

Article

# DIAGONALS OF RATIONAL FUNCTIONS: FROM DIFFERENTIAL ALGEBRA TO EFFECTIVE ALGEBRAIC GEOMETRY

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**Abstract:** We show that the results we had previously obtained on diagonals of nine- and ten-parameter families of rational functions in three variables  $x, y, z$ , using creative telescoping, yielding modular forms expressed as pullbacked  ${}_2F_1$  hypergeometric functions, can be obtained, much more efficiently, by calculating the  $j$ -invariant of an elliptic curve canonically associated with the denominator of the rational functions. These results can be drastically generalized by changing the parameters into arbitrary rational functions of the product  $p = xyz$ . In other cases where creative telescoping yields pullbacked  ${}_2F_1$  hypergeometric functions, we extend this algebraic geometry approach to other families of rational functions in three or more variables. In particular, we generalize this approach to rational functions in more than three variables when the denominator can be associated to an algebraic variety corresponding to products of elliptic curves, or foliations in elliptic curves. We also extend this approach to rational functions in three variables when the denominator is associated with a genus-two curve such that its Jacobian is a split Jacobian, corresponding to the product of two elliptic curves. We sketch the situation where the denominator of the rational function is associated with algebraic varieties that are not of the general type, having an infinite set of birational automorphisms. We finally provide some examples of rational functions in more than three variables, where the telescoper has pullbacked  ${}_2F_1$  hypergeometric solutions, because the denominator corresponds to an algebraic variety that has a selected elliptic curve.

**Keywords:** Diagonal of a rational function, pullbacked hypergeometric function, modular form, Hauptmodul, creative telescoping, telescoper, elliptic curve,  $j$ -invariant, K3 surface, split Jacobian, extremal rational surface, birational automorphism, algebraic variety of general type.

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## 1. Introduction

In previous papers [1,2], using creative telescoping [3], we have obtained *diagonals*<sup>1</sup> of nine- and ten-parameter families of rational functions, given by (classical) modular forms expressed as pullbacked  ${}_2F_1$  hypergeometric functions [12]. The natural emergence

<sup>1</sup> For the introduction of the concept of diagonals of rational functions, see [4–11].

of diagonals of rational functions<sup>2</sup> in lattice statistical mechanics is explained in [19,20]. This can be seen as the reason of the frequent occurrence of *modular forms* and *Calabi-Yau operators* in lattice statistical mechanics [21–27]. In other previous papers [17,18], dedicated to Heun functions that are either diagonals of simple rational functions, or only solutions of *telescopers* [28,29] of simple rational functions of three or four variables, we have obtained many order-three telescopers having squares of Heun functions as solutions that turn out to be squares of pullbacked  ${}_2F_1$  hypergeometric functions corresponding to *classical modular forms* and even *Shimura automorphic forms* [30,31], strongly reminiscent of periods of *extremal rational surfaces* [32,33], and other foliations of K3 surfaces in elliptic curves. In other words, one finds experimentally that the  ${}_2F_1$  hypergeometric functions emerging in the calculation of diagonals of rational functions, or of solutions of the telescopers of rational functions, seem to be only special  ${}_2F_1(a, b; c; x)$  hypergeometric functions with a selected set of parameters  $a, b, c$  (see the list (B.1) in Appendix B of [17], corresponding to classical modular forms<sup>3</sup>, together with a finite set of parameters, like  $7/24, 11/24, 5/4$ , corresponding to Shimura automorphic forms [30,31]), pullbacked by selected pullbacks. This last paper [17] also underlined the difference between the diagonal of a rational function  $Diag(R)$ , and the solutions of the telescoper of the same rational function.

These results strongly suggested to find an algebraic geometry interpretation for all these exact results, and, more generally, suggested to provide an alternative algebraic geometry approach of the results emerging from creative telescoping<sup>4</sup>.

This is the purpose of the present paper. In particular, we are going to show that most of these pullbacked  ${}_2F_1$  hypergeometric functions can be obtained efficiently through algebraic geometry calculations, thus providing a more intrinsic algebraic geometry interpretation of the creative telescoping calculations which are typically differential algebra calculations [28,29,34,35].

Creative telescoping [28,29,34,36] is a methodology to deal with parametrized symbolic sums and integrals that yields differential/recurrence equations for such expressions. This methodology became popular in computer algebra in the past twenty-five years. By “telescoper” of a rational function, say  $R(x, y, z)$ , we here refer to the output of the creative telescoping program [3], applied to the *transformed* rational function  $\tilde{R} = R(x/y, y/z, z)/(yz)$ . Such a telescoper is a linear differential operator  $T$  in  $x$  and  $\frac{\partial}{\partial x}$ , such that  $T + \frac{\partial}{\partial y} \cdot U + \frac{\partial}{\partial z} \cdot V$  annihilates  $\tilde{R}$ , where the so-called “certificates”  $U, V$  are rational functions in  $x, y, z$ . In other words, the telescoper  $T$  represents a linear ODE that is satisfied by  $Diag(R)$ .

The paper is essentially dedicated to solutions of telescopers of rational functions which are *not necessarily* diagonals of rational functions. These solutions correspond to *periods* [37] of algebraic varieties over some cycles which are not necessarily vanishing [38] cycles<sup>5</sup> like in the case of diagonals of rational functions. The reader interested in the connection between the process of taking diagonals, calculating telescopers, and the notion of periods, de Rham cohomology (i.e. differential forms) and other Picard-Fuchs equations can read in detail the thesis of Pierre Lairez [35] (see also [41]). We just sketch some of these ideas in Appendix A.

The purpose of this paper is not to give an introduction to creative telescoping [28,29], but to provide many pedagogical (non-trivial) examples of telescopers using<sup>6</sup> extensively the `HolonomicFunctions` Mathematica package [3].

The paper is organized as follows. We first recall in section 2 the exact results of [1,2] for nine- and ten-parameter families of rational functions using creative telescoping, yielding

<sup>2</sup> The lattice Green functions are the simplest examples of such diagonals of rational functions [13–18].

<sup>3</sup> See Felix Klein’s connection of the  ${}_2F_1(1/12, 5/12; 1; x)$  Gauss hypergeometric function with modular forms, for instance in the very pedagogical and heuristic paper [12].

<sup>4</sup> The reader may refer to [34] for an extensive survey of “creative telescoping” approaches.

<sup>5</sup> In french “cycles évanescents” [39,40].

<sup>6</sup> One can obtain these telescopers using Chyzak’s algorithm [42] or Koutschan’s semi-algorithm [3,43] (the termination is not proven). For the examples displayed in this paper, Koutschan’s package [3] is more efficient.

modular forms expressed as pullbacked  ${}_2F_1$  hypergeometric functions. We show in section 3 that these exact results can be obtained, much more efficiently, by calculating the  $j$ -invariant of an elliptic curve that is canonically associated with the denominator of the rational function, and we underline the fact that one can drastically generalize these results, the parameters becoming quite arbitrary rational functions. Section 4 generalizes the previous calculations to denominators of the rational functions of more than three variables, corresponding to products (or foliations) of elliptic curves. In section 5 we show how modular forms expressed as pullbacked  ${}_2F_1$  hypergeometric functions occur for rational functions in three variables when the denominator is associated with a genus-two curve such that its Jacobian is a split Jacobian corresponding to the product of two elliptic curves. In section 6 we sketch the situation where the denominator of the rational function is associated with algebraic varieties of low Kodaira dimension, having an infinite set of birational automorphisms. We finally provide some examples of rational functions in more than three variables, where the telescopers have pullbacked  ${}_2F_1$  hypergeometric solutions, the denominator corresponding to an algebraic variety having a selected elliptic curve in the variety explaining these pullbacked  ${}_2F_1$  solutions.

## 2. Classical modular forms and diagonals of nine- and ten-parameter families of rational functions

In previous papers [1,2], using creative telescoping [3], we have obtained diagonals of nine- and ten-parameter families of rational functions, given by (classical) modular forms expressed as pullbacked  ${}_2F_1$  hypergeometric functions. Let us recall these results.

### 2.1. Nine-parameter rational functions giving pullbacked ${}_2F_1$ hypergeometric functions for their diagonals

Let us recall the nine-parameter rational function in three variables  $x$ ,  $y$  and  $z$ :

$$\frac{1}{a + b_1 x + b_2 y + b_3 z + c_1 yz + c_2 xz + c_3 xy + d y^2 z + e z x^2}. \tag{1}$$

Calculating<sup>7</sup> the telescoper of this rational function (1), one gets an order-two linear differential operator annihilating the diagonal of the rational function (1). The diagonal of the rational function (1) can be written [1,2] as a pullbacked hypergeometric function

$$\frac{1}{P_4(x)^{1/4}} \cdot {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; 1 - \frac{P_6(x)^2}{P_4(x)^3}\right), \tag{2}$$

where  $P_4(x)$  and  $P_6(x)$  are two polynomials of degree four and six in  $x$ , respectively. The Hauptmodul pullback in (2) has the form

$$\mathcal{H} = \frac{1728}{j} = 1 - \frac{P_6(x)^2}{P_4(x)^3} = \frac{1728 \cdot x^3 \cdot P_8(x)}{P_4(x)^3}, \tag{3}$$

where  $j$  is the  $j$ -invariant and  $P_8(x)$  is a polynomial of degree eight in  $x$ . Such a pullbacked  ${}_2F_1$  hypergeometric function (2) corresponds to a classical modular form [1,2].

### 2.2. Ten-parameter rational functions giving pullbacked ${}_2F_1$ hypergeometric functions for their diagonals.

Let us recall the ten-parameter rational function in three variables  $x$ ,  $y$  and  $z$ :

$$R(x, y, z) = \frac{1}{a + b_1 x + b_2 y + b_3 z + c_1 yz + c_2 xz + c_3 xy + d_1 x^2 y + d_2 y^2 z + d_3 z^2 x}. \tag{4}$$

<sup>7</sup> Using the HolonomicFunctions Mathematica package [3].

Calculating the telescoper of this rational function (4), one gets an order-two linear differential operator annihilating the diagonal of the rational function (4). The diagonal of the rational function (4) can be written [1,2] as a pullbacked hypergeometric function

$$\frac{1}{P_3(x)^{1/4}} \cdot {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; 1 - \frac{P_6(x)^2}{P_3(x)^3}\right), \tag{5}$$

where  $P_3(x)$  and  $P_6(x)$  are two polynomials of degree three and six in  $x$ , respectively. Furthermore, the Hauptmodul pullback in (5) is seen to be of the form:

$$\mathcal{H} = \frac{1728}{j} = 1 - \frac{P_6(x)^2}{P_3(x)^3} = \frac{1728 \cdot x^3 \cdot P_9(x)}{P_3(x)^3}. \tag{6}$$

where  $P_9(x)$  is a polynomial of degree nine in  $x$ . Again, (5) corresponds to a classical modular form [1,2].

### 3. Deducing creative telescoping results from effective algebraic geometry

Obtaining the previous pullbacked hypergeometric results (2) and (5) required [1,2] an accumulation of creative telescoping calculations, and a lot of “guessing” using all the symmetries of the diagonals of these rational functions (1) and (4). We are looking for a more efficient and intrinsic way of obtaining these exact results. These two pullbacked hypergeometric results (2) and (5), are essentially “encoded” by their *Hauptmodul* pullbacks (3) and (6), or, equivalently, their corresponding *j*-invariants. The interesting question, which will be addressed in this paper, is whether it is possible to canonically associate elliptic curves whose *j*-invariants correspond precisely to these Hauptmoduls  $\mathcal{H} = \frac{1728}{j}$ .

#### 3.1. Revisiting the pullbacked hypergeometric results in an algebraic geometry perspective.

One expects such an elliptic curve to correspond to the singular part of the rational function, namely the denominator of the rational function. Let us recall that the diagonal of a rational function in (for example) three variables is obtained through its multi-Taylor expansion [19,20]

$$R(x, y, z) = \sum_m \sum_n \sum_l a_{m,n,l} \cdot x^m y^n z^l, \tag{7}$$

by extracting the “diagonal” terms, i.e. the powers of the product  $p = xyz$ :

$$Diag\left(R(x, y, z)\right) = \sum_m a_{m,m,m} \cdot x^m. \tag{8}$$

Consequently, it is natural to consider the algebraic curve corresponding to the intersection of the surface defined by the vanishing condition  $D(x, y, z) = 0$  of the denominator  $D(x, y, z)$  of these rational functions (1) and (4), with the hyperbola  $p = xyz$  (where  $p$  is seen, here, as a constant). This amounts, for instance, to eliminating the variable  $z$ , substituting  $z = \frac{p}{xy}$  in  $D(x, y, z) = 0$ .

