{H}_\pm^{(2)}\big)^2 \,\,+256\,x\cdot\,(64\,x^2\,-30\,x\,+1)\cdot\,\mathcal{H}_\pm^{(2)} +65536\,x^4 \,\,=\,\, 0, \qquad (56)$$

the algebraic function $\mathcal{A}_\pm^{(1)}$ being solution of

$$512 \,\,\, -27\cdot\,(1\,-20\,x)\cdot\,(19\,-312\,x\,-6000\,x^2\,-80000\,x^3)\cdot\,Y +(1\,+4\,x)^3\cdot\,(1\,-50\,x)^6\cdot\,Y^2 \,\,=\,\, 0, \qquad (57)$$

where $Y \, = \, \big(\mathcal{A}_\pm^{(1)}\big)^{18}$, and the algebraic function $\mathcal{A}_\pm^{(2)}$ being solution of

$$1\,\,+\,2\cdot\,q_8(x)\cdot Y \,\,+\,3^{32}\cdot\,(1\,-96\,x)^{16}\cdot\,Y^2 \,\,=\,\, 0, \qquad \text{where:} \qquad (58)$$

$$q_8(x) \,\,=\,\, 9239327393023110047334\,x^8\,-1823967923835879156613\,12\,x^7 +7442201965961886564352\,x^6\,+1056452765570247006617\,6\,x^5 -1994146206485388984320\,x^4\,+1544084662968304271\,36\,x^3 -6048257896412868608\,x^2\,+118593292086518528\,x\,-926510094425921, \qquad (59)$$

where $Y \, = \, \big(\mathcal{A}_\pm^{(2)}\big)^{64}$. The series expansions of the Hauptmoduls $\mathcal{H}_\pm^{(1)}$ read:

$$\mathcal{H}_-^{(1)} \,\,=\,\, -108\,x\,-8208\,x^2\,-547776\,x^3\,-34193664\,x^4\,-2048523264\,x^5 -119335292928\,x^6\,-6811411267584\,x^7\,-382782182326272\,x^8\,+\,\cdots \qquad (60)$$

and:

$$\mathcal{H}_+^{(1)} \,\,=\,\, -108\,x^2\,-2160\,x^3\,-56592\,x^4\,-1475712\,x^5\,-39711168\,x^6 -1088716032\,x^7\,-30317739264\,x^8\,-854924599296\,x^9\,+\,\cdots \qquad (61)$$

The relation between these two Hauptmoduls corresponds to the genus zero *modular equation*:

$$625\,A^3\,B^3\,\,-525\,A^2\,B^2\cdot\,(A+B)\,\,-96\,A\,B\,\cdot\,(A^2+B^2)\,\,-3\,A^2\,B^2 -4\cdot\,(A^3+B^3)\,\,+528\cdot\,A\,B\cdot\,(A+B)\,\,-432\cdot\,A\,B \,\,=\,\, 0, \qquad (62)$$

which can (for instance) be rationally parametrised as follows:

$$A(v) \,\,=\,\, \frac{108\cdot\,v\cdot\,(1\,+v)^2}{(16\,+15\,v)\cdot\,(2\,+3\,v)^2}, \qquad B(v) \,\,=\,\, -\frac{108\cdot\,(1\,+v)\cdot\,v^2}{(4\,+3\,v)\cdot\,(32\,+33\,v)^2}, \qquad (63)$$

where $A(v)$ and $B(v)$ are related by an involution:

$$B(v) \, = \, A\Big(-\frac{64 \cdot (1+v)}{63\,v+64}\Big), \qquad A(v) \, = \, B\Big(-\frac{64 \cdot (1+v)}{63\,v+64}\Big). \qquad (64)$$

The series expansions of the Hauptmoduls $\mathcal{H}_{\pm}^{(2)}$ read:

$$\mathcal{H}_{-}^{(2)} \, = \, -56\,x \, -9072\,x^2 \, -1229256\,x^3 \, -152418672\,x^4 \, -17935321320\,x^5 \\ -2038883437584\,x^6 \, -226173478925520\,x^7 \, + \, \cdots \qquad (65)$$

and

$$\mathcal{H}_{+}^{(2)} \, = \, -56\,x^3 \, -1680\,x^4 \, -46872\,x^5 \, -1291248\,x^6 \, -35752752\,x^7 \\ -998627616\,x^8 \, -28151491032\,x^9 \, -800518405680\,x^{10} \, + \, \cdots \qquad (66)$$

The relation between these last two Hauptmoduls $\mathcal{H}_{\pm}^{(2)}$ corresponds to a genus zero *modular equation*:

$$640000 \cdot \, A^2\,B^2 \cdot \, (9\,A^2 \, + 14\,A\,B \, + 9\,B^2) \\ + 4800\,A\,B \cdot \, (A+B) \cdot \, (A^2 \, - 1954\,A\,B \, + B^2) \\ + A^4 + B^4 \, - 56196\,A\,B \cdot \, (A^2 + B^2) \, + 3512070\,A^2\,B^2 \\ + 116736 \cdot \, A\,B \cdot \, (A+B) \, - 65536 \cdot \, A\,B \, = \, 0, \qquad (67)$$

which is the same *modular equation* as (62). Now the modular equation (22) of example 1, is actually the same as the modular equation (67) of example 4! This is a consequence of identity (51).

• Example 5. The rational function in four variables

$$R(x,\, y,\, z,\, w) \, = \, \frac{1}{1 \, - \, (y \, + z \, + x\,y \, + x\,z \, + w\,y \, + w\,z \, + w\,x\,z)}, \qquad (68)$$

has a diagonal that reads:

$$Diag\Big(R(x,\, y,\, z,\, w)\Big) \, = \, 1 \, + 6\,x \, + 114\,x^2 \, + 2940\,x^3 \, + 87570\,x^4 \, + \, \cdots \qquad (69)$$

The operator annihilating the diagonal (69) of this rational function in four variables (68) reads:

$$6 \, + 12 \cdot x \, - (1 \, - 144 \cdot x \, - 108 \cdot x^2) \cdot \, D_x \, - x \cdot \, (3 \, - 198 \cdot x \, - 96 \cdot x^2) \cdot \, D_x^2 \\ - x^2 \cdot \, (1 \, - 44 \cdot x \, - 16 \cdot x^2) \cdot \, D_x^3, \qquad (70)$$

It is the *symmetric square* of a linear differential operator of order two which admits a Heun solution analytic at $x \, = \, 0$. Consequently the rational function of order three (70) has a solution that is the square of a Heun function, and admits the series expansion with *integer coefficients*:

$$Heun\Big(-\frac{123}{2} \, + \frac{55}{2} \cdot 5^{1/2}, -\frac{33}{8} \, + \frac{15}{8} \cdot 5^{1/2}, \, \frac{1}{4}, \, \frac{3}{4}, \, 1, \, \frac{1}{2}, \, 2 \cdot (11 \, - 5 \cdot \, 5^{1/2}) \cdot \, x\Big)^2 \\ = \, 1 \, + 6\,x \, + 114\,x^2 \, + 2940\,x^3 \, + 87570\,x^4 \, + 2835756\,x^5 \, + 96982116\,x^6 \\ + 3446781624\,x^7 \, + 126047377170\,x^8 \, + \, \cdots \qquad (71)$$