##### 3.1.1. Nine-parameter case:

In the case of the rational functions (1) this corresponds to the (planar) algebraic curve

$$a + b_1 x + b_2 y + b_3 \frac{p}{xy} + c_1 y \frac{p}{xy} + c_2 x \frac{p}{xy} + c_3 xy + d y^2 \frac{p}{xy} + e \frac{p}{xy} x^2 = 0, \tag{9}$$

which can be rewritten as a (general, nine-parameter) *biquadratic*:

$$a x y + b_1 x^2 y + b_2 x y^2 + b_3 p + c_1 p y + c_2 p x + c_3 x^2 y^2 + d p y^2 + e p x^2 = 0. \quad (10)$$

Using formal calculations<sup>8</sup> one can easily calculate the genus of the planar algebraic curve (10), and find that it is actually an elliptic curve (of genus one). Furthermore, one can (almost instantaneously) find the exact expression of the  $j$ -invariant of this elliptic curve as a rational function of the nine parameters  $a, b_1, b_2, \dots, e$  in (1). One actually finds that this  $j$ -invariant is *precisely* the  $j$  such that the Hauptmodul  $\mathcal{H} = \frac{1728}{j}$  is the exact expression (3). In other words, the classical modular form result (2) could have been obtained, almost instantaneously, by calculating the  $j$ -invariant of an elliptic curve canonically associated with the denominator of the rational function (1). The algebraic planar curve (10) corresponds to the most general biquadratic of two variables, which depends on nine homogeneous parameters. Such a general biquadratic is known to be an elliptic curve for *generic values* of the nine parameters<sup>9</sup>.

Thus, the nine-parameter exact result (2) can be seen as a simple consequence of the fact that the most general nine-parameter biquadratic is an elliptic curve.

### 3.1.2. Ten-parameter case:

In the case of the rational function (4), substituting  $z = \frac{p}{xy}$  in  $D(x, y, z) = 0$ , one obtains the ten-parameter bicubic:

$$a x y^2 + b_1 x^2 y^2 + b_2 x y^3 + b_3 p y + c_1 p y^2 + c_2 p x y + c_3 x^2 y^3 + d_1 x^3 y^3 + d_2 y^3 + d_3 p^2 = 0. \quad (11)$$

As before, we find that this planar algebraic curve is actually an elliptic curve<sup>10</sup> and that the exact expression of its  $j$ -invariant is precisely the  $j$  of the Hauptmodul  $H = 1728/j$  in (6).

Thus, this ten-parameter result (5) can again be seen as a simple consequence of the fact that there exists a family of ten-parameter bicubics (see (11)) which are elliptic curves for generic values of the ten parameters.

These preliminary calculations are a strong incentive to try to replace the differential algebra calculations of creative telescoping, by more intrinsic algebraic geometry calculations, or, at least to perform effective algebraic geometry calculations to provide an algebraic geometry interpretation of the exact results obtained from creative telescoping.

### 3.2. Finding creative telescoping results from $j$ -invariant calculations.

One might think that these results are a consequence of the simplicity of the denominators of the rational functions (1) or (4), being associated with biquadratics or selected bicubics. In fact, these results are very general. Let us, for instance, consider a nine-parameter family of planar algebraic curves that are not biquadratics or (selected) bicubics:

$$a_1 x^4 + a_2 x^3 + a_3 x^2 + a_4 x + a_5 + a_6 x^2 y + a_7 y^2 + a_8 y + a_9 x y = 0. \quad (12)$$

One can easily calculate the genus of this planar curve and see that this genus is actually one for *arbitrary* values of the  $a_n$ 's. Thus the planar curve (12) is an elliptic curve for generic

<sup>8</sup> Namely using `with(algcurves)` in Maple, and, in particular, the command `j_invariant`.

<sup>9</sup> So many results in integrable models correspond to this most general biquadratic: the Bethe ansatz of the Baxter model [44,45], the elliptic curve foliating the sixteen-vertex model [45], so many QRT birational maps [46], etc.

<sup>10</sup> Generically, the most general planar bicubic is *not* a genus-one algebraic curve. It is a genus-four curve.

values of the nine parameters  $a_1, \dots, a_9$ . It is straightforward to see that the algebraic surface  $S(x, y, z) = 0$ , corresponding to

$$z \cdot (a_1 x^4 + a_2 x^3 + a_3 x^2 + a_4 x + a_5 + a_6 x^2 y + a_7 y^2 + a_8 y) + a_9 p = 0, \quad (13)$$

will automatically be such that its intersection with the hyperbola  $p = xyz$  gives back the elliptic curve (12).

Using this kind of “reverse engineering” suggests to consider the rational function in three variables  $x, y$  and  $z$

$$R(x, y, z) = \frac{1}{1 + z \cdot (a_1 x^4 + a_2 x^3 + a_3 x^2 + a_4 x + a_5 + a_6 x^2 y + a_7 y^2 + a_8 y)}, \quad (14)$$

which will be such that its denominator is canonically associated with an elliptic curve. Again we can immediately calculate the  $j$ -invariant of that elliptic curve. If one calculates the telescoper of this eight-parameter family of rational functions (14), one finds that this telescoper is an order-two linear differential operator with pullbacked hypergeometric solutions of the form

$$\mathcal{A}(x) \cdot {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; \mathcal{H}\right), \quad (15)$$

where  $\mathcal{A}(x)$  is an algebraic function and, where again, the pullback-Hauptmodul  $\mathcal{H} = 1728/j$ , precisely corresponds to the  $j$ -invariant of the elliptic curve.

More generally, seeking for planar elliptic curves, one can, for given values of two integers  $M$  and  $N$ , look for planar algebraic curves

$$\sum_{n=0}^{n=N} \sum_{m=0}^{m=M} a_{m,n} \cdot x^n y^m = 0, \quad (16)$$

defined by the set of  $a_{m,n}$ 's which are equal to zero, apart of  $\mathcal{N}$  homogeneous parameters  $a_{m,n}$  being, as in (10) or (11) or (13), *independent parameters*. Finding such an  $\mathcal{N}$ -parameter family of (planar) elliptic curves automatically provides an  $\mathcal{N}$ -parameter family of rational functions such that their telescopers have a pullbacked  ${}_2F_1$  hypergeometric solution which we can simply deduce from the  $j$ -invariant of that elliptic curve.

Recalling the results of section 2.2, the quite natural question to ask now is whether it is possible to find families of such (planar) elliptic curves which depend on more than ten independent parameters?

Before addressing this question, let us recall the concept of birationally equivalent elliptic curves. Let us consider the monomial transformation:

$$(x, y) \longrightarrow (x^M y^N, x^P y^Q), \quad (17)$$

where  $M, N, P, Q$  are integers such that  $M \cdot Q - P \cdot N = 1$ , then its compositional inverse is the monomial transformation:

$$(x, y) \longrightarrow \left(\frac{x^Q}{y^N}, \frac{y^M}{x^P}\right). \quad (18)$$

This monomial transformation (17) is thus a *birational*<sup>11</sup> transformation. A birational transformation transforms an elliptic curve, like (12), into another elliptic curve with the

<sup>11</sup> This transformation is rational and its compositional inverse is also rational (here monomial).

same  $j$ -invariant: these two elliptic curves are called birationally equivalent. In the case of the birational and monomial transformation (17), the elliptic curve (12) is changed into<sup>12</sup>:

$$a_1 \cdot x^{4M} y^{4N} + a_2 \cdot x^{3M} y^{3N} + a_3 \cdot x^{2M} y^{2N} + a_4 \cdot x^M y^N + a_5 + a_6 \cdot x^{2M+P} y^{2N+Q} + a_7 \cdot x^{2P} y^{2Q} + a_8 \cdot x^P y^Q + a_9 \cdot x^{M+P} y^{N+Q} = 0. \quad (19)$$

With this kind of birational monomial transformation (17), we see that one can obtain families of elliptic curves (19) of arbitrary large degrees in  $x$  and  $y$ . Consequently one can find nine- or ten-parameter families of rational functions of arbitrary large degrees yielding pullbacked  ${}_2F_1$  hypergeometric functions. There is no constraint on the degree of the planar algebraic curves (19): the only relevant question concerns the maximum number of (linearly) independent parameters in families of planar elliptic curves which is shown to be ten. The demonstration<sup>13</sup> is sketched in Appendix B.

### 3.3. Pullbacked ${}_2F_1$ functions for higher genus curves: monomial transformations.

Let us recall another important point. We have already remarked in [1,2] that once we have an exact result for a diagonal of a rational function of three variables  $R(x, y, z)$ , we immediately get another exact result for the diagonal of the rational function  $R(x^n, y^n, z^n)$  for any positive integer  $n$ . As a result we obtain a new expression for the diagonal changing  $x$  into  $x^n$ . In fact, this is also a result on the telescoper of the rational function  $R(x, y, z)$ : the telescoper of the rational function  $R(x^n, y^n, z^n)$  is the  $x \rightarrow x^n$  pullback of the telescoper of the rational function  $R(x, y, z)$ . Having a pullbacked  ${}_2F_1$  solution for the telescoper of the rational function  $R(x, y, z)$  (resp. the diagonal of the rational function  $R(x, y, z)$ ), we will immediately deduce a pullbacked  ${}_2F_1$  solution for the telescoper of the rational function  $R(x^n, y^n, z^n)$  (resp. the diagonal of the rational function  $R(x^n, y^n, z^n)$ ).

Along this line, let us change in the rational function (1),  $(x, y, z)$  into  $(x^2, y^2, z^2)$ :

$$R_2(x, y, z) = \frac{1}{a + b_1 x^2 + b_2 y^2 + b_3 z^2 + c_1 y^2 z^2 + c_2 x^2 z^2 + c_3 x^2 y^2 + d y^4 z^2 + e z^2 x^4}. \quad (20)$$

The diagonal of this new rational function (20) will be the pullbacked  ${}_2F_1$  exact expression (2) where we change  $x \rightarrow x^2$ . The intersection of the algebraic surface corresponding to the vanishing condition of the denominator of the new rational function (20), with the hyperbola  $p = xyz$  (i.e.  $z = \frac{p}{xy}$ ), is nothing but equation (10) where we have changed  $(x, y; p)$  into  $(x^2, y^2; p^2)$

$$a x^2 y^2 + b_1 x^4 y^2 + b_2 x^2 y^4 + b_3 p^2 + c_1 p^2 y^2 + c_2 p^2 x^2 + c_3 x^4 y^4 + d p^2 y^4 + e p^2 x^4 = 0, \quad (21)$$

which is *no longer*<sup>14</sup> an elliptic curve but a curve of genus 9.

With that example we see that classical modular form results, or pullbacked  ${}_2F_1$  exact expressions like (2), can actually emerge from higher genus curves like (21). As far as these diagonals, or telescopers, of rational function calculations are concerned, higher genus curves like (21) must in fact be seen as “almost” elliptic curves up to an  $x \rightarrow x^n$  covering.

Such results for monomial transformations like  $(x, y, z) \rightarrow (x^n, y^n, z^n)$  can, in fact, be generalized to more general (not birational<sup>15</sup>) monomial transformations. This is sketched in Appendix C.

<sup>12</sup> One can easily verify for particular values of the  $M, N, P, Q$  and  $a_k$ 's, using with(algcurves) in Maple, that the  $j$ -invariants of (12) and (19) are actually equal.

<sup>13</sup> We thank Josef Schicho for providing this demonstration.

<sup>14</sup> If we perform the same calculations with the ten-parameter rational function (4) we get an algebraic curve of genus 10 instead of 9.

<sup>15</sup> In contrast with transformations like (17).

### 3.4. Changing the parameters into functions of the product $p = xyz$ .

All these results for parametrized families of rational functions can be drastically generalized when one remarks that allowing any of these parameters to be a rational function of the product  $p = xyz$  also yields the previous pullbacked  ${}_2F_1$  exact expression, like (2), where the parameter is changed into that rational function of  $x$  (see [1]). Let us consider a simple (two-parameter) illustration of this general result. Let us consider a subcase of the previous nine- or ten-parameter families, introducing, for example, the two-parameter rational function:

$$\frac{1}{1 + 2x + b_2 \cdot y + 5yz + xz + c_3 \cdot xy}. \quad (22)$$

The diagonal of this rational function (22) is the pullbacked hypergeometric function:

$$\frac{1}{P_2(x)^{1/4}} \cdot {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; 43200 \cdot x^4 \cdot \frac{P_4(x)}{P_2(x)^3}\right), \quad (23)$$

where

$$P_2(x) = 1 - 8 \cdot (b_2 + 10) \cdot x + 8 \cdot (2b_2^2 - 20b_2 + 15c_3 + 200) \cdot x^2, \quad (24)$$

and

$$\begin{aligned} P_4(x) = & -675c_3^4 \cdot x^4 + 4c_3^2 \cdot (b_2 + 10) \cdot (4b_2^2 - 100b_2 + 45c_3 + 400) \cdot x^3 \\ & + (64b_2^4 - 32b_2^3c_3 - 8b_2^2c_3^2 - 1280b_2^3 + 1280b_2^2c_3 \\ & - 460b_2c_3^2 - 5c_3^3 + 6400b_2^2 - 3200b_2c_3 - 800c_3^2) \cdot x^2 \\ & - (b_2 + 10) \cdot (32b_2^2 - 16b_2c_3 - c_3^2) \cdot x + 2b_2 \cdot (2b_2 - c_3), \end{aligned} \quad (25)$$

Let us now consider the previous rational function (22) where the two parameters  $b_2$  and  $c_3$  become some rational functions of the product  $p = xyz$ , for instance:

$$b_2(p) = \frac{1 + 3p}{1 + 7p^2}, \quad c_3(p) = \frac{1 + p^2}{1 + 2p} \quad \text{where: } p = xyz. \quad (26)$$

The new corresponding rational function becomes more involved but one can easily calculate the telescoper of this new rational function of three variables  $x$ ,  $y$  and  $z$ , and find that it is, *again*, an order-two linear differential operator having the pullbacked hypergeometric solution (23) where  $b_2$  and  $c_3$  are, now, replaced by ( $p$  is now  $x$ ) the functions:

$$b_2(x) = \frac{1 + 3x}{1 + 7x^2}, \quad c_3(x) = \frac{1 + x^2}{1 + 2x}. \quad (27)$$

In that case, one gets a diagonal which is the pullbacked hypergeometric solution

$$\begin{aligned} & (1 + 2x)^{1/4} \cdot (1 + 7x^2)^{1/4} \cdot q_8^{-1/4} \\ & \times {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; \frac{43200 \cdot x^4 \cdot (1 + 7x^2)^2 \cdot q_{20}}{(1 + 2x) \cdot q_8^3}\right), \end{aligned} \quad (28)$$

where  $q_8$  and  $q_{20}$  are two polynomials with integer coefficients of degree eight and twenty in  $x$ . The exact expression (28) is nothing but (23) (with (24) and (25)) where  $b_2$  and  $c_3$  have been replaced by the rational functions (27). Similar calculations can be performed for more general rational functions (1) or (4), when all the (nine or ten) parameters are more involved rational functions.

When performing our creative telescoping symbolic calculations using the Holonomic-Functions package [3], such results may look quite impressive. From the algebraic geometry



viewpoint, it is almost tautological<sup>16</sup>, if one takes for granted the result of our previous subsections 3.1 and 3.2, namely that the pullbacked hypergeometric solution of the telescoper corresponds to the Hauptmodul  $1728/j$ , where  $j$  is the  $j$ -invariant of the elliptic curve corresponding to the intersection of the algebraic surface corresponding to the vanishing condition of the denominator, with the hyperbola  $p = xyz$ : this calculation of the  $j$ -invariant is performed for  $p$  fixed, and arbitrary (nine or ten) parameters  $a, b_1, \dots$ . It is clearly possible to force the parameters to be functions<sup>17</sup> of  $p$ , the  $j$ -invariant being changed accordingly. Of course, in that case, the parameters in the rational function are the same functions but of the product  $p = xyz$ .

One thus gets pullbacked hypergeometric solutions (classical modular forms) for an (unreasonably) large set of rational functions in three variables, namely the families of rational functions (1) or (4), but where, now, the nine or ten parameters are nine, or ten, totally arbitrary rational functions (with Taylor series expansions) of the product  $p = xyz$ .

We see experimentally that changing the parameters of the rational function into functions, actually works for diagonals of rational functions, as well as for solutions of telescopers of rational functions depending on parameters.

#### 4. Creative telescoping on rational functions of more than three variables associated with products or foliations of elliptic curves

Let us show that such an algebraic geometry approach to creative telescoping can be generalized to rational functions of *more than three* variables, when the vanishing condition of the denominator can be associated with products of elliptic curves, or more generally, algebraic varieties with foliations in elliptic curves.

- The telescoper of the rational function in the four variables  $x, y, z$  and  $w$

$$\frac{xyz}{(1+z)^2 - x \cdot (1-x) \cdot (x - xyzw) \cdot y \cdot (1-y) \cdot (y - xyzw)}, \quad (29)$$

gives an order-three *self-adjoint* linear differential operator which is, thus, the *symmetric square* of an order-two linear differential operator. The latter has the pullbacked hypergeometric solution:

$$\begin{aligned} \mathcal{S}_1(x) &= (1-x+x^2)^{-1/4} \cdot {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; \frac{27}{4} \cdot \frac{x^2 \cdot (1-x)^2}{(x^2-x+1)^3}\right) \\ &= {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right). \end{aligned} \quad (30)$$

In [18] we underlined the difference between the diagonal of a rational function and solutions of the telescoper of the same rational function. In this case, the diagonal of the rational function (29) is zero<sup>18</sup> and is thus different from the pullbacked hypergeometric solution (30), which is a “period” [37] of the algebraic variety corresponding to the denominator over some (*non-vanishing*<sup>19</sup>) cycle. From now on, we will have a similar situation in most of the following examples of this paper.

<sup>16</sup> An algebraic geometer will probably see this as a trivial remark: diagonalization is an algebraic procedure and nothing really happens to the coefficients. Therefore if one replaces the coefficients by anything else, one will find those replaced coefficients in the end result.

<sup>17</sup> The functions should be rational functions if one wants to stick with diagonals and telescopers of rational functions, but the result remains valid for algebraic functions, or even transcendental functions with reasonable Taylor series expansions at  $x = 0$ : for instance, for  ${}_2F_1$  hypergeometric functions, one gets a differentially algebraic function corresponding to the composition of  ${}_2F_1$  hypergeometric functions.

<sup>18</sup> The reason is that the integration takes place over a cycle homologically equivalent to the trivial cycle. The cycle becomes trivial after taking the limit  $p \rightarrow 0$ . Integrals over non-vanishing cycles usually give logarithms of  $p$ , like the second solution to the hypergeometric function  ${}_2F_1(1/2, 1/2; 1; x)$ .

<sup>19</sup> Diagonals of rational functions correspond to periods over vanishing cycles [38,40].

This example is a simple illustration of what we expect for products of elliptic curves, or algebraic varieties with foliations in elliptic curves. Introducing the product  $p = xyzw$ , the vanishing condition of the denominator of the rational function (29) yields the surface  $S(x, y, z) = 0$ :

$$(1+z)^2 - x \cdot (1-x) \cdot (x-p) \cdot y \cdot (1-y) \cdot (y-p) = 0. \quad (31)$$

For fixed  $p$  and fixed  $y$ , equation (31) can be seen as an algebraic curve

$$(1+z)^2 - \lambda \cdot x \cdot (1-x) \cdot (x-p) = 0 \quad (32)$$

with:  $\lambda = y \cdot (1-y) \cdot (y-p).$

For fixed  $p$  and fixed  $y$ ,  $\lambda$  can be considered as a constant, the algebraic curve (32) being an elliptic curve with an obvious Weierstrass form:

$$Z^2 - x \cdot (1-x) \cdot (x-p) = 0 \quad \text{where:} \quad Z = \frac{1+z}{\sqrt{\lambda}}. \quad (33)$$

The  $j$ -invariant of (32), or<sup>20</sup> (33), is well-known and yields the Hauptmodul  $\mathcal{H}$ :

$$\mathcal{H} = \frac{1728}{j} = \frac{27}{4} \cdot \frac{p^2 \cdot (1-p)^2}{(p^2 - p + 1)^3} \quad (34)$$

For fixed  $p$  and fixed  $x$ , equation (31) can be seen as an algebraic curve

$$(1+z)^2 - \mu \cdot y \cdot (1-y) \cdot (y-p) = 0 \quad (35)$$

for:  $\mu = x \cdot (1-x) \cdot (x-p),$

which is also an elliptic curve with an obvious Weierstrass form and the *same* Hauptmodul (34). This Hauptmodul is precisely the one occurring in the pullbacked hypergeometric solution (30).

More generally, the rational function of the four variables  $x, y, z$  and  $w$

$$\frac{xyz}{(1+z)^2 - x \cdot (1-x) \cdot (x - R_1(p)) \cdot y \cdot (1-y) \cdot (y - R_2(p))}, \quad (36)$$

where  $p = xyzw$ , and where  $R_1(p)$  and  $R_2(p)$  are two arbitrary rational functions (with Taylor series expansions) of the product  $p = xyzw$ , yields a telescoper which has an order-four linear differential operator which is the symmetric product<sup>21</sup> of two order-two linear differential operators having respectively the pullbacked hypergeometric solutions (30) where  $x$  is replaced by  $R_1(x)$  and  $R_2(x)$ . These two hypergeometric solutions thus have the two Hauptmodul pullbacks

$$\mathcal{H}_1 = \frac{1728}{j_1} = \frac{27}{4} \cdot \frac{R_1(p)^2 \cdot (1 - R_1(p))^2}{(R_1(p)^2 - R_1(p) + 1)^3}, \quad (37)$$

$$\mathcal{H}_2 = \frac{1728}{j_2} = \frac{27}{4} \cdot \frac{R_2(p)^2 \cdot (1 - R_2(p))^2}{(R_2(p)^2 - R_2(p) + 1)^3}, \quad (38)$$

obtained by calculations similar to the ones previously performed on (31) but, now, for the Weierstrass form corresponding to the denominator (36).

<sup>20</sup> A shift  $z \rightarrow z+1$  or a rescaling  $z^2 \rightarrow \frac{z^2}{\lambda}$  does not change the  $j$ -invariant of the Weierstrass elliptic form.

<sup>21</sup> The present paper belonging to the symbolic computation literature and not pure mathematics for algebraic geometers, we use the standard Maple (DEtools) terminology of symmetric powers and symmetric products of linear differential operators [47]. Note that "symmetric product" is not a proper mathematical name for this construction on the solution space; it is a homomorphic image of the tensor product. The (Maple/DEtools) reason for choosing the name `symmetric_product` is the resemblance with the function `symmetric_power`.

A solution of the telescoper of (36) is thus the product of these two pullbacked hypergeometric functions. Let us give a simple illustration of this general result, with the next example.

- The telescoper of the rational function in the four variables  $x, y, z$  and  $w$

$$\frac{xyz}{(1+z)^2 - x \cdot (1-x) \cdot (x - xyzw) \cdot y \cdot (1-y) \cdot (y - 3xyzw)}, \tag{39}$$

corresponding to (36) with  $R_1(p) = p$  and  $R_2(p) = 3p$ , gives an order-four linear differential operator which is the symmetric product of two order-two operators having respectively the pullbacked hypergeometric solution (30) and the solution (30) where the variable  $x$  has been changed into  $3x$ :

$$\begin{aligned} \mathcal{S}_2(x) &= \mathcal{S}_1(3x) \\ &= (1 - 3x + 9x^2)^{-1/4} \cdot {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; \frac{243}{4} \cdot \frac{x^2 \cdot (1 - 3x)^2}{(1 - 3x + 9x^2)^3}\right). \end{aligned} \tag{40}$$

#### 4.1. Creative telescoping on rational functions of five variables associated with products or foliations of three elliptic curves

Let us now introduce the rational function in five variables  $x, y, z, v$  and  $w$

$$\frac{xyzv}{D(x, y, z, v, w)}, \tag{41}$$

where the denominator  $D(x, y, z, v, w)$  reads:

$$\begin{aligned} D_p &= \\ &(1+v)^2 - x \cdot (1-x) \cdot (x-p) \cdot y \cdot (1-y) \cdot (y-3p) \cdot z \cdot (1-z) \cdot (z-5p), \\ \text{where:} & \quad p = xyzvw. \end{aligned} \tag{42}$$

The telescoper of the rational function (41) of five variables gives<sup>22</sup> an order-eight linear differential operator which is the symmetric product of three order-two linear differential operators having respectively the pullbacked hypergeometric solution (30), the solution (30) where  $x$  has been changed into  $3x$ , namely (40), and the solution (30), where  $x$  has been changed into  $5x$ :

$$\begin{aligned} \mathcal{S}_3(x) &= \mathcal{S}_1(5x) \\ &= (1 - 5x + 25x^2)^{-1/4} \cdot {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; \frac{675}{4} \cdot \frac{x^2 \cdot (1 - 5x)^2}{(1 - 5x + 25x^2)^3}\right). \end{aligned} \tag{43}$$

In other words, the order-eight telescoper of the rational function (41) has the product  $\mathcal{S} = \mathcal{S}_1 \cdot \mathcal{S}_2 \cdot \mathcal{S}_3$ , of (30), (40) and (43) as a solution. From an algebraic geometry viewpoint, this is a consequence of the fact that, for fixed  $p$ , the algebraic variety  $D_p = 0$ , where  $D_p$  is given by (42), can be seen, for fixed  $y$  and  $z$ , as an elliptic curve  $\mathcal{E}_1$  of equation  $D_{y,z,p}(v, x) = 0$ , for fixed  $x$  and  $z$  as an elliptic curve  $\mathcal{E}_2$  of equation  $D_{x,z,p}(v, y) = 0$ , and for fixed  $x$  and  $y$  also as an elliptic curve  $\mathcal{E}_3$  of equation  $D_{x,y,p}(v, z) = 0$ , the  $j$ -invariants  $j_k, k = 1, 2, 3$  of these three elliptic curves  $\mathcal{E}_k$  yielding (in terms of  $p$ ), precisely, the three Hauptmoduls  $\mathcal{H}_k = \frac{1728}{j_k}$

$$\frac{27}{4} \cdot \frac{x^2 \cdot (1-x)^2}{(x^2 - x + 1)^3}, \quad \frac{243}{4} \cdot \frac{x^2 \cdot (1-3x)^2}{(1-3x+9x^2)^3}, \quad \frac{675}{4} \cdot \frac{x^2 \cdot (1-5x)^2}{(1-5x+25x^2)^3}, \tag{44}$$

<sup>22</sup> Such a creative telescoping calculation requires “some” computing time to achieve the result.

occurring as pullbacks in the three  $\mathcal{S}_k$ 's of the solution  $\mathcal{S} = \mathcal{S}_1 \cdot \mathcal{S}_2 \cdot \mathcal{S}_3$ , of the telescoper of (41).