The Heun solution (71) can also be rewritten as a pullbacked ${}_2F_1$ hypergeometric function:

$$\mathcal{A}(x) \cdot \, {}_2F_1\Big([\frac{1}{12}, \, \frac{5}{12}], \, [1], \, \mathcal{H}\Big)^2, \qquad (72)$$

where $\mathcal{A}(x)$ is an algebraic function and where the Hauptmodul $\mathcal{H}$ is solution of the quadratic relation:

$$(144\,x^2 + 216\,x + 1)^3 \cdot \mathcal{H}^2 \\ - 1728\,x \cdot (3456\,x^5 + 7776\,x^4 - 12600\,x^3 + 1890\,x^2 - 80\,x + 1) \cdot \mathcal{H} \\ + 2985984\,x^6 \; = \; 0. \tag{73}$$

The two Hauptmoduls read

$$\mathcal{H}_\pm \; = \; \frac{864\,x \cdot (3456\,x^5 + 7776\,x^4 - 12600\,x^3 + 1890\,x^2 - 80\,x + 1)}{(144\,x^2 + 216\,x + 1)^3} \\ \pm \; \frac{864\,(1 - 36\,x) \cdot (1 - 18\,x)\,(1 - 4\,x)\,x}{(144\,x^2 + 216\,x + 1)^3} \cdot (1 - 44\,x - 16\,x^2)^{1/2}, \tag{74}$$

and admit the respective expansions:

$$\mathcal{H}_+ \; = \; 1728\,x \; - 1257984\,x^2 \; + 575828352\,x^3 \; - 214274336256\,x^4 \; + \; \cdots \\ \mathcal{H}_- \; = \; 1728\,x^5 \; + 138240\,x^6 \; + 7793280\,x^7 \; + 383961600\,x^8 \; + \; \cdots \tag{75}$$

These two Hauptmoduls series (74) are related by a *genus zero* modular equation which admits the following rational parametrization‡ as:

$$\mathcal{H}_+ \; = \; \frac{1728\,z}{(z^2 + 10\,z + 5)^3}, \qquad \mathcal{H}_- \; = \; \frac{1728\,z^5}{(z^2 + 250\,z + 3125)^3}. \tag{76}$$

• Example 6. The rational function in four variables

$$R(x,\, y,\, z,\, w) \; = \; \frac{1}{1 \; - \; (y + z + x\,y + x\,z + w\,z + w\,x\,y + w\,x\,y\,z)}, \tag{77}$$

has a diagonal that reads:

$$Diag\Big(R(x,\, y,\, z,\, w)\Big) \; = \; 1 \; + 5\,x \; + 73\,x^2 \; + 1445\,x^3 \; + 33001\,x^4 \; + \; \cdots \tag{78}$$

The operator annihilating the diagonal of the rational function (77) reads:

$$L_3 \; = \; x^2 \cdot (1 - 34\,x + x^2) \cdot D_x^3 \; + 3\,x \cdot (1 - 51\,x + 2\,x^2) \cdot D_x^2 \\ + (1 - 112\,x + 7\,x^2) \cdot D_x \; + x - 5. \tag{79}$$

It is the *symmetric square* of an order-two linear differential operator with a Heun solution, analytic at $x = 0$. Consequently the diagonal of (77), solution of (79), can be written in terms of the square of two Heun functions¶ which have series expansions with integer coefficients:

$$(1 - 34\,x + x^2) \; \times \\ Heun\Big(577 + 408 \cdot 2^{1/2}, \; \frac{663}{2} + 234 \cdot 2^{1/2}, \; \frac{3}{2}, \; \frac{3}{2}, \; 1, \; \frac{3}{2}, \; (17 + 12 \cdot 2^{1/2}) \cdot x\Big)^2 \\ = \; (1 - 34\,x + x^2) \; \times \\ Heun\Big(577 - 408 \cdot 2^{1/2}, \; \frac{663}{2} - 234 \cdot 2^{1/2}, \; \frac{3}{2}, \; \frac{3}{2}, \; 1, \; \frac{3}{2}, \; (17 - 12 \cdot 2^{1/2}) \cdot x\Big)^2 \\ = \; 1 + 5\,x + 73\,x^2 + 1445\,x^3 + 33001\,x^4 + 819005\,x^5 + 21460825\,x^6 + \; \cdots \tag{80}$$

It can also be written as a pullbacked ${}_2F_1$ hypergeometric function

$$\mathcal{A}_- \cdot {}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], \; \mathcal{H}_-\Big)^2, \tag{81}$$

‡ It corresponds to $N = 5$ in Table 4 and Table 5 of [28].
¶ These two Heun functions are Galois conjugates.

where the Hauptmodul $\mathcal{H}_\pm$ reads

$$\mathcal{H}_\pm \; = \; {{1 \, -24\,x \, +30\,x^2 \, +x^3} \over {2 \cdot \, (1 \, +x)^3}} \quad \pm \; {{1 \, -7\,x \, +x^2} \over {2 \cdot \, (1 \, +x)^3}} \cdot \, (1 \, -34\,x \, +x^2)^{1/2}, \qquad (82)$$

with the expansions:

$$\mathcal{H}_- \; = \; 27\,x^2 + 648\,x^3 + 15471\,x^4 + 389016\,x^5 + 10234107\,x^6 + 278861616\,x^7$$
$$+ \, 7808397759\,x^8 \, + 223397228880\,x^9 \; + \; \cdots \qquad (83)$$

and where the algebraic factor $\mathcal{A}_-$ reads:

$$\mathcal{A}_- \; = \; {{3} \over {2}} \cdot {{1 \, -x} \over {(1 \, + \, x)^2}} \; - \; {{(1 \, -34\,x \, + \, x^2)^{1/2}} \over {2 \cdot \, (1 \, + \, x)^2}}. \qquad (84)$$

2.2. Periods of extremal rational surfaces

The rational function in *three* variables:

$$R(x, \, y, \, z) \; = \; {{1} \over {1 \; +x \, +y \, +z \; +x\,y \, +y\,z \; -x^3\,y\,z}}, \qquad (85)$$

has a diagonal given by the following series expansion:

$$Diag\Big(R(x, \, y, \, z)\Big) \; = \; 1 \; -2\,x \; +6\,x^2 \; -11\,x^3 \; -10\,x^4 \; +273\,x^5 \, -1875\,x^6$$
$$+ \, 9210\,x^7 \; -34218\,x^8 \; +78721\,x^9 \; +108581\,x^{10} \; + \; \cdots \qquad (86)$$

In order to find the diagonal of this rational function of three variables, one gets the telescoper annihilating this diagonal using creative telescoping [6]. This telescoper is a linear differential operator of order four L_4, which is the *direct sum* of two† linear differential operators having order two $L_4 \, = \, L_2 \oplus M_2$. These two operators read respectively

$$L_2 \; = \; (1 \, +9\,x \, +27\,x^2) \cdot \, x^2 \cdot \, D_x^2 \; + (1 \, +9\,x)^2 \cdot \, x \cdot \, D_x \; +3\,x \cdot \, (1 \, +9\,x),$$

and:

$$M_2 \; = \; (1 \, +9\,x \, +27\,x^2) \cdot \, (5 \, +18\,x) \cdot \, (1 \, -2\,x) \cdot \, x^2 \cdot \, D_x^2$$
$$+ \, (5 \, +70\,x \, +261\,x^2 \, -756\,x^3 \, -2916\,x^4) \cdot \, x \cdot \, D_x$$
$$+ \; x \cdot \, (1 \, -9\,x) \cdot \, (5 \, + \, 60\,x \, +108\,x^2). \qquad (87)$$