**Remark:** Other examples of rational functions of three, four, five variables where the denominators also correspond to Weierstrass (resp. Legendre) forms, are displayed in Appendix D. They provide simple illustrations of rational functions where the denominator is associated with *K3 surfaces*<sup>23</sup>, or Calabi-Yau threefolds. In these cases the algebraic varieties have simple foliations in terms of two or three families of elliptic curves, and the solutions of the corresponding telescopers can be selected  ${}_3F_2$  and  ${}_4F_3$  hypergeometric functions (see (A28) in Appendix D), naturally associated with K3 surfaces and Calabi-Yau operators [27].

## 5. Creative telescoping of rational functions in three variables associated with genus-two curves with split Jacobians

In papers [17,18], dedicated to Heun functions that are solutions of telescopers of simple rational functions of three and four variables, we have obtained<sup>24</sup> an order-four telescoper of a rational function of three variables, which is the direct sum of two order-two linear differential operators, each having classical modular form solutions which can be written as pullbacked  ${}_2F_1$  hypergeometric solutions. Unfortunately, the intersection of the algebraic surface corresponding to the denominator of the rational function with the  $p = xyz$  hyperbola, yields a genus-two algebraic curve.

Let us try to understand, in this section, how a genus-two curve can yield two classical modular forms. Let us first recall the results in section 2.2 of [18].

### 5.1. Periods of extremal rational surfaces

Let us recall the rational function in just three variables [18]:

$$R(x, y, z) = \frac{1}{1 + x + y + z + xy + yz - x^3 yz}. \quad (45)$$

Its telescoper is actually an order-four linear differential operator  $L_4$  which, not only factorizes into two order-two linear differential operators, but is actually the *direct sum* (LCLM) of two<sup>25</sup> order-two linear differential operators  $L_4 = L_2 \oplus M_2$ . These two (non homomorphic) order-two linear differential operators have, respectively, the two pullbacked hypergeometric solutions:

$$\begin{aligned} \mathcal{S}_1 = & (1 + 9x)^{-1/4} \cdot (1 + 3x)^{-1/4} \cdot (1 + 27x^2)^{-1/4} \\ & \times {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; \frac{1728 \cdot x^3 \cdot (1 + 9x + 27x^2)^3}{(1 + 3x)^3 \cdot (1 + 9x)^3 \cdot (1 + 27x^2)^3}\right), \end{aligned} \quad (46)$$

and:

$$\begin{aligned} \mathcal{S}_2 = & \frac{1}{(1 + 4x - 2x^2 - 36x^3 + 81x^4)^{1/4}} \\ & \times {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; \frac{1728 \cdot x^5 \cdot (1 + 9x + 27x^2) \cdot (1 - 2x)^2}{(1 + 4x - 2x^2 - 36x^3 + 81x^4)^3}\right). \end{aligned} \quad (47)$$

The diagonal of (45) is actually the half-sum of the two series (46) and (47):

$$\text{Diag}\left(R(x, y, z)\right) = \frac{\mathcal{S}_1 + \mathcal{S}_2}{2}. \quad (48)$$

<sup>23</sup> See the emergence of product of elliptic curves from a Shioda-Inose structure on surfaces with Picard number 19 in [48]. In [48], Ling Long considers one-parameter families of K3 surfaces with generic Picard number 19. The existence of a Shioda-Inose structure implies that there is a one-parameter family of elliptic curves.

<sup>24</sup> See equation (83) in section 2.2 of [18].

<sup>25</sup> These two order-two linear differential operators  $L_2$  and  $M_2$  are not homomorphic.

As far as our algebraic geometry approach is concerned, the intersection of the algebraic surface corresponding to the denominator of the rational function (45) with the hyperbola  $p = xyz$  gives the planar algebraic curve (corresponding to the elimination of the  $z$  variable by the substitution  $z = \frac{p}{xy}$ ):

$$1 + x + y + \frac{p}{xy} + xy + y\frac{p}{xy} - x^3y\frac{p}{xy} = 0. \quad (49)$$

One easily finds that this algebraic curve is (for  $p$  fixed) a genus-two curve, and that this higher genus situation *does not* correspond to the “almost elliptic curves” described in subsection 3.2, namely an elliptic curve transformed by a monomial transformation. How can a “true” genus-two curve give two  $j$ -invariants, namely a telescope with two Hauptmodul pullbacked  ${}_2F_1$  solutions? We are going to see that the answer is that the Jacobian of this genus-two curve<sup>26</sup> is in fact isogenous to a product  $\mathcal{E} \times \mathcal{E}'$  of two elliptic curves (split Jacobian).

### 5.2. Split Jacobians

Let us first recall the concept of *split Jacobian* with a simple example. In [49], one has a crystal-clear example of a genus-two curve  $C$

$$y^2 - (x^3 + 420x - 5600) \cdot (x^3 + 42x^2 + 1120) = 0, \quad (50)$$

such that its Jacobian  $J(C)$  is isogenous to a product of elliptic curves with  $j$ -invariants  $j_1 = -2^7 \cdot 7^2 = -6272$  and  $j_2 = -2^5 \cdot 7 \cdot 17^3 = -1100512$ , corresponding to the following two values of the Hauptmodul  $\mathcal{H} = \frac{1728}{j}$ :  $\mathcal{H}_1 = -27/98$  and  $\mathcal{H}_2 = -54/34391$ . Let us consider the genus-one elliptic curve

$$v^2 = u^3 + 4900u^2 + 7031500u + 2401000000, \quad (51)$$

of  $j$ -invariant  $j = j_2 = -2^5 \cdot 7 \cdot 17^3$ . We consider the following transformation<sup>27</sup>:

$$u = -\frac{882000 \cdot (x - 14)}{x^3 + 420x - 5600}, \quad v = \frac{49000 \cdot (x^3 - 21x^2 - 140)}{(x^3 + 420x - 5600)^2} \cdot y. \quad (52)$$

This change of variables (52) actually transforms the elliptic curve (51) in  $u$  and  $v$  into the genus-two curve (50) in  $x$  and  $y$ . This provides a simple example of a genus-two curve with split Jacobian through K3 surfaces.

More generally, let us consider the Jacobian of a genus-two curve  $C$ . The Jacobian is simple if it does not contain a proper abelian subvariety, otherwise the Jacobian is reducible, or decomposable or “split”. For this latter case, the only possibility for a genus-two curve is that its Jacobian is isogenous to a product  $\mathcal{E} \times \mathcal{E}'$  of two elliptic curves<sup>28</sup>. Equivalently, there is a degree- $n$  map  $C \rightarrow \mathcal{E}$  to some elliptic curves. Classically such pairs<sup>29</sup>  $C, \mathcal{E}$  arose in the reduction of hyperelliptic integrals to elliptic ones [49]. The  $j$ -invariants correspond, here, to the two elliptic subfields: see [49].

<sup>26</sup> An algebraic geometer will probably recall that it is very well-known that a genus two curve *may* have Jacobian isogenous to a product of elliptic curves. This is not the case in general. The genus two curves that have a (non constant) map to an elliptic curve have this property. Our purpose in section (5.3) is to perform a creative telescoping calculation in such a selected situation.

<sup>27</sup> This transformation is rational but not birational. If it were birational, then it would preserve the genus. Here, one goes from genus one to genus two.

<sup>28</sup> Along these lines, see also the concepts of Igusa-Clebsch invariants and Hilbert modular surfaces [49–52].

<sup>29</sup> One also has an anti-isometry Galois invariant  $\mathcal{E}' \simeq \mathcal{E}$  under Weil pairing. The decomposition corresponds to real multiplication by quadratic ring of discriminant  $n^2$ .

5.3. Creative telescoping on rational functions in three variables associated with genus-two curves with split Jacobians: a two-parameter example. 396  
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Let us now consider the example with two parameters,  $a$  and  $b$ , given in section 4.5 on page 12 of [49]. Let us substitute the rational parametrization<sup>30</sup> 398  
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$$u = \frac{x^2}{x^3 + ax^2 + bx + 1}, \quad v = \frac{y \cdot (x^3 - bx - 2)}{(x^3 + ax^2 + bx + 1)^2}, \quad (53)$$

in the elliptic curve 400

$$R \cdot v^2 = R \cdot u^3 + 2 \cdot (ab^2 - 6a^2 + 9b) \cdot u^2 + (12a - b^2) \cdot u - 4, \quad (54)$$

where 401

$$R = 4 \cdot (a^3 + b^3) - a^2b^2 - 18ab + 27. \quad (55)$$

This gives the genus-two curve  $C_{a,b}(x, y) = 0$  with: 402

$$C_{a,b}(x, y) = R \cdot y^2 + (4x^3 + b^2x^2 + 2bx + 1) \cdot (x^3 + ax^2 + bx + 1). \quad (56)$$

The  $j$ -invariant of the elliptic curve (54) gives the following exact expression for the Hauptmodul  $\mathcal{H} = \frac{1728}{j}$ : 403  
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$$\mathcal{H} = \frac{108 \cdot (b - 3)^3 \cdot (4a^3 + 4b^3 - a^2b^2 - 18ab + 27)^2 \cdot (b^2 + 3b + 9)^3}{(a^2b^4 + 12b^5 - 126ab^3 + 216ba^2 + 405b^2 - 972a)^3}. \quad (57)$$

Let us consider the telescoper of the rational function of three variables  $xy/D_a(x, y, z)$  where the denominator  $D_a(x, y, z)$  is  $C_{a,b}(x, y)$  given by (56), but for  $b = 3 + xyz$ : 405  
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$$\begin{aligned} D_a(x, y, z) &= C_{a,3+xyz}(x, y) \\ &= x^6y^3z^3 + x^7y^2z^2 + 4x^3y^5z^3 + 9x^5y^2z^2 + 6x^6yz + 3x^4y^2z^2 + 36y^4x^2z^2 \\ &\quad + 6x^5yz + 4x^6 + 27x^4yz + 9x^5 + 18x^3yz + 108xy^3z + 18x^4 + 3x^2yz \\ &\quad + 32x^3 + 27x^2 + 135y^2 + 9x + 1 \\ &\quad + (x^6y^2z^2 + 6x^5yz + 2x^4yz + 4x^5 - 18xy^3z + 9x^4 + 6x^3 + x^2 - 54y^2) \cdot a \\ &\quad - y^2 \cdot (xyz + 3)^2 \cdot a^2 + 4y^2 \cdot a^3. \end{aligned} \quad (58)$$

This telescoper of the rational function 407

$$R_a(x, y, z) = \frac{xy}{D_a(x, y, z)}, \quad (59)$$

is an order-four linear differential operator  $L_4$  which is actually the direct sum,  $L_4 = LCLM(L_2, M_2) = L_2 \oplus M_2$ , of two order-two linear differential operators, having two pullbacked hypergeometric solutions. One finds out that one of the two pullbacks precisely corresponds to the Hauptmodul  $\mathcal{H}$  given by (57) for  $b = 3 + x$ . 408  
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Let us consider the  $a = 3$  subcase<sup>31</sup>. For  $a = 3$ , the Hauptmodul  $\mathcal{H} = \frac{1728}{j}$ , given by (57) becomes for  $b = 3 + x$ : 412  
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$$\mathcal{H} = \frac{4 \cdot x \cdot (27 + 4x)^2 \cdot (x^2 + 9x + 27)^3}{(9 + x)^3 \cdot (4x^2 + 27x + 27)^3}. \quad (60)$$

<sup>30</sup> See also [53] section 6 page 48.

<sup>31</sup> The discriminant in  $b$  of  $4a^3 + 4b^3 - a^2b^2 - 18ab + 27$  reads:  $(a - 3)^3 \cdot (a^2 + 3a + 9)^3$ , consequently the exact expressions are simpler at  $a = 3$ .

The telescoper of the rational function (59) with  $D_a(x, y, z)$  given by (58) for  $a = 3$ , is an order-four linear differential operator which is the direct sum of two order-two linear differential operators  $L_4 = LCLM(L_2, M_2) = L_2 \oplus M_2$ , these two order-two linear differential operators having the pullbacked hypergeometric solutions

$$(27 + 4x)^{-1/2} \cdot x^{-5/4} \cdot {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; 1 + \frac{27}{4x}\right), \tag{61}$$

for  $L_2$ , and

$$\frac{3 + x}{(9 + x)^{1/4} \cdot (4x^2 + 27x + 27)^{1/4} \cdot x^{3/2} \cdot (27 + 4x)^{1/2}} \times {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; \frac{4 \cdot x \cdot (27 + 4x)^2 \cdot (x^2 + 9x + 27)^3}{(9 + x)^3 \cdot (4x^2 + 27x + 27)^3}\right), \tag{62}$$

for  $M_2$ , where we see clearly that the Hauptmodul in (62) is precisely the Hauptmodul (60). The Jacobian of the genus-two curve is a split Jacobian corresponding to the product  $\mathcal{E}_1 \times \mathcal{E}_2$  of two elliptic curves, the  $j$ -invariant of the second elliptic curve corresponds to the Hauptmodul  $\mathcal{H} = \frac{1728}{j}$  given by (57) when the  $j$ -invariant of the first elliptic curve reads

$$j_1 = \frac{6912x}{27 + 4x}, \tag{63}$$

corresponding to the Hauptmodul  $\frac{1728}{j_1} = 1 + \frac{27}{4x}$  in (61). This second invariant is, as it should, exactly the  $j$ -invariant of the second elliptic curve  $\mathcal{E}'$ , given page 48 in [53]:

$$j(\mathcal{E}') = \frac{256 \cdot (3b - a^2)^3}{4a^3c - a^2b^2 - 18abc + 4b^3 + 27c^2}, \tag{64}$$

for  $c = 1, a = 3$  and  $b = 3 + x$ .

#### 5.4. Creative telescoping on rational functions of three variables associated with genus-two curves with split Jacobians: a simple example

Another simpler example of a genus-two curve with pullbacked  ${}_2F_1$  solution (not product of pullbacked  ${}_2F_1$ ) of the telescoper can be given if one considers the genus-two algebraic curve  $C_p(x, y) = 0$  given in Lemma 7 of [54] (see also [55,56])

$$C_p(x, y) = x^5 + x^3 + p \cdot x - y^2. \tag{65}$$

Let us introduce the rational function  $x y / D(x, y, z)$  where the denominator  $D(x, y, z)$  is given by:

$$D(x, y, z) = C_{(p=xyz)}(x, y) = x^5 + x^3 + x^2 y z - y^2. \tag{66}$$

The telescoper of this rational function is an order-two linear differential operator which has the two hypergeometric solutions

$$x^{-1/4} \cdot {}_2F_1\left(\frac{1}{8}, \frac{5}{8}; \frac{3}{4}; 4x\right) \tag{67}$$

which is a Puiseux series at  $x = 0$  and:

$$x^{-1/4} \cdot {}_2F_1\left(\frac{1}{8}, \frac{5}{8}; 1; 1 - 4x\right). \tag{68}$$

These two hypergeometric solutions can be rewritten as<sup>32</sup>

$$\mathcal{A}(x) \cdot {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; \frac{1728}{J}\right), \quad (69)$$

where the  $j$ -invariant  $J$ , in the Hauptmodul  $\frac{1728}{J}$  in (69), corresponds exactly to the degree-two elliptic subfields

$$j^2 - 128 \cdot \frac{(2000x^2 + 1440x + 27)}{(1 - 4x)^2} \cdot J - 4096 \cdot \frac{(100x - 9)^3}{(1 - 4x)^3} = 0, \quad (70)$$

given in the first equation of page 6 of [54].

**Remark:** In contrast to the previous example of subsection 5.3 where we had two  $j$ -invariants corresponding to the two order-two linear differential operators  $L_2$  and  $M_2$  of the direct-sum decomposition of the order-four telescoper, we have, here, just one order-two telescoper, which is enough to “encapsulate” the two  $j$ -invariants (70), since they are Galois-conjugate.

## 6. Rational functions with tri-quadratic denominator and $N$ -quadratic denominator.

We try to find telescopers of rational functions corresponding to (factors of) linear differential operators of “small” orders, for instance order-two linear differential operators with pullbacked  ${}_2F_1$  hypergeometric functions, classical modular forms, or their modular generalizations (order-four Calabi-Yau linear differential operators [27], etc.). As we saw in the previous sections, this corresponds to the fact that the denominator of these rational functions is associated with an elliptic curve, or products of elliptic curves, with K3 surfaces or with threefold Calabi-Yau manifolds corresponding to algebraic varieties with foliations in elliptic curves<sup>33</sup>. Since this paper tries to reduce the *differential algebra* creative telescoping calculations to *effective algebraic geometry* calculations<sup>34</sup> and structures, we want to focus on rational functions with denominators that correspond to *selected* algebraic varieties [45,59], beyond algebraic varieties corresponding to products of elliptic curves or foliations in elliptic curves<sup>35</sup>, namely algebraic varieties with an infinite number of birational automorphisms<sup>36</sup>. This infinite number of birational symmetries excludes algebraic varieties of the “general type” (with a finite number<sup>37</sup> of birational symmetries). For algebraic surfaces, this amounts to discarding the surfaces of “general type” which have Kodaira dimension 2, focusing on Kodaira dimension one (elliptic surfaces), or Kodaira dimension zero (abelian surfaces, hyperelliptic surfaces, K3 surfaces, Enriques surfaces), or even Kodaira dimension  $-\infty$  (ruled surfaces, rational surfaces).

In contrast with algebraic curves where one can easily, and very efficiently, calculate the genus of the curves to discard algebraic curves of higher genus and, in the case of

<sup>32</sup> The fact that  ${}_2F_1\left(\frac{1}{8}, \frac{5}{8}; 1; z\right)$  can be rewritten as  ${}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; \mathcal{H}(z)\right)$  where the Hauptmodul  $\mathcal{H}(z)$  is solution of a quadratic equation is given in equation (H.14) of Appendix H of [18].

<sup>33</sup> Even though K3 surfaces or threefold Calabi-Yau manifolds are *not* abelian varieties, the Weierstrass-Legendre forms introduced in Appendix D amount to saying that K3 surfaces can be “essentially viewed” (as far as creative telescoping is concerned) as foliations in two elliptic curves, and threefold Calabi-Yau manifolds as foliations in three elliptic curves.

<sup>34</sup> One has birational automorphisms in projective spaces [57,58], but since this paper is dedicated to (efficient) formal calculations we work exclusively in affine coordinates (see for instance (A41), (A42), (A43) below). For algebraic geometers an elliptic curve is a smooth complete genus 1 curve with a choice of a base point. Here our elliptic curves are, in fact, an affine piece of a genus 1 curve with no base point, but this does not really matter because the  $j$ -invariant which is all we care about in this kind of creative telescoping calculations, is determined by that much information.

<sup>35</sup> K3 surfaces, threefold Calabi-Yau manifolds, higher curves with split Jacobian corresponding to products of elliptic curves, ...

<sup>36</sup> The best explicit illustration of this situation emerges in integrable models [45,59–61]

<sup>37</sup> There are even precise bounds for the number of automorphisms. The upper bound is  $84(g - 1)$  for curves of genus  $g$  and these bounds have been extensively studied in higher dimensions [62–64].



genus one, obtain the  $j$ -invariant using formal calculations<sup>38</sup>, it is, in practice, quite difficult to see for higher dimensional algebraic varieties, that the algebraic variety is not of the “general type”, because it has an infinite number of birational symmetries. For these (low Kodaira dimension) “selected cases” we are interested in, calculating the generalization of the  $j$ -invariant (Igusa-Shioda invariants, etc.) is quite hard.

Along this line we want to underline that there exists a remarkable set of algebraic surfaces, namely the algebraic surfaces corresponding to tri-quadratic equations:

$$\sum_{m=0,1,2} \sum_{n=0,1,2} \sum_{l=0,1,2} a_{m,n,l} \cdot x^m y^n z^l = 0, \quad (71)$$

depending on  $27 = 3^3$  parameters  $a_{m,n,l}$ . More generally, one can introduce algebraic varieties corresponding to  $N$ -quadratic equations:

$$\sum_{m_1=0,1,2} \sum_{m_2=0,1,2} \cdots \sum_{m_N=0,1,2} a_{m_1, m_2, \dots, m_N} \cdot x_1^{m_1} x_2^{m_2} \cdots x_N^{m_N} = 0. \quad (72)$$

With these tri-quadratic (71), or  $N$ -quadratic (72) equations, we will see, in E.1 and E.2, that we have *automatically* (selected) algebraic varieties that are not of the “general type” having an infinite number of birational symmetries, which is precisely our requirement for the denominator of rational functions with remarkable telescopers<sup>39</sup>.

Let us first, as a warm-up, consider, in the next subsection, a remarkable example of tri-quadratic (71), where the underlying foliation in elliptic curves is crystal clear.

### 6.1. Rational functions with tri-quadratic denominator simply corresponding to elliptic curves.

Let us first recall the tri-quadratic equation in three variables  $x$ ,  $y$  and  $z$

$$\begin{aligned} x^2 y^2 z^2 - 2 \cdot M \cdot xyz \cdot (x + y + z) + 4 \cdot M \cdot (M + 1) \cdot xyz \\ + M^2 \cdot (x^2 + y^2 + z^2) - 2 M^2 \cdot (xy + xz + yz) = 0, \end{aligned} \quad (73)$$

already introduced in Appendix C of [65]. This algebraic surface, symmetric in  $x$ ,  $y$  and  $z$ , can be seen for  $z$  (resp.  $x$  or  $y$ ) fixed, as an elliptic curve whose  $j$ -invariant is independent of  $z$ , yielding the corresponding Hauptmodul:

$$\mathcal{H} = \frac{1728}{j} = \frac{27 \cdot M^2 \cdot (M - 1)^2}{4 \cdot (M^2 - M + 1)^3}. \quad (74)$$

This corresponds to the fact that this algebraic surface (73) can be seen as a product of two times the same elliptic curve with the Hauptmodul (74). This is a consequence of the fact that, introducing  $x = sn(u)^2$ ,  $y = sn(v)^2$  and  $z = sn(u + v)^2$ , and  $M = 1/k^2$ , this algebraic surface (73) corresponds to the well-known formula for the addition on elliptic sine<sup>40</sup>:

$$sn(u + v) = \frac{sn(u) cn(v) dn(v) + sn(v) cn(u) dn(u)}{1 - k^2 sn(u)^2 sn(v)^2}. \quad (75)$$

<sup>38</sup> Use `with(algcurves)` in Maple and the commands “genus” and “j\_invariant”.

<sup>39</sup> Telescopers with factors of “small enough” order, possibly yielding classical modular forms, Calabi-Yau operators, etc. Rational functions with denominators of the “general type” will yield telescopers of very large orders.

<sup>40</sup> See equation (C.3) in Appendix C of [65].

For  $M = xyzw$ , the LHS of the tri-quadratic equation (73) yields a polynomial of four variables  $x, y, z$  and  $w$ , that we denote  $T(x, y, z, w)$ :

$$T(x, y, z, w) = x^2y^2z^2 - 2 \cdot x^2y^2z^2w \cdot (x + y + z) + 4 \cdot (xyzw + 1) \cdot x^2y^2z^2w + x^2y^2z^2w^2 \cdot (x^2 + y^2 + z^2) - 2x^2y^2z^2w^2 \cdot (xy + xz + yz). \quad (76)$$

The telescoper of the rational function in four variables  $x, y, z$  and  $w$ ,

$$\frac{xyz}{T(x, y, z, w)}, \quad (77)$$

is an order-three (self-adjoint) linear differential operator which is the symmetric square of the order-two linear differential operator having the following pullbacked  ${}_2F_1$  hypergeometric solution:

$$x^{-1/2} \cdot (x^2 - x + 1)^{-1/4} \times {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; \frac{27 \cdot x^2 \cdot (x-1)^2}{4 \cdot (x^2 - x + 1)^3}\right). \quad (78)$$

As it should, the Hauptmodul in (78) is the same as the Hauptmodul (74). The algebraic surface (73) can be seen as the product of two times the same elliptic curve with the Hauptmodul (74): as expected, the solution of the order-three telescoper is the square of the pullbacked  ${}_2F_1$  hypergeometric function (78) with that Hauptmodul.