The solution of the order-two linear differential operator L_2 has the following Heun function‡ solution, analytic at $x \, = \, 0$:

$$\mathcal{S}_1 \; = \; Heun\Big({{1} \over {2}} \, - \, {{i\sqrt{3}} \over {2}}, \, {{1} \over {2}} \, - \, {{i\sqrt{3}} \over {6}}, \, 1, \, 1, \, 1, \, 1, \; {{3} \over {2}} \cdot \, \Big(-3 \, + \, i\sqrt{3}\Big) \cdot \, x\Big) \qquad (88)$$
$$= \; 1 \; -3\,x \; \; +9\,x^2 \; -21\,x^3 \; +9\,x^4 \; +297\,x^5 \; -2421\,x^6 \; +12933\,x^7 \; + \; \cdots \qquad (89)$$

This Heun function (88) can also be written alternatively in terms of other ${}_2F_1$ hypergeometric functions:

$$Heun\Big({{1} \over {2}} \, - \, {{i\sqrt{3}} \over {2}}, \, {{1} \over {2}} \, - \, {{i\sqrt{3}} \over {6}}, \, 1, \, 1, \, 1, \, 1, \; {{3} \over {2}} \cdot \, \Big(-3 \, + \, i\sqrt{3}\Big) \cdot \, x\Big)$$

† These two operators L_2 and M_2 are not homomorphic because they do not have the same singularities.
‡ This Heun function $Heun(a, q, \alpha, \beta, \, \gamma, \, \delta, \, \rho\,x)$ is such that $q \; = \; a/(1 \, + \, a)$, $q/\rho \; = \; -1/9$, $a/\rho^2 = 1/27$, $1/\rho$ and a/ρ are complex conjugates.

$$= \Big({{1} \over {1\, +3\, x}}\Big)^{1/4} \cdot \, \Big({{1} \over {1\, +9\, x\, +27\, x^2\, +3\, x^3}}\Big)^{1/4} \qquad (90)$$

$$\times \, _2F_1\Big([{{1} \over {12}}, {{5} \over {12}}], [1], \, {{1728 \cdot \, x^9 \cdot \, (1\, +9\, x\, +27\, x^2)} \over {(1\, +3\, x)^3 \cdot \, (1\, +9\, x\, +27\, x^2\, +3\, x^3)^3}}\Big)$$

$$= \, (1\, +9\, x)^{-1/4} \cdot \, (1\, +243\, x\, +2187\, x^2\, +6561\, x^3)^{-1/4}$$

$$\times \, _2F_1\Big([{{1} \over {12}}, {{5} \over {12}}], [1], \, {{1728 \cdot \, x \cdot \, (1\, +9\, x\, +27\, x^2)} \over {(1\, +9\, x)^3 \cdot \, (1\, +243\, x\, +2187\, x^2\, +6561\, x^3)^3}}\Big). \qquad (91)$$

The modular equation relating the Hauptmoduls of the two Gauss hypergeometric functions in (91) corresponds to $q \, \leftrightarrow \, q^9$ in the nome q (see also Table 4 and Table 5 in [28]).

The Heun function (90) is in fact the *period* of an *extremal rational surface* [25], and was shown to be related¶ to *classical modular forms* in table 15 in [28] for $N \, = \, 9$:

$$Heun\Big({{-9 \mp 3\, \sqrt{3}\, i} \over {-9 \pm 3\, \sqrt{3}\, i}}, \, {{9 \pm \, 3\, \sqrt{3}\, i} \over {18}}, \, 1, \, 1, \, 1, \, 1, \, \, {{2\, x} \over {-9 \pm 3\, \sqrt{3}\, i}}\Big)$$

$$= \, \, Heun\Big({{1 \pm \, \sqrt{3}\, i} \over {2}}, \, {{3 \pm \, \sqrt{3}\, i} \over {6}}, \, 1, \, 1, \, 1, \, 1, \, \, {{-3 \mp \, \sqrt{3}\, i} \over {18}} \cdot \, x\Big). \qquad (92)$$

The other operator M_2 has the following (classical modular form, see Appendix B) pullbacked $_2F_1$ hypergeometric solution analytic at $x \, = \, 0$:

$$\mathcal{S}_2 \, \, = \, \, \, {{1} \over {(1\, +4\, x\, -2\, x^2\, -36\, x^3\, +81\, x^4)^{1/4}}}$$

$$\times \, _2F_1\Big([{{1} \over {12}}, {{5} \over {12}}], [1], \, {{1728 \cdot \, x^5 \, \cdot \, (1 + 9\, x\, +27\, x^2) \cdot \, (1\, -2\, x)^2} \over {(1\, +4\, x\, -2\, x^2\, -36\, x^3\, +81\, x^4)^3}}\Big)$$

$$= \, \, 1 \, \, -x \, \, +3\, x^2 \, \, -x^3 \, \, -29\, x^4 \, \, +249\, x^5 \, \, -1329\, x^6 \, \, +5487\, x^7 \, \, -16029\, x^8 \, \, +12149\, x^9 \, \, +252253\, x^{10} \, \, + \, \, \cdots \qquad (93)$$

Thus the diagonal of (85) *is the half-sum of the two series* (88) and (93) corresponding to *classical modular forms*:

$$Diag\Big(R(x, \, y, \, z)\Big) \, \, = \, \, \, {{\mathcal{S}_1 \, +\mathcal{S}_2} \over {2}}. \qquad (94)$$

2.3. Derivatives of classical modular forms

We give here an example of a diagonal of a rational function in three variables yielding a derivative of a classical modular form (or a derivative of a Heun function). Let us consider the following rational function in three variables:

$$R(x, \, y, \, z) \, \, = \, \, \, {{3\, x^3\, y} \over {1 \, \, + \, x \, + \, y \, \, + \, z}}. \qquad (95)$$

The diagonal of (95) has the following series expansion with *integer coefficients*:

$$-\, 30\, x^3 \, \, +840\, x^4 \, \, -20790 x^5 \, \, +504504\, x^6 \, \, -12252240\, x^7 \, \, +299304720\, x^8 \, \, -7362064710\, x^9 \, \, +182298745200\, x^{10} \, \, + \, \, \cdots \qquad (96)$$

¶ Change $x \, \rightarrow \, x/27$ to match $\mathcal{S}_1$, given by (88), with (92).