More generally, we can also consider another tri-quadratic equation of three variables  $x, y$  and  $z$  and two parameters  $M$  and  $N$ :

$$x^2y^2z^2 - 2M \cdot xyz \cdot (x + y + z) + N \cdot xyz + M^2 \cdot (x^2 + y^2 + z^2) - 2M^2 \cdot (xy + xz + yz) = 0. \quad (79)$$

This surface, symmetric in  $x, y$  and  $z$ , can be seen for  $z$  (resp.  $x$  or  $y$ ) fixed as an elliptic curve whose  $j$ -invariant is, again, independent of  $z$  yielding the corresponding Hauptmodul:

$$\mathcal{H} = \frac{1728}{j} = \frac{1728 \cdot M^6 \cdot (64M^3 - N^2)}{(48M^3 - N^2)^3}. \quad (80)$$

Let us consider the following change of variables  $M = m^2$  and  $N = 8 \cdot m^3 + p$  in (79). For  $p = xyzw$ , the LHS of the tri-quadratic equation (79) yields a polynomial in four variables  $x, y, z$  and  $w$ , that we denote  $\mathcal{T}_m(x, y, z, w)$ :

$$\mathcal{T}_m(x, y, z, w) = x^2y^2z^2 - 2m^2 \cdot xyz \cdot (x + y + z) + (8 \cdot m^3 + xyzw) \cdot xyz + m^4 \cdot (x^2 + y^2 + z^2) - 2m^4 \cdot (xy + xz + yz). \quad (81)$$

For  $z$  (resp.  $x$  or  $y$ ) fixed, the corresponding Hauptmodul (80) reads:

$$\mathcal{H} = \frac{1728 \cdot m^{12} \cdot p \cdot (16m^3 + p)}{(16m^6 + 16m^3 \cdot p + p^2)^3}. \quad (82)$$

The telescoper of the rational function in four variables  $x, y, z$  and  $w$ ,

$$\frac{xyz}{\mathcal{T}_m(x, y, z, w)}, \quad (83)$$

is an order-three (self-adjoint) linear differential operator which is the symmetric square of an order-two linear differential operator having the following pullbacked  ${}_2F_1$  hypergeometric solution:

$$(16m^6 + 16m^3 \cdot x + x^2)^{-1/4} \cdot {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; \frac{1728 \cdot m^{12} \cdot x \cdot (16m^3 + x)}{(16m^6 + 16m^3 \cdot x + x^2)^3}\right). \quad (84)$$

As it should, the Hauptmodul in (84) is the same as the Hauptmodul (82). The algebraic surface (79) can be seen as the product of two times the same elliptic curve with the Hauptmodul (80) (or (82)). As expected, the solution of the order-three telescoper is the square of the pullbacked  ${}_2F_1$  hypergeometric function (84) with the Hauptmodul (82).

**Remark:** Let us perform some (slight) deformation of the rational function (77), changing the first  $-2$  coefficient in (76) into a  $-3$  coefficient. One thus considers the polynomial  $T(x, y, z, w)$ :

$$T(x, y, z, w) = x^2 y^2 z^2 - 3 \cdot x^2 y^2 z^2 w \cdot (x + y + z) + 4 \cdot (xyzw + 1) \cdot x^2 y^2 z^2 w + x^2 y^2 z^2 w^2 \cdot (x^2 + y^2 + z^2) - 2 \cdot x^2 y^2 z^2 w^2 \cdot (xy + xz + yz). \quad (85)$$

The telescoper of the rational function in four variables,

$$\frac{xyz}{T(x, y, z, w)}, \quad (86)$$

is an (irreducible) linear differential operator  $L_4$  of (only) order four which is non-trivially homomorphic to its adjoint<sup>41</sup>. A priori, we cannot exclude the fact that  $L_4$  could be homomorphic to the symmetric cube of a second-order linear differential operator, or to a symmetric product of two second-order operators. Furthermore, it could also be, in principle, that these second-order operators admit classical modular forms as solutions (pullbacks of special  ${}_2F_1$  hypergeometric functions). However, these options can both be excluded by using some results from differential Galois theory [68], specifically from [69, Prop. 7, p. 50] for the symmetric cube case, and from [69, Prop. 10, p. 69] for the symmetric product case, see also [70, §3]. Indeed, if  $L_4$  were either a symmetric cube or a symmetric product of order-two operators, then its symmetric square would contain a (direct) factor of order 3 or 1. This is ruled out by a factorization procedure which shows that the symmetric square of  $L_4$  is (LCLM-)irreducible.

This example does not correspond to an addition formula like (75), but the polynomial  $T(x, y, z, w)$  still corresponds to a tri-quadratic. Consequently it is an algebraic variety with an infinite number of birational automorphisms, as shown in Appendix E.1.

## 6.2. Rational functions with tri-quadratic denominator: Fricke cubics examples associated with Painlevé VI equations

Let us consider other simple examples of tri-quadratic surfaces that occur in different domains of mathematics and physics.

Among the *Fricke families* of cubic surfaces, the family [71–73]

$$xyz + x^2 + y^2 + z^2 + b_1 x + b_2 y + b_3 z + c = 0, \quad (87)$$

of affine cubic surfaces parametrized by the four constants  $(b_1, b_2, b_3, c)$  is known [72] to be a deformation of a  $D_4$  singularity which occurs at the symmetric (Manin's) case  $b_1 = b_2 = b_3 = -8, c = 28$ .

<sup>41</sup> Its exterior square has a rational solution. However this order-four linear differential operator is not MUM (maximum unipotent monodromy [27,66,67]).

Among the symmetric  $b_1 = b_2 = b_3$  cases some selected sets of the four constants  $(b_1, b_2, b_3, c)$  emerge: the Markov cubic  $b_1 = b_2 = b_3 = c = 0$ , Cayley’s nodal cubic  $b_1 = b_2 = b_3 = 0, c = -4$ , Clebsch’s diagonal cubic  $b_1 = b_2 = b_3 = 0, c = -20$ , and Klein’s cubic  $b_1 = b_2 = b_3 = -1, c = 0$ .

Some of these symmetric cubics can be seen as the monodromy manifold of the Painlevé VI equation (see equation (1.7) in [74], see also equations (1.2) and (1.4) in [73]): the Picard-Hitchin cases  $(0, 0, 0, 4), (0, 0, 0, -4), (0, 0, 0, -32)$ , Kitaev’s cases  $(0, 0, 0, 0), (-8, -8, -8, -64)$ , and especially Manin’s case  $(-8, -8, -8, 28)$ .

Let us consider the (symmetric) rational function in three variables  $x, y$  and  $z$  [72]:

$$R(x, y, z) = \frac{1}{x^2 + y^2 + z^2 + xyz + c} \tag{88}$$

that takes into account the other Picard-Hitchin cases<sup>42</sup>  $(0, 0, 0, 4), (0, 0, 0, -4), (0, 0, 0, 32)$ . The rational function (88) has an order-two telescoper which has a simple pullbacked hypergeometric solution:

$$\begin{aligned} & \frac{1}{x+c} \cdot {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; -\frac{27x^2}{(x+c)^3}\right) \\ &= (x+c)^{-1/4} \cdot q_3(x)^{-1/4} \cdot {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; -\frac{1728 \cdot x^6 \cdot p_3(x)}{(x+c)^3 \cdot q_3(x)^3}\right), \end{aligned} \tag{89}$$

where<sup>43</sup>:

$$\begin{aligned} p_3(x) &= x^3 + 3 \cdot (c+9) \cdot x^2 + 3 \cdot c^2 \cdot x + c^3, \\ q_3(x) &= x^3 + 3 \cdot (c+8) \cdot x^2 + 3 \cdot c^2 \cdot x + c^3, \end{aligned}$$

Eliminating  $z = \frac{p}{xy}$  in the denominator of (88) gives the genus-four algebraic curve:

$$x^2y^2 \cdot (x^2 + y^2) + (p+c) \cdot x^2y^2 + p^2 = 0. \tag{90}$$

Again, the question is to see whether the Jacobian of this genus-four algebraic curve (88) could also correspond to a split Jacobian, with a  $j$ -invariant corresponding to the Hauptmodul in (89).

### 7. Telescopers of rational functions of several variables

Let us consider the rational function in four variables  $x, y, z, u$ :

$$R(x, y, z, u) = \frac{1}{1 + 3y + z + 9yz + 11z^2y + 3ux}. \tag{91}$$

The telescoper of this rational function of four variables is an order-two linear differential operator  $L_2$  which has the pullbacked hypergeometric solution:

$$\begin{aligned} & (1 - 2592x^2)^{-1/4} \\ & \times {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; -\frac{419904 \cdot x^3 \cdot (5 - 12x - 19440x^2 + 2665872x^3)}{(1 - 2592x^2)^3}\right). \end{aligned} \tag{92}$$

<sup>42</sup> As well as the Markov cubic  $b_1 = b_2 = b_3 = c = 0$ , Cayley’s nodal cubic  $b_1 = b_2 = b_3 = 0, c = -4$ , and Clebsch’s diagonal cubic  $b_1 = b_2 = b_3 = 0, c = -20$  cases.

<sup>43</sup> The values  $c = 0$  and  $c = -4$  are the only values such that the discriminant in  $x$  of  $p_3(x)$  can be zero.

The diagonal of (91) is the expansion of this pullbacked hypergeometric function (92):

$$1 + 648x^2 - 72900x^3 + 1224720x^4 - 330674400x^5 + 23370413220x^6 - 1276733858400x^7 + 180019474034400x^8 - 12013427240614800x^9 + \dots \quad (93)$$

If one considers the intersection of the vanishing condition of the denominator of (91) with the hyperbola  $p = xyz u$ , eliminating for instance  $u = \frac{p}{xyz}$  in the vanishing condition of the denominator of (91), one gets a condition, independent of  $x$ , which corresponds to a genus-one curve

$$11y^2z^3 + 9y^2z^2 + 3y^2z + yz^2 + yz + 3p = 0. \quad (94)$$

The Hauptmodul of this elliptic curve (94) reads:

$$\mathcal{H} = -\frac{419904 \cdot p^3 \cdot (5 - 12p - 19440p^2 + 2665872p^3)}{(1 - 2592p^2)^3}, \quad (95)$$

which corresponds precisely to the Hauptmodul pullback in (92).

**Remark :** The expansion (93) of (92) is not only the diagonal of the rational function  $R(x, y, z, u)$  in four variables (91), it is also the diagonal of the rational function of three variables  $R(x, y, z, 1)$ . Actually, using section 3, one sees easily that eliminating  $x = \frac{p}{yz}$  in the vanishing condition of the denominator of  $R(x, y, z, 1)$  gives exactly the same elliptic curve (94).

Let us now generalize the rational function (91) of four variables  $x, y, z, u$ , by introducing the rational function of  $N + 3$  variables  $x, y, z, u_1, u_2, \dots, u_N$ :

$$R(x, y, z, u_1, u_2, \dots, u_N) = \frac{1}{1 + 3y + z + 9yz + 11z^2y + 3x \cdot u_1 u_2 \dots u_N}. \quad (96)$$

The telescoper of this rational function of  $N + 3$  variables is the same order-two telescoper as for (91), which has the pullbacked hypergeometric solution (92). Again one can verify that the diagonal of (96) is the expansion (93) of the pullbacked hypergeometric function<sup>44</sup> (92). If one considers the intersection of the vanishing condition of the denominator of (96) with the hyperbola  $p = xyz u_1 u_2 \dots u_N$ , eliminating for instance  $u_N = \frac{p}{xyz u_1 \dots u_{N-1}}$  in the vanishing condition of the denominator of (96), one gets again a condition, independent of  $x$  but also of  $u_1, \dots, u_N$ , which corresponds to the genus-one curve (94):

$$11y^2z^3 + 9y^2z^2 + 3y^2z + yz^2 + yz + 3p = 0. \quad (97)$$

The Hauptmodul of this elliptic curve (97) is again equal to the Hauptmodul (95) which corresponds precisely to the Hauptmodul pullback in (92).

Other examples, corresponding to simple polynomial deformations of (91), such that their diagonal is the pullbacked  ${}_2F_1$  hypergeometric function (92), are displayed in Appendix F. This (infinite) family of rational functions corresponds to a different algebraic geometry scenario: the “canonical” algebraic surface corresponding to the intersection of the vanishing condition of the denominator of the rational function with the hyperbola  $p = xyz$ , is foliated in (generically high genus) algebraic curves depending on the variable  $x$ . One sees (experimentally) that the Hauptmodul of the pullbacked  ${}_2F_1$  hypergeometric functions corresponds to the Hauptmodul of the  $x = 0$  algebraic curve, which is an elliptic

<sup>44</sup> A pure algebraic geometer will probably consider this result as trivial from the computational point of view, saying that the variety is a fiber bundle over a family of elliptic curves with constant fiber (see also below).

curve<sup>45</sup>. In contrast to the other examples and results of this paper, we have no algebraic geometry interpretation of this experimental result yet.

## 8. Conclusion

Diagonals of rational functions emerge quite naturally in lattice statistical mechanics [19,20]. This explains the frequent occurrence of modular forms, represented as pullbacked  ${}_2F_1$  hypergeometric functions [1,2] in lattice statistical mechanics [21–27].

We have shown that the results we had obtained on diagonals of nine- and ten-parameter families of rational functions in three variables, using creative telescoping yielding classical modular forms expressed as pullbacked  ${}_2F_1$  hypergeometric functions [1, 2], can be obtained much more efficiently by calculating the  $j$ -invariant of an elliptic curve canonically associated with the denominator of the rational functions. In the case where creative telescoping yields pullbacked  ${}_2F_1$  hypergeometric functions, we generalize this result to other families of rational functions of three, and even more than three, variables, when the denominator can be associated with products of elliptic curves or foliations in terms of elliptic curves, or when the denominator is associated with a genus-two curve with a split Jacobian corresponding to products of elliptic curves.

We have seen different scenarii. In the first cases, we have considered denominators corresponding to products of elliptic curves: in these cases the solutions of the telescoper were products of pullbacked  ${}_2F_1$  hypergeometric functions. We have also considered denominators corresponding to genus-two curves with split Jacobians isogenous to products of two elliptic curves, and in these cases the solutions of the telescoper were sums of two pullbacked  ${}_2F_1$  hypergeometric functions, sometimes one pullbacked  ${}_2F_1$  hypergeometric function being enough to describe the two Galois-conjugate  $j$ -invariants (see 5.4). We also considered denominators corresponding to algebraic varieties where the Hauptmodul pullback in the pullbacked  ${}_2F_1$  hypergeometric functions emerges from a selected ( $x = 0$ , see Appendices F.1, F.2) elliptic curve of the algebraic variety. We also encountered denominators corresponding to algebraic manifolds with an infinite set of birational automorphisms and elliptic curves foliation, yielding no longer classical modular forms represented as pullbacked  ${}_2F_1$  hypergeometric functions, but more general modular structures associated with selected linear differential operators like Calabi-Yau linear differential operators [27,66] and their generalizations.

The creative telescoping method on a rational function is an efficient way to find the periods of an algebraic variety over *all possible cycles*<sup>46</sup>. The fact that the solution of the telescoper corresponds to “periods” [37] over all possible cycles is a simple consequence of the fact that creative telescoping corresponds to purely differential algebraic manipulations on the integrand independently of the cycles, thus being blind to analytical details. In this paper, we show that the final result emerging from such differential algebra procedures (which can be cumbersome when the result depends on nine or ten parameters), can be obtained almost instantaneously from a more fundamental intrinsic pure algebraic geometry approach, calculating, for instance, the  $j$ -invariant of some canonical elliptic curve. This corresponds to a shift Analysis  $\rightarrow$  Differential Algebra  $\rightarrow$  Algebraic Geometry. Algebraic geometry studies of more involved algebraic varieties than product of elliptic curves, foliation in elliptic curves (Calabi-Yau manifolds, etc.) is often a tedious and/or difficult task (finding Igusa-Shioda invariants, etc.), and formal calculation tools are not always available or user-friendly. Ironically, for such involved algebraic varieties the creative telescoping may then become a simple and efficient tool to perform effective algebraic geometry studies.

Some work remains to be done. While in the present paper, we have focused on studying the relationship between the denominator of a rational function and its diagonal,

<sup>45</sup> The algebraic curves for other values of  $x$  are not necessarily elliptic curves, they can be algebraic curves of quite large genus.

<sup>46</sup> Not only the vanishing cycles [38,40] corresponding to diagonals of rational functions.

it would be interesting to explore the impact that variations of the numerator have on the telescoper or on the diagonal. Then, one may expect that all these algebraic-geometric findings could be exploited in the design of novel and more efficient creative telescoping algorithms for multivariate rational functions.

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## References

1. Y. Abdelaziz, S. Boukraa, C. Koutschan, J-M. Maillard, *Diagonals of rational functions, pullbacked  ${}_2F_1$  hypergeometric functions and modular forms*, *J. Phys. A* **51**, Number 45 (2018) 455201 (30 pages)
2. Y. Abdelaziz, S. Boukraa, C. Koutschan, J-M. Maillard, *Diagonals of rational functions, pullbacked  ${}_2F_1$  hypergeometric functions and modular forms (unabridged version)*, [arXiv:1805.04711v1\[math-ph\]](https://arxiv.org/abs/1805.04711v1) (2018)
3. HolonomicFunctions Package version 1.7.1 (09-Oct-2013) written by Christoph Koutschan, Copyright 2007-2013, Research Institute for Symbolic Computation (RISC), Johannes Kepler University, Linz, Austria
4. G. Christol, 1984, Diagonales de fractions rationnelles et équations différentielles, *Study group on ultrametric analysis, 10th year: 1982/83, No. 2, Exp. No. 18* (Paris: Inst. Henri Poincaré) pp 1–10. [http://archive.numdam.org/article/GAU\\_1982-1983\\_\\_10\\_2\\_A4\\_0.pdf](http://archive.numdam.org/article/GAU_1982-1983__10_2_A4_0.pdf)
5. G. Christol, 1985, Diagonales de fractions rationnelles et équations de Picard-Fuchs, *Study group on ultrametric analysis, 12th year, 1984/85, No. 1, Exp. No. 13* (Paris: Secrétariat Math.) pp 1–12. [http://archive.numdam.org/article/GAU\\_1984-1985\\_\\_12\\_1\\_A8\\_0.pdf](http://archive.numdam.org/article/GAU_1984-1985__12_1_A8_0.pdf)
6. G. Christol, 1988, Diagonales de fractions rationnelles, *Séminaire de Théorie des Nombres, Paris 1986–87 (Progr. Math. vol 75)* (Boston, MA: Birkhäuser Boston) pp. 65-90
7. G. Christol, 1990 *Globally bounded solutions of differential equations, Analytic number theory (Tokyo, 1988) (Lecture Notes in Math. 1434* (Berlin: Springer) pp. 45-64. <http://dx.doi.org/10.1007/BFb0097124>
8. L. Lipshitz and A.J. van der Poorten, 1990, Rational functions, diagonals, automata and arithmetic, *Number theory (Banff, AB, 1988)* (Berlin: de Gruyter) pp. 339-358
9. L. Lipshitz, 1988, The diagonal of a  $D$ -finite power series is  $D$ -finite, *J. Algebra* **113** 373–378. [http://dx.doi.org/10.1016/0021-8693\(88\)90166-4](http://dx.doi.org/10.1016/0021-8693(88)90166-4)
10. J. Denef and L. Lipshitz, 1987, Algebraic power series and diagonals, *J. Number Theory* **26** 46–67. [http://dx.doi.org/10.1016/0022-314X\(87\)90095-3](http://dx.doi.org/10.1016/0022-314X(87)90095-3)
11. A. Bostan, S. Boukraa, J-M. Maillard, J-A. Weil, *Diagonal of rational functions and selected differential Galois groups*, (2015), *J. Phys. A* **48**: Math. Theor. 504001 (29 pages). [arXiv:1507.03227v2 \[math-ph\]](https://arxiv.org/abs/1507.03227v2)
12. R. S. Maier (2009), *On rationally parametrized modular equations*, *J. Ramanujan Math. Soc.* **24** pp. 1-73. <http://arxiv.org/abs/math/0611041>
13. M.L. Glasser and A.J. Guttmann, 1994, Lattice Green function (at 0) for the 4D hypercubic lattice, *J. Phys. A* **27** 7011–7014. <http://arxiv.org/abs/cond-mat/9408097>
14. A.J. Guttmann, 2010, Lattice Green’s functions in all dimensions, *J. Phys. A* **43** 305205, 26. <http://arxiv.org/abs/1004.1435>
15. N. Zenine, S. Hassani, J-M. Maillard, *Lattice Green Functions: the seven-dimensional face-centred cubic lattice*, (2015), *J. Phys. A* **48**: Math. Theor. 035205 (18 pages). [arXiv:1409.8615v1 \[math-ph\]](https://arxiv.org/abs/1409.8615v1)
16. S. Hassani, C. Koutschan, J-M. Maillard and N. Zenine, *Lattice Green Functions: the  $d$ -dimensional face-centred cubic lattice,  $d = 8, 9, 10, 11, 12$* , (2016), *J. Phys. A* **49**: Math. Theor. 164003 (30 pages). [arXiv:1601.05657v2 \[math-ph\]](https://arxiv.org/abs/1601.05657v2)
17. Y. Abdelaziz, S. Boukraa, C. Koutschan and J-M. Maillard, *Heun functions and diagonals of rational functions*, 2020 *J. Phys. A* **53**: Math. Theor. 075206 (24 pages).
18. Y. Abdelaziz, S. Boukraa, C. Koutschan and J-M. Maillard, *Heun functions and diagonals of rational functions (unabridged version)*, (2019). <https://arxiv.org/abs/1910.10761>