The telescoper of this rational function of three variables (95) gives a linear differential operator of order three $L_3 = L_1 \oplus L_2$ that is the direct sum of a linear differential operator of order one L_1, and a linear differential of operator of order two L_2, where:

$$L_1 = x \cdot D_x - 1, \tag{97}$$
$$L_2 = (1 + 27\,x) \cdot (1 + 30\,x) \cdot x \cdot D_x^2 - 3 \cdot x \cdot D_x + 180\,x + 3.$$

The operator L_1 admits the solution $y(x) = x$, while the operator L_2 has the following Heun solution:

$$x \cdot Heun\Big(\frac{9}{10}, 0, \frac{1}{3}, \frac{2}{3}, 2, 1, -27 \cdot x\Big) = x - 30\,x^3 + 840\,x^4 - 20790x^5$$
$$+ 504504\,x^6 - 12252240\,x^7 + 299304720\,x^8 + \cdots \tag{98}$$

$$= -x \cdot {}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], -27 \cdot x\Big) \tag{99}$$
$$+ 2 \cdot x \cdot (1 + 27\,x) \cdot {}_2F_1\Big([\frac{4}{3}, \frac{5}{3}], [2], -27 \cdot x\Big)$$

$$= \mathcal{L}_1\Big({}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], -27 \cdot x\Big)\Big) \tag{100}$$

where:
$$\mathcal{L}_1 = -x - \frac{1 + 27\,x}{3} \cdot x \cdot \frac{d}{dx}. \tag{101}$$

With this example we see that a Heun function which has a series expansion with *integer coefficients* (or more generally a globally bounded series), may *not* necessarily be a classical modular form¶, and can instead be *a linear differential operator of order one acting on a classical modular form.*

2.4. Generalization of the previous result

Let us recall example 6, and let us consider, instead of the rational function (9), its *homogeneous partial derivative* with respect to one of its four variables:

$$x \cdot \frac{\partial R(x, y, z, w)}{\partial x} = \frac{x \cdot (y + z + w)}{(1 + x\,y + y\,z + z\,w + w\,x + y\,w + x\,z)^2}. \tag{102}$$

The telescoper of this rational function (102) is a linear differential operator of order three M_3 which is homomorphic to the operator of order three L_3 which was the telescoper of the rational function (9). This homomorphism reads:

$$M_3 \cdot \theta = L_1 \cdot L_3 \qquad \text{where:} \qquad L_1 = (1 - 18\,x) \cdot \theta + 18\,x, \tag{103}$$

where θ is the *homogeneous derivative* $\theta = x \cdot D_x$. Consequently the solutions of the order-three linear differential operator M_3 are simply obtained by taking the homogeneous derivative $\theta = x \cdot D_x$ of the solutions of the order-three linear differential operator L_3. In particular, the diagonal of the rational function (102) is the *homogeneous derivative* of the diagonal of the rational function (9):

$$Diag\Big(x \cdot \frac{\partial R(x, y, z, w)}{\partial x}\Big) = x \cdot \frac{d}{dx}\Big(Diag\Big(R(x, y, z, w)\Big)\Big), \tag{104}$$

The diagonal of (102) will thus be the *homogeneous derivative of the classical modular form* (11). We do not provide a proof, but in the experimental framework the following

¶ In the sense defined in Appendix A and Appendix B.

identity seems to hold for any order- N linear differential operator L:

$$Diag\Big(\mathcal{L}\Big(R(x,\, y,\, z,\, w)\Big)\Big) \;\; = \;\; L\Big(\, Diag\Big(R(x,\, y,\, z,\, w)\Big)\Big), \qquad (105)$$

$$\text{where:} \qquad L \; = \; \sum_{n=0}^{N} P_n(x) \cdot \theta^n, \qquad \mathcal{L} \; = \; \sum_{n=0}^{N} P_n(x\,y\,z\,w) \cdot \Theta^n, \qquad (106)$$

$$\text{with:} \qquad \theta \; = \; x \cdot \frac{d}{dx}, \qquad \cdots \qquad \Theta \; = \; w \cdot \frac{\partial}{\partial w}, \qquad (107)$$

where the P_n's are polynomials. This identity can, of course be generalized to the diagonal of rational functions of an arbitrary number of variables. For any Heun function or classical modular form of this paper obtained as a diagonal of a rational function, we can use these identities (104), (105) to get other rational functions that will be *derivatives of Heun functions or classical modular forms*†.

3. Heun function solutions of telescopers of rational functions related to Shimura curves

The rational function in four variables

$$R(x,\, y,\, z,\, w) \;\; = \;\; \frac{x\,y\,z}{1 \;\; - x\,y\,z\,w \;\; + x\,y\,z \cdot (x\, + y\, + z) \;\; + x\,y + \,y\,z\, + \,x\,z}, \qquad (108)$$

has a telescoper that is a linear differential operator of order three:

$$L_3 \;\; = \;\; 8\,x \cdot (1\, - x) \cdot (1\, - 4\,x) \cdot D_x^3 \;\; + 12 \cdot (1\, - 10\,x\, + 12\,x^2) \cdot D_x^2 \\ \qquad - 6 \cdot (7\, - 17 \cdot x) \cdot D_x \;\; + 3, \qquad (109)$$

which corresponds to the *symmetric square* of a linear differential operator of order two. The solutions of L_3 are thus expressed in terms of the following Heun functions:

$$Heun\Big(\frac{1}{4},\, \frac{1}{64},\, \frac{1}{8},\, \frac{3}{8},\, \frac{1}{2},\, \frac{1}{2},\; x\Big)^2, \qquad x \cdot\, Heun\Big(\frac{1}{4},\, \frac{21}{64},\, \frac{5}{8},\, \frac{7}{8},\, \frac{3}{2},\, \frac{1}{2},\; x\Big)^2, \qquad (110)$$

or:

$$x^{1/2} \cdot\, Heun\Big(\frac{1}{4},\, \frac{1}{64},\, \frac{1}{8},\, \frac{3}{8},\, \frac{1}{2},\, \frac{1}{2},\; x\Big) \cdot\, Heun\Big(\frac{1}{4},\, \frac{21}{64},\, \frac{5}{8},\, \frac{7}{8},\, \frac{3}{2},\, \frac{1}{2},\; x\Big). \qquad (111)$$

The series expansion of the first expression in (110) reads:

$$1 \, + \frac{1}{4}\, x \;\; + \frac{5}{16}\, x^2 \;\; + \frac{5}{8}\, x^3 \;\; + \frac{2795}{1792}\, x^4 \;\; + \frac{15691}{3584}\, x^5 \;\; + \frac{1039363}{78848}\, x^6 \;\; + \;\; \cdots \qquad (112)$$

While the other Heun functions obtained in this paper are diagonals of rational functions and have globally bounded series expansions, the series expansion (112) is *not*† globally bounded: it cannot be recast into a series with integer coefficients. Hence (112) *cannot be a diagonal* of a rational function since diagonals of rational functions are necessarily globally bounded [1]: it is instead a solution of the telescoper of a rational function. In fact the diagonal of the rational function (108) is *zero*.

The operator (109) is the *symmetric square* of the linear differential operator of order two L_2:

$$L_2 \;\; = \;\; D_x^2 \;\; + \frac{1 - 10\,x + 12\,x^2}{2\,x \cdot (1\, - 4\,x) \cdot (1\, - x)} \cdot D_x \;\; - \frac{1\, - 3\,x}{16 \cdot x \cdot (1\, - 4\,x) \cdot (1\, - x)}, \qquad (113)$$

† Derivatives of modular forms are not modular forms
† After a rescaling of the variable.

whose (formal) series expansions at 0, 1, and ∞ do not contain *logarithms*. This linear differential operator of order two L_2 admits the solutions:

$$x^{1/2} \cdot (1-x)^{-7/8} \cdot {}_2F_1\Big([\frac{7}{24}, \frac{11}{24}], [\frac{5}{4}], \frac{27}{4} \cdot \frac{x^2}{(1-x)^3}\Big), \tag{114}$$

$$(1-x)^{-1/8} \cdot {}_2F_1\Big([\frac{1}{24}, \frac{5}{24}], [\frac{3}{4}], \frac{27}{4} \cdot \frac{x^2}{(1-x)^3}\Big).$$

The precise correspondence with the Heun functions in (110) reads:

$$Heun\Big(\frac{1}{4}, \frac{1}{64}, \frac{1}{8}, \frac{3}{8}, \frac{1}{2}, \frac{1}{2}, x\Big)$$
$$= (1-x)^{-1/8} \cdot {}_2F_1\Big([\frac{1}{24}, \frac{5}{24}], [\frac{3}{4}], \frac{27}{4} \cdot \frac{x^2}{(1-x)^3}\Big), \tag{115}$$

$$Heun\Big(\frac{1}{4}, \frac{21}{64}, \frac{5}{8}, \frac{7}{8}, \frac{3}{2}, \frac{1}{2}, x\Big)$$
$$= (1-x)^{-7/8} \cdot {}_2F_1\Big([\frac{7}{24}, \frac{11}{24}], [\frac{5}{4}], \frac{27}{4} \cdot \frac{x^2}{(1-x)^3}\Big). \tag{116}$$

The two solutions of the linear differential operator (113) can be used to construct a basis for the space of *automorphic forms*, which can then be used to construct Hecke operators relative to this basis¶. The second solution in (114) corresponds to an *automorphic form associated with a Shimura curve* with signature (0, 4, 2, 6) which appears in Table 1 in [30]. Hence one obtains *Shimura curves associated to telescopers of rational functions*. More details on Heun functions or ${}_2F_1$ *automorphic forms* associated to *Shimura curves* [31, 32, 33, 34, 35] are given in Appendix C.

4. Conclusion

The examples of diagonals of rational functions in three or four variables, that we presented here, illustrate cases where the diagonal of the rational functions are given by Heun functions having series with *integer coefficients*, and can be expressed either in terms of pullbacked hypergeometric functions that are *classical modular forms*, or *derivatives* of classical modular forms. Furthermore, we constructed in subsection 2.2, a rational function whose diagonal is given by a Heun function that has already been identified as a "period" of an *extremal rational elliptic surface* [25], and that has also emerged in the context of pullbacked ${}_2F_1$ hypergeometric functions [28]. Finally we have also seen a case where the rational function has a telescoper with Heun function solutions, that can be expressed as pullbacked ${}_2F_1$ hypergeometric functions that are *not* globally bounded, and happen to be associated with one of the 77 cases of *Shimura curves* [30]. Such remarkable ${}_2F_1$ hypergeometric functions solutions of a telescoper of a rational function are *not* diagonals of that rational function since their series are not globally bounded. They can be interpreted as "periods" [36, 37] of an algebraic variety over some non-evanescent† cycles.

These examples suggest an *algebraic geometrical* link between the diagonals/solutions of the telescopers, and the original rational functions, and this link should be investigated. This study should help shed light on the geometrical nature of the algebraic varieties associated with the denominators of the rational functions

¶ See example 9 in [29] for more details.
† Diagonals are periods over evanescent cycles.

(K3, Calabi-Yau threefolds, extremal rational elliptic surfaces, Shimura varieties). In a forthcoming paper which is a work in progress at the current stage, we intend to introduce an *algebraic geometry approach* that proves to be efficient in explaining this link.

Acknowledgments. Y. A. and J-M. M. would like to thank the Stat. and Math. Department of Melbourne University for hospitality, where part of this paper was written. Y. A. and J-M. M. would like to thank Tony Guttmann for numerous discussions on lattice Green functions. Y. A. would like to thank Andreas Malmendier for several enlightening discussions on the periods of rational extremal surfaces. Y. A. would like to thank Dahlia Abdelaziz, Rashida Abdelaziz and Siwa Abdelaziz for the great time spent together while working on this paper, and for being the joyful and loving cousins they are. Y. A. would like to thank Mark van Hoeij for the explanations he provided him on Heun to hypergeometric functions pullbacks. Y. A. would like to thank John Voight for enlightening explanations on Shimura curves and for pointing out the reference [30]. Y. A. would like to thank Yifang Yang for several very explanatory e-mails on automorphic functions and Shimura curves. Y. A. would like to thank Wadim Zudilin for an enlightening exchange on the modular parametrization of hypergeometric functions. J-M. M. would like to thank Pierre Charollois for many enlightening discussions on modularity and automorphic forms. J-M. M. would like to thank Kilian Raschel for enlightening discussions on walks on the quarter plane. S. B. would like to thank the LPTMC and the CNRS for kind support. Y. A and C. K. were supported by the Austrian Science Fund (FWF): F5011-N15. We thank the Research Institute for Symbolic Computation for access to the RISC software packages. We thank M. Quaggetto for technical support.

Appendix A. A nome necessary condition to be a classical modular form: why ${}_2F_1([1/5, 1/5], [1], x)$ is not a classical modular form.

Consider the identity:

$$
\begin{aligned}
&{}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], \; x\Big) \\
&\qquad = \; (1 \, + \, 8\,x)^{-1/4} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \; 64 \cdot \frac{x \cdot (1-x)^3}{(1+8\,x)^3}\Big).
\end{aligned} \tag{A.1}
$$

The nome associated to the linear differential operator of order two having ${}_2F_1([1/3, \, 2/3], \, [1], \, x)$ as a solution is given by:

$$
Q(x) \; = \; x \, + \frac{5}{9}\,x^2 \, + \frac{31}{81}\,x^3 + \frac{5729}{19683}\,x^4 + \frac{41518}{177147}\,x^5 + \frac{312302}{1594323}\,x^6 \; + \; \cdots \tag{A.2}
$$

and the nome associated to the operator of order two having ${}_2F_1([1/12, 5/12], \, [1], \, x)$ as a solution expands as follows:

$$
q(x) \; = \; x \, + \frac{31}{72}\,x^2 + \frac{20845}{82944}\,x^3 + \frac{27274051}{161243136}\,x^4 + \frac{183775457147}{1486016741376}\,x^5 \; + \; \cdots \tag{A.3}
$$

The two ${}_2F_1$ hypergeometric series are *globally bounded*, the series of the corresponding nomes (A.2) and (A.3) *are also globally bounded*, as one expects for a classical modular form. The identity (A.1) on the other solutions of the linear differential operators annihilating ${}_2F_1([1/3,\, 2/3],\, [1],\, x)$ and ${}_2F_1([1/12, 5/12],\, [1],\, p(x))$, gives the following identity on their respective ratio τ

$$\tau\Big([\frac{1}{3},\, \frac{2}{3}],\, [1],\, x\Big) \quad = \quad \mu \cdot \tau\Big([\frac{1}{12},\, \frac{5}{12}],\, [1],\, 64 \cdot \frac{x \cdot (1-x)^3}{(1+8\,x)^3}\Big), \tag{A.4}$$

where μ is a constant, which gives after exponentiation:

$$64 \cdot Q(x) \quad = \quad q\Big(64 \cdot \frac{x \cdot (1-x)^3}{(1+8\,x)^3}\Big). \tag{A.5}$$

Now, the RHS of (A.5) is *necessarily globally bounded*, which agrees with the globally bounded character of the nome (A.2).

In contrast, let us consider ${}_2F_1([1/5, 1/5], [1],\, x)$. The corresponding series *is globally bounded*¶, however the corresponding nome which reads

$$Q_{[1/5,1/5]}(x) \quad = \quad x \quad + \frac{8}{25}\,x^2 \quad + \frac{102}{625}\,x^3 \quad + \frac{4744}{46875}\,x^4 \quad + \frac{81914}{1171875}\,x^5$$
$$+ \frac{63094248}{1220703125}\,x^6 \quad + \frac{11003093386\,x^7}{274658203125} \quad + \quad \cdots \tag{A.6}$$

is *not* globally bounded. Therefore, it is not possible to find any *algebraic (or rational) pullback* $p(x)$ such that

$$\mu \cdot Q_{[1/5,1/5]}(x) \quad = \quad q\Big(p(x)\Big), \tag{A.7}$$

since the RHS of (A.7) is *necessarily globally bounded* when $\mu \cdot Q_{[1/5,1/5]}(x)$ *cannot be globally bounded* regardless of the constant μ. In Appendix B we give the exhaustive list of these 28 hypergeometric ${}_2F_1$'s related to *classical modular curves* that were obtained using the necessary condition on the nome explained here.

Appendix B. Special ${}_2F_1$ hypergeometric functions associated with classical modular forms

The Heun functions of this paper can all be rewritten in terms of pullbacked ${}_2F_1$ hypergeometric functions which turn out to correspond to *classical modular curves* (with the exception of the "Shimura" Heun functions of section (3). These ${}_2F_1$ hypergeometric functions correspond in fact to *classical modular forms* because they can be rewritten [38] as $\mathcal{A} \cdot {}_2F_1([1/12, 5/12], [1], p(x))$ where the pullback $p(x)$ is in general more involved than simple rational pullbacks, being often *algebraic* functions. Using the globally bounded nome condition of Appendix A, we looked for all possible ${}_2F_1$ hypergeometric functions related§ to pullbacked ${}_2F_1([1/12, 5/12], [1], x)$ (see (A.7)). We give here a *list* of 28 hypergeometric functions that have series with

¶ Any ${}_2F_1(a,\, b], [c],\, x)$ with $c = 1$ is globally bounded since it is of *weight* zero: it is of the form ${}_nF_{n-1}$, and has c given by an integer and not a fractional number.
§ See [1, 2], and the hypergeometric functions in the previous sections in this paper.

integer coefficients, that are related to *modular forms*.

$$
\begin{aligned}
&{}_2F_1\Big([\frac{1}{2},\, \frac{1}{2}],\, [1],\, 16\, x\Big), \quad {}_2F_1\Big([\frac{1}{2},\, \frac{1}{3}],\, [1],\, 36\, x\Big), \quad {}_2F_1\Big([\frac{1}{3},\, \frac{1}{3}],\, [1],\, 27\, x\Big), \\
&{}_2F_1\Big([\frac{1}{3},\, \frac{2}{3}],\, [1],\, 27\, x\Big), \quad {}_2F_1\Big([\frac{1}{6},\, \frac{1}{2}],\, [1],\, 432\, x\Big), \quad {}_2F_1\Big([\frac{1}{6},\, \frac{1}{3}],\, [1],\, 108\, x\Big), \\
&{}_2F_1\Big([\frac{1}{6},\, \frac{2}{3}],\, [1],\, 108\, x\Big), \quad {}_2F_1\Big([\frac{1}{6},\, \frac{1}{6}],\, [1],\, 432\, x\Big), \quad {}_2F_1\Big([\frac{1}{6},\, \frac{5}{6}],\, [1],\, 432\, x\Big), \\
&{}_2F_1\Big([\frac{1}{4},\, \frac{1}{4}],\, [1],\, 64\, x\Big), \quad {}_2F_1\Big([\frac{1}{4},\, \frac{1}{2}],\, [1],\, 32\, x\Big), \quad {}_2F_1\Big([\frac{1}{4},\, \frac{3}{4}],\, [1],\, 64\, x\Big), \\
&{}_2F_1\Big([\frac{1}{8},\, \frac{3}{8}],\, [1],\, 256\, x\Big), \quad {}_2F_1\Big([\frac{1}{8},\, \frac{5}{8}],\, [1],\, 256\, x\Big), \quad {}_2F_1\Big([\frac{3}{8},\, \frac{7}{8}],\, [1],\, 256\, x\Big), \\
&{}_2F_1\Big([\frac{2}{3},\, \frac{5}{6}],\, [1],\, 108\, x\Big), \quad {}_2F_1\Big([\frac{1}{3},\, \frac{5}{6}],\, [1],\, 108\, x\Big), \quad {}_2F_1\Big([\frac{1}{2},\, \frac{3}{4}],\, [1],\, 32\, x\Big), \\
&{}_2F_1\Big([\frac{3}{4},\, \frac{3}{4}],\, [1],\, 64\, x\Big), \quad {}_2F_1\Big([\frac{5}{8},\, \frac{7}{8}],\, [1],\, 256\, x\Big), \quad {}_2F_1\Big([\frac{2}{3},\, \frac{2}{3}],\, [1],\, 27\, x\Big), \\
&{}_2F_1\Big([\frac{5}{6},\, \frac{5}{6}],\, [1],\, 432\, x\Big), \quad {}_2F_1\Big([\frac{1}{2},\, \frac{5}{6}],\, [1],\, 144\, x\Big), \quad {}_2F_1\Big([\frac{1}{2},\, \frac{2}{3}],\, [1],\, 36\, x\Big) \\
&{}_2F_1\Big([\frac{1}{12},\, \frac{7}{12}],\, [1],\, 1728\, x\Big), \qquad {}_2F_1\Big([\frac{1}{12},\, \frac{5}{12}],\, [1],\, 1728\, x\Big), \qquad\qquad \text{(B.1)} \\
&{}_2F_1\Big([\frac{5}{12},\, \frac{11}{12}],\, [1],\, 1728\, x\Big), \qquad {}_2F_1\Big([\frac{7}{12},\, \frac{11}{12}],\, [1],\, 1728\, x\Big).
\end{aligned}
$$

Using this globally bounded condition of the nome criterion, we wrote a program that went through all the values of a and b in $[-1,\, 1]$ (with small increments like 1/200), with $c\,=\,1$, singling out the ${}_2F_1$ hypergeometric functions that have series with integer coefficients (or more generally globally bounded), both for the ${}_2F_1$ hypergeometric functions, and *for the nome*. Running this program returned to us exactly the ${}_2F_1$ hypergeometric functions in the above list (B.1).

Appendix C. Heun functions solutions of telescopers of rational functions related to Shimura curves

The rational function in four variables

$$
R(x,\, y,\, z,\, u) \;=\; \frac{x\, y\, z\, u}{uxy\, +uxz\, +uyz\, -xyz\, +ux^2yz\, +uxy^2z\, +uxyz^2\, +ux^2y^2z^2}, \qquad \text{(C.1)}
$$

has a telescoper that is a linear differential operator of order three which actually corresponds to the *symmetric square* of a linear differential operator of order two. The solutions of this operator of order two are expressed in terms of the following Heun functions:

$$
x^{3/8} \cdot Heun\Big(4,\, \frac{49}{64},\, \frac{3}{8},\, \frac{7}{8},\, \frac{5}{4},\, \frac{1}{2},\, x\Big), \qquad x^{1/8} \cdot Heun\Big(4,\, \frac{9}{64},\, \frac{1}{8},\, \frac{5}{8},\, \frac{3}{4},\, \frac{1}{2},\, x\Big),
$$

or equivalently, the pullbacked ${}_2F_1$ solutions:

$$
\begin{aligned}
&x^{3/8} \cdot (1\,-x)^{-7/8} \cdot {}_2F_1\Big([\frac{7}{24},\, \frac{11}{24}],\, [\frac{5}{4}],\, \frac{-27}{4} \cdot \frac{x}{(1\,-x)^3}\Big), \qquad \text{(C.2)} \\
&x^{1/8} \cdot (1\,-x)^{-1/8} \cdot {}_2F_1\Big([\frac{1}{24},\, \frac{5}{24}],\, [\frac{3}{4}],\, \frac{-27}{4} \cdot \frac{x}{(1\,-x)^3}\Big).
\end{aligned}
$$

similar to (114) but with a different pullback. One recovers the same "Shimura" ${}_2F_1$ hypergeometric function as the one in (114), but with a *different* pullback.

Like the pullback in (114), this last pullback $\frac{-27}{4} \cdot \frac{x}{(1-x)^3}$ is "special" as can be seen in Appendix C.1 with equations (C.5) and (C.6).

Appendix C.1. The pullbacks in ${}_2F_1([\frac{1}{24}, \frac{7}{24}], [\frac{5}{6}], x)$ and ${}_2F_1([\frac{5}{24}, \frac{11}{24}], [\frac{7}{6}], x)$ are special.

Like all the *Belyi coverings* [39], the pullback $\frac{27}{4} \cdot \frac{x^2}{(1-x)^3}$ in (114) is "special". It has already been seen to occur in another framework [40], namely† the walk in a Weyl chamber of the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$. It actually occurs in the well-known "kernel equation" for that particular walk described in [40]

$$G(x, y) \, + G(0, 0) \, = \, G(x, 0) \, + G(0, y), \tag{C.3}$$

$$\text{where:} \qquad G(x, y) \, = \, L(x, y) \cdot H(x, y), \tag{C.4}$$

and where the generating function $H(x, y)$ of the walk and the kernel of the walk $L(x, y)$, read respectively:

$$H(x, y) \, = \, \frac{1 - x\,y}{(1-x)^3 \cdot (1-y)^3}, \qquad L(x, y) \, = \, \frac{27}{4} \cdot (y \, + \, x\,y^2 \, + x^2 \, - 3\,x\,y).$$

Noticeably, $G(x, y)$ is the sum of the particular rational function pullback $w(x) = \frac{27}{4} \cdot \frac{x^2}{(1-x)^3}$ and of another rational function of y:

$$G(x, y) \, = \, \frac{27}{4} \cdot \frac{x^2}{(1-x)^3} \, + \frac{27}{4} \cdot \frac{y}{(1-y)^3}. \tag{C.5}$$

Note that this additional rational function of y corresponds to the duality $x \leftrightarrow 1/x$:

$$G(x, y) \, = \, L(x, y) \cdot H(x, y) \, = \, w(x) \, - w\Big(\frac{1}{y}\Big). \tag{C.6}$$

Appendix C.2. Identities on Shimura ${}_2F_1$ hypergeometric functions and modular equations.

There exists *an algebraic series* $y(x)$ such that the two hypergeometric (114) (or (C.2)) *verify the two following identities*:

$$\begin{aligned} & w^{3/8} \cdot \rho \cdot y'(x)^{1/2} \cdot x^{3/8} \cdot (1-x)^{1/4} \cdot {}_2F_1\Big([\frac{1}{24}, \frac{5}{24}], [\frac{3}{4}], x\Big) \\ & \quad = \, y(x)^{3/8} \cdot (1 - y(x))^{1/4} \cdot {}_2F_1\Big([\frac{1}{24}, \frac{5}{24}], [\frac{3}{4}], y(x)\Big), \end{aligned} \tag{C.7}$$

and (with the *same* ρ and w)

$$\begin{aligned} & w^{5/8} \cdot \rho \cdot y'(x)^{1/2} \cdot x^{5/8} \cdot (1-x)^{1/4} \cdot {}_2F_1\Big([\frac{7}{24}, \frac{11}{24}], [\frac{5}{4}], x\Big) \\ & \quad = \, y(x)^{5/8} \cdot (1 - y(x))^{1/4} \cdot {}_2F_1\Big([\frac{7}{24}, \frac{11}{24}], [\frac{5}{4}], y(x)\Big), \end{aligned} \tag{C.8}$$

where the two complex constants ρ and w are given by $\rho \, = \, (7 \, - \, 24\,i)/25$ and $w \, = \, 1/\rho^2$. These two complex numbers w and ρ *are on the unit circle* $|w| \, = \, |\rho| \, = \, 1$

† See equation $w(x) \, = \, \frac{27}{4} \cdot \frac{x^2}{(1-x)^3}$, page 3165 in [40].

but are not N-th root of unity. The algebraic series $y(x)$ is given by the symmetric genus zero *modular equation of level five* $P(x, y) \, = \, 0$ which is parametrised by:

$$x \, = \, x(v) \, \, = \, \, -\frac{225\, v^2 + 18\, v + 1}{1350000 \cdot \, v^6}, \qquad y \, = \, y(v) \, \, = \, \, x \Bigl(\frac{11\, v + 2}{252\, v - 11} \Bigr) \qquad \text{(C.9)}$$

The algebraic series $y(x)$ in (C.7) or (C.8), given by the *modular equation* of level five $P(x, y) \, = \, 0$ reads:

$$y(x) = \, w \cdot \, x \, + \Bigl(\frac{172937\, w}{168750} + \frac{103}{270} \Bigr) \cdot \, x^2 \, + \Bigl(\frac{338124694601\, w}{398671875000} + \frac{270081319}{637875000} \Bigr) \cdot \, x^3 + \, \cdots \, .$$

References

[1] A. Bostan, S. Boukraa, G. Christol, S. Hassani, J-M. Maillard, *Ising n-fold integrals as diagonals of rational functions and integrality of series expansions* J. Phys. **A 46**: Math. Theor. 185202, 44 pages, (2013) `http://arxiv.org/abs/1211.6645v2`

[2] A. Bostan, S. Boukraa, G. Christol, S. Hassani and J-M. Maillard *Ising n-fold integrals as diagonals of rational functions and integrality of series expansions: integrality versus modularity* `http://arxiv.org/abs/1211.6031`

[3] A. Bostan, S. Boukraa, J-M. Maillard, J-A. Weil, *Diagonal of rational functions and selected differential Galois groups* J. Phys. **A 48**: Math. Theor. 504001, 29 pages, (2015) arXiv:1507.03227v2 [math-ph]

[4] Y. Abdelaziz, S. Boukraa, C. Koutschan, J-M. Maillard, *Diagonals of rational functions, pullbacked $_2F_1$ hypergeometric functions and modular forms (unabridged version)*, (2018) `arXiv:1805.04711v1`

[5] Y. Abdelaziz, S. Boukraa, C. Koutschan, J-M. Maillard, *Diagonals of rational functions, pullbacked $_2F_1$ hypergeometric functions and modular forms*, J. Phys. **A 51**: Math. Theor. 455201, 30 pages, (2018)

[6] C. Koutschan, HolonomicFunctions Package version 1.7.1 (09-Oct-2013), Copyright 2007-2013, Research Institute for Symbolic Computation (RISC), Johannes Kepler University, Linz, Austria

[7] R. Vidunas, *Algebraic transformations of Gauss hypergeometric functions*, Funkcialaj Ekvacioj, 52, pp. 139-180, (2009) `arXiv:math/0408269`

[8] K. Takemura, *The Heun equation and the Calogero-Moser-Sutherland system I: the Bethe Ansatz method*, Comm. Math. Phys. **258**, Issue 2, pages 367-403, (2001) `arXiv: math/0103077v2`

[9] G. Valent, *Heun functions versus elliptic functions*, "Difference Equations, Special Functions and Orthogonal Polynomials", Proceedings of the International Conference, Munich, Germany, 25-30 July 2005, World Scientific `arXiv:math-ph/0512006v1`

[10] A. O. Smirnov, *Elliptic solitons and Heun equations* "The Kowalevski Property", Vadim B. Kuznetsov editor, CRM Proceedings and Lecture Notes, (2002) `arXiv:math/0109149v2`

[11] M. L. Glasser and A. J. Guttmann, *Lattice Green function (at* 0*) for the* 4*D hypercubic lattice*, *J. Phys. A* **27** 7011–7014, (1994) `http://arxiv.org/abs/cond-mat/9408097`

[12] A. Malmendier *The eigenvalue equation on the Eguchi-Hanson space* J. Math. Phys., 44:9, 4308-4343, (2003) `arXiv:math/0210081`

[13] M. Hortaçsu, *Heun Functions and Some of Their Applications in Physics* Advances in High Energy Physics, vol. 2018, Article ID 8621573, 14 pages, (2018) `arXiv:1101.0471`

[14] R. S. Maier, On reducing the Heun equation to the hypergeometric equation, Journal of Differential Equations **213**, Issue 1, pp. 171-203, 1 June 2005 `arXiv:math/0203264v4`

[15] A. Ronveaux (1995) *Factorization of Heun's differential operator*, Applied Mathematics and Computation **141**, pp. 177-184 (2003).

[16] M. N. Hounkonnou, A. Ronveaux *Generalized Heun and Lamé equations: factorization*, Commun. Math. Anal., **11**, Number 1 (2011), pp. 121-136 arXiv: 0902.2991v1

[17] N. D. Elkies, *Shimura curve computations via K3 surfaces of Néron-Severi rank at least 19*, Algorithmic Number Theory: 8th International Symposium, ANTS-VIII Banff, May 2008, Proceedings (pp. 196-2111) `arXiv:0802.1301v1`

[18] J. Voight, *Shimura curves of genus at most two*, Math. Comp. 78 pp 1155-1172, (2009).

[19] J. Voight, *Three lectures on Shimura curves*, 16th april (2006).

[20] T. Shaska, *Genus 2 fields with degree 3 elliptic subfields*, Forum Math. **16**, pp. 263-280, (2004) `arXiv:math/0109155`

[21] E. Hallouin, *Computation of a cover of Shimura curves using a Hurwitz space*, November 16, Journal of Algebra, Elsevier, 2009, 351 (2), pp.558-566, (2013)

[22] A. Kurihara, *On some examples of equations defining Shimura curves and the Mumford uniformization*, J. Fac. Sci. Univ. Tokyo **25**, pp. 277-301, (1979)

[23] R. Vidunas, G. Filipuk, *Parametric transformations between the Heun and Gauss hypergeometric functions*, Funkcialaj Ekvacioj, 56, pp. 271-321, (2013) `arXiv:0910.3087v2`

[24] G. S. Joyce, *On the Simple Cubic Lattice Green Function*, Phil. Trans. of the Royal Soc. of London, Series A, Math. and Phys. Sciences, **273**, (1973) pp. 583-610.

[25] C. F. Doran, A. Malmendier *Calabi-Yau Manifolds Realizing Symplectically Rigid Monodromy Tuples*, (2019) `arXiv:1503.07500v4`

[26] C. Peters and J. Stienstra, *A pencil of K3-surfaces related to Apéry's recurrence for* $\zeta(3)$ *and Fermi surfaces for potential zero*, 1989, In book: Arithmetic of Complex Manifolds, Proc. Erlangen 1988, Edition: Lect. Notes Math. 1399, Publisher: Springer Verlag, Editors: Barth, Lange, pp.110-127

[27] http://www.unilim.fr/pages_perso/jacques-arthur.weil/diagonals/ and http://www.unilim.fr/pages_perso/jacques-arthur.weil/diagonals/4var/4var_ORDER_3.txt

[28] R. S. Maier, *On rationally parametrized modular equations* J. Ramanujan Math. Soc., **24**, pp. 1-73, (2009) `arXiv:math/0611041v4`

[29] Yifan Yang, *Schwarzian differential equations and Hecke eigenforms on Shimura curves* `arXiv:1110.6284v1`

[30] K. Takeuchi, *Commensurability classes of arithmetic triangle groups*, J. Fac. Science Univ. Tokyo Sec. IA. Math. **24**, pp. 201-212., (1977)

[31] Fang-Ting Tu, *Algebraic Transformations of hypergeometric functions arising from theory of Shimura curves*, RIMS Kôkyûroku Bessatsu, **B 44**, pp.223-245 (2013).

[32] Fang-Ting Tu, Yifan Yang, *Algebraic Transformations of Hypergeometric Functions and automorphic Forms on Shimura Curves*, Transactions of the American Mathematical Society, **365**, Number 12, (2013), pp. 6697-6729 `arXiv:1112.1001v1`

[33] R. Vidunas, G. Filipuk, *A Classification of coverings yielding Heun-to-hypergeometric Reductions*, Osaka J. Math., **51**, Number 4, pp. 867-905, (2014) `arXiv:1204.2730v1`

[34] M. van Hoeij, R. Vidunas, *Belyi functions for hyperbolic Hypergeometric-to-Heun transformations*, Journal of Algebra, Volume **441**, 1 November 2015, 609-659, (2015) `arXiv:1212.3803v3`

[35] N. D. Elkies, *Shimura curve computations* In: Buhler J.P. (eds) Algorithmic Number Theory. ANTS 1998. Lecture Notes in Computer Science, vol 1423. Springer, Berlin, Heidelberg `arXiv:math/0005160v1`

[36] M. Kontsevich, D. Zagier, *Periods*, IHES/M/01/22 2001, https://www.maths.ed.ac.uk/~v1ranick/papers/kontzagi.pdf

[37] D. Zagier, *Arithmetic and Topology of Differential Equations*, https://people.mpim-bonn.mpg.de/zagier/files/tex/ECMHirzebruchLecture/ECMHirzebruchLectureCorrected.pdf

[38] Y. Abdelaziz, J-M. Maillard, *Modular forms, Schwarzian conditions, and symmetries of differential equations in physics*, J. Phys. **A 49**: Math. Theor. 215203 (44 pages), (2016) `arXiv:1611.08493v3` [math-ph]

[39] M. Musty, S. Shiavone, J. Slisling and J. Voight, *A database of Belyi maps*, Proceedings of the Thirteenth Algorithmic Number Theory Symposium (2019) `arXiv:1805.07751v3`

[40] K. Raschel, *Random walks in the quarter plane, discrete harmonic functions and conformal mappings*, Stochastic Processes and their Applications **124** pp. 3147-3178, (2014)