19. A. Bostan, S. Boukraa, G. Christol, S. Hassani, J-M. Maillard, *Ising  $n$ -fold integrals as diagonals of rational functions and integrality of series expansions*, (2013), *J. Phys. A* **46**: Math. Theor. 185202 (44 pages). <http://arxiv.org/abs/1211.6645v2> 700
20. A. Bostan, S. Boukraa, G. Christol, S. Hassani and J-M. Maillard, *Ising  $n$ -fold integrals as diagonals of rational functions and integrality of series expansions: integrality versus modularity* Preprint, <http://arxiv.org/abs/1211.6031> 702
21. M. Assis, S. Boukraa, S. Hassani, M. van Hoeij, J-M. Maillard and B.M. McCoy, 2012, *Diagonal Ising susceptibility: elliptic integrals, modular forms and Calabi-Yau equations* *J. Phys. A* **45** 075205, 32. <http://arxiv.org/abs/1110.1705> 704
22. S. Boukraa, S. Hassani, J-M. Maillard, J-A. Weil, *Differential algebra on lattice Green functions and Calabi-Yau operators*, (2014), *J. Phys. A* **48**: Math. Theor. 095203 (37 pages) 706
23. A. Bostan, S. Boukraa, A.J. Guttmann, S. Hassani, I. Jensen, J-M. Maillard and N. Zenine, 2009, *High order Fuchsian equations for the square lattice Ising model:  $\chi^{(5)}$*  *J. Phys. A* **42** 275209, 32, <http://arxiv.org/abs/0904.1601> 708
24. S. Boukraa, S. Hassani, I. Jensen, J-M. Maillard and N. Zenine, 2010, *High-order Fuchsian equations for the square lattice Ising model:  $\chi^{(6)}$*  *J. Phys. A* **43** 115201, 22. <http://arxiv.org/abs/0912.4968> 710
25. S. Boukraa, S. Hassani, J-M. Maillard and N. Zenine, 2007, *Singularities of  $n$ -fold integrals of the Ising class and the theory of elliptic curves* *J. Phys. A* **40** pp. 11713-11748. <http://arxiv.org/abs/0706.3367> 712
26. M. Assis, M. van Hoeij, J-M. Maillard, *The perimeter generating functions of three-choice, imperfect, and one-punctured staircase polygons*, (2016), *J. Phys. A* **49**: Math. Theor. 214002 (29 pages). <https://arxiv.org/abs/1602.00868> 714
27. D. van Straten, *Calabi-Yau operators*, (2017). [arXiv:1704.00164v1\[math.AG\]](https://arxiv.org/abs/1704.00164v1) 716
28. A. Bostan, P. Lairez and B. Salvy, *Creative telescoping for rational functions using the Griffiths-Dwork method*, Proceedings ISSAC'13, pp. 93-100, ACM Press, 2013. <http://specfun.inria.fr/bostan/publications/BoLaSa13.pdf> 717
29. S. Chen, M. Kauers and M.F. Singer, *Telescopers for Rational and Algebraic Functions via Residues*, 2012, Proceedings of the 37th International Symposium on Symbolic and Algebraic Computation, ed. by Joris van der Hoeven and Mark van Hoeij, 2012 pp. 130-137. <https://arxiv.org/abs/1201.1954> 720
30. K. Takeuchi, *Commensurability classes of arithmetic triangle groups*, *J. Fac. Science Univ. Tokyo Sec. IA. Math.* **24**, (1977) pp. 201-212. 722
31. J. Voight, *Shimura curves of genus at most two*, *Math. Comp.* **78** pp 1155-1172, (2009). 723
32. C. F. Doran and A. Malmendier, *Calabi-Yau Manifolds Realizing Symplectically Rigid Monodromy Tuples*. <https://arxiv.org/pdf/1503.07500.pdf> 724
33. A. Malmendier, T. Shaska, editors *Higher Genus Curves in Mathematical Physics and Arithmetic Geometry*, Contemporary Mathematics, AMS Special Session Higher Genus Curves and Fibrations in Mathematical Physics and Arithmetic Geometry January 2016. 726
34. F. Chyzak, *The ABC of Creative Telescoping - Algorithms, Bounds, Complexity*, 2014, <https://hal.inria.fr/tel-01069831> 728
35. Pierre Lairez, *Périodes d'intégrales rationnelles: algorithmes et applications*, Thèse de doctorat, <https://pierre.lairez.fr/these.pdf> 729
36. D. Zeilberger, *The Method of Creative Telescoping*, *J. Symbolic Computation* (1991) **11**, pp. 195-204 730
37. M. Kontsevich and D. Zagier, *Periods*, IHES/M/01/22 2001. <https://www.maths.ed.ac.uk/~v1ranick/papers/kontzagi.pdf> 731
38. J. I. Igusa, *Abstract vanishing cycle theory*, *Proc. Japan Acad.*, **34**, (1958) pp. 589-593. 732
39. P. Deligne, *Intégration sur un cycle évanescant*, (1983), *Inventiones Math.* **76**, 1-29-1-43, Springer-Verlag. 733
40. G. Christol, *Diagonales de fractions rationnelles et équations de Picard-Fuchs*, (1984), Groupe de travail d'analyse ultramétrique, Tome 12 (1984-1985) no. 1, Exposé no. 13, 12 p. [http://www.numdam.org/item/GAU\\_1984-1985\\_\\_12\\_1\\_A8\\_0/](http://www.numdam.org/item/GAU_1984-1985__12_1_A8_0/) 734
41. P. Lairez, *Computing periods of rational integrals*, *Mathematics of Computation*, **85**, Number 300, pp. 1719-1752 and [arXiv:1404.5069v3](https://arxiv.org/abs/1404.5069v3) [cs.SC] 736
42. F. Chyzak, *An extension of Zeilberger's fast algorithm to general holonomic functions*, *Discrete Math.* **217** 1-3 (2000) Formal power series and algebraic combinatorics (Vienna 1997) pp. 115-134 738
43. C. Koutschan, *A fast approach to creative telescoping*, *Math. Comput. sci.* **4**, (2-3) pp. 259-266. 740
44. S. Boukraa and J-M. Maillard, *Symmetries of lattice models in statistical mechanics and effective algebraic geometry*, *J. Phys. I France*, (1993) pp. 293-258. 742
45. M.P. Bellon, J-M. Maillard and C-M. Viallet, *Quasi integrability of the sixteen-vertex model*, *Phys. Letters B* **281**, (1992) pp. 315-319. 743
46. S. Boukraa, J-M. Maillard and G. Rollet, *Determinantal identities on integrable mappings*, *International Journal of Modern Physics B* **8**, (1994) pp. 2157-2201. 744
47. M. Bronstein, T. Mudders and J-A. Weil, *On Symmetric Powers of Differential Operators*, ISSAC '97: Proceedings of the 1997 international symposium on Symbolic and algebraic computation 746
48. L. Long, *On Shioda-Inose structures of one-parameter families of K3 surfaces*, *Jour. Number Theory* **109** (2004) pp. 299-318. 748
49. A. Kumar, *Hilbert Modular Surfaces for square discriminants and elliptic subfields of genus 2 function fields*, *Mathematical Sciences* (2015) 2:24, Research in the mathematical Sciences, a Springer Open Journal. [arXiv:1412.2849v2\[math.AG\]](https://arxiv.org/abs/1412.2849v2) (2016). 749
50. A. Kumar, *Elliptic Fibrations on a Generic Jacobian Kummer Surface* (2014). [arXiv:1105.1715v3\[math.AG\]](https://arxiv.org/abs/1105.1715v3) 751
51. A. Kumar and R. Mukamel, *Algebraic models and arithmetic geometry of Teichmüller curves in genus two*. <https://arxiv.org/pdf/1406.7057.pdf> 752
52. N. Elkies and A. Kumar, *K3 surfaces and equations for Hilbert modular surfaces*, *Algebra and Number Theory* **8** (2014), no. 10, 2297-2411. [arXiv: 1209.3527](https://arxiv.org/abs/1209.3527) 754
53. R.M. Kuhn, *Curves of genus 2 with split Jacobian* (2014), *Transactions of the American Mathematical Society* Vol. **307**, No. 1 (May, 1988), pp. 41-49 Published by: American Mathematical Society. 756
54. T. Shaska, *Genus 2 curves with (3,3)-split Jacobian and large automorphism group*. [arXiv:math/0201008v1\[math.AG\]](https://arxiv.org/abs/math/0201008v1), 1 Jan 2002. 757



55. T. Shaska, *Genus 2 fields with degree 3 elliptic subfields*, *Forum Math.* **16**, (2004) pp. 263-280. [arXiv:math/0109155\[math.AG\]](https://arxiv.org/abs/math/0109155) 759
56. T. Shaska and H. Völklein, *Elliptic subfields and automorphisms of genus 2 function fields*. [arXiv:math/0107142v1\[math.AG\]](https://arxiv.org/abs/math/0107142v1), 19 Jul 760  
2001. 761
57. E. Bedford, K. Kim, T. T. Truong, N. Abarenkova and J-M. Maillard, *Degree Complexity of a Family of Birational Maps*, *Math. Phys. Anal. Geom.* Springer 762
58. J-Ch Anglès d'Auriac, J-M Maillard and C M Viallet, *On the complexity of some birational transformations*, *J. Phys. A* **39**: Math. Gen. (2006) pp. 3641-3654 763
59. J-M. Maillard, *Automorphisms of algebraic varieties and Yang-Baxter equations*, *J. Math. Phys.* **27**, 2776, (1986). [doi:10.1063/1.527303](https://doi.org/10.1063/1.527303) 766
60. M.P. Bellon, J-M. Maillard and C-M. Viallet, *Infinite discrete symmetry group for the Yang-Baxter equations: spin models*, *Phys. Lett. A* **157** (1991) 343-353. 767
61. M.P. Bellon, J-M. Maillard and C-M. Viallet, *Infinite discrete symmetry group for the Yang-Baxter equations: vertex models*, *Phys. Lett. B* **260** (1991) 87-100. 768
62. A. Corti, *Polynomial bounds for the number of automorphisms of a surface of general type*, *Ann. Sci. Ecole Norm. Sup.* **24** (1991) pp. 113-137 769
63. E. Szabó, *Bounding automorphism groups*, *Math. Ann.* **304** (1996) pp. 801-811. 770
64. C.D. Hacon, J. McKernan and C. Xu, *On the birational automorphisms of varieties of general type*, *Annals of Mathematics* **177**, (2013) pp. 1077-1111 and [arXiv:1011.1464v2 \[math.AG\]](https://arxiv.org/abs/1011.1464v2) 771
65. Y. Abdelaziz, J-M. Maillard, *Modular forms, Schwarzian conditions, and symmetries of differential equations in physics*, (2017), *J. Phys. A* **50**: Math. Theor. 215203 (44 pages). [arXiv:1611.08493v3\[math-ph\]](https://arxiv.org/abs/1611.08493v3) 772
66. G. Almkvist, Ch. van Enckevort, D. van Straten and W. Zudilin, *Tables of Calabi-Yau equations*, (2010). [arXiv:math/0507430v2\[math.AG\]](https://arxiv.org/abs/math/0507430v2) 773
67. A. Bostan, S. Boukraa, S. Hassani, M. van Hoeij, J-M. Maillard, J-A. Weil, N. J. Zenine, *The Ising model: from elliptic curves to modular forms and Calabi-Yau equations*, *J. Phys. A* **44**: Math. Theor. (2011) (43 pp) 045204. and [arXiv:1007.69804v1\[math-ph\]](https://arxiv.org/abs/1007.69804v1). and [hal-00684883,version1](https://hal.archives-ouvertes.fr/hal-00684883) 774
68. M.F. Singer, *Solving homogeneous linear differential equations in terms of second order linear differential equations*, *Amer. J. Math.* **107** (1985), no. 3, 663-696. 775
69. A.C. Person, *Solving Homogeneous Linear Differential Equations of Order 4 in Terms of Equations of Smaller Order*, PhD thesis, Raleigh, North Carolina, 2002. <http://www.lib.ncsu.edu/resolver/1840.16/3059> 776
70. M. van Hoeij, *Solving third order linear differential equations in terms of second order equations*, *Proc. ISSAC'07*, 355-360, ACM, 2007. 777
71. R. Fricke and F. Klein, *Vorlesungen über die Theorie der automorphen Funktionen. I*, Druck und Verlag von B. G. Teubner, Leipzig, 1897, p. 366 778
72. P. Boalch and R. Paluba, *Symmetric cubic surfaces and  $G_2$  character varieties*. [arXiv:1305.6594v2\[math.AG\]](https://arxiv.org/abs/1305.6594v2) 779
73. S. Cantat and F. Loray, *Holomorphic dynamics, Painlevé VI equation and Character Varieties*. <https://hal.archives-ouvertes.fr/hal-00186558v2> 780
74. M. Mazzocco and R. Vidunas, *Cubic and Quartic Transformations of the Sixth Painlevé Equation in Terms of Riemann-Hilbert Correspondence*. [arXiv:1305.6594v2\[math.AG\]](https://arxiv.org/abs/1305.6594v2) 781
75. P. A. Griffiths, *On the periods of certain rational integrals I, II*. *Ann. of Math.* (2) **90** (1969), pp. 460-495 and pp. 496-541. 782
76. J.H. Silverman and J. Tate, *Rational Points on elliptic Curves*, Undergraduate Texts in Mathematics, Springer 1992 783
77. M. Sadek, *Minimal genus one curves*, *Functiones et Approximatio*, **46.1** (2012) 117-131. [arXiv:1002.0451v1\[math.NT\]](https://arxiv.org/abs/1002.0451v1) 784
78. B. Poonen, *An explicit algebraic family of genus-one curves violating the Hasse principle*. [arXiv:math/9910124v1\[math.NT\]](https://arxiv.org/abs/math/9910124v1) 785
79. N. Elkies, *Three lectures on elliptic surfaces and curves of high rank*, (2007). [arXiv:0709.2908v1\[math.NT\]](https://arxiv.org/abs/0709.2908v1) 786
80. A.G. Khovanskii, *Newton polyhedra and the genus of complete intersections*, *Funct. Anal. i Ego Pril.* English translation: *Functional Anal. Appl.*, **12** (1978), 38-46. 787
81. <https://mathoverflow.net/questions/16615/calculating-the-genus-of-a-curve-using-the-newton-polygon> 788
82. S. Rabinowitz, *A census of convex lattice polygons with at most one interior lattice point*, *Ars Combinatoria* **28** (1989), pp. 83-96. 789
83. J. Schicho, *Simplification of surface parametrizations—a lattice polygon approach*, *Journal of Symbolic Computation* **36** (2003), pp. 535-554. 790
84. S. Algreen *The Point of a Certain Fivefold over Finite Fields and the Twelfth Power of the eta Function* *Finite Fields and Their applications* **8**, 18-33 (2002). 791
85. S. Boukraa, S. Hassani, J.-M. Maillard, *Noetherian mappings*, *Physica D* **185** (2003) pp. 3-44 792
86. E. Picard, *Sur les intégrales doubles de fonctions rationnelles dont les résidus sont nuls*, *Bulletin des Sciences Mathématiques, série 2* **26** (1902). 793

## Appendix A. Diagonals of rational functions and Picard-Fuchs equations

For simplicity, let us consider rational functions of three variables and double integrals [86]. The diagonal of a rational function of three variables is obtained through its multi-Taylor expansion [19,20]

$$R(x, y, z) = \sum_m \sum_n \sum_l a_{m,n,l} \cdot x^m y^n z^l, \quad (\text{A1})$$

by extracting the “diagonal” terms, i.e. the powers of the product  $p = xyz$ :

$$\text{Diag}(R(x, y, z)) = \sum_m a_{m,m,m} \cdot p^m. \quad (\text{A2})$$

Such diagonals are closely related to the integrals of rational functions. For example  $\text{Diag}(R(x, y, z))$  is the constant term (in  $y, z$ ) in the infinite expansion

$$R\left(\frac{p}{yz}, y, z\right) = \sum_{m,n,l \geq 0} a_{m,n,l} \cdot p^m y^{n-m} z^{l-m}, \quad (\text{A3})$$

which can be represented by the integral [35]

$$\frac{1}{(2\pi i)^2} \oint \oint R\left(\frac{p}{yz}, y, z\right) \frac{dy}{y} \wedge \frac{dz}{z}. \quad (\text{A4})$$

The diagonal (A2) is also the constant term (in  $y, z$ ) of

$$R\left(\frac{p}{y}, \frac{y}{z}, z\right) = \sum_{m,n,l \geq 0} a_{m,n,l} \cdot p^m y^{n-m} z^{l-n}, \quad (\text{A5})$$

which is of the form

$$\frac{1}{(2\pi i)^2} \oint \oint \frac{N_p(y, z)}{D_p(y, z)} \frac{dy}{y} \wedge \frac{dz}{z}, \quad (\text{A6})$$

where the numerator  $N_p(y, z)$  and the denominator  $D_p(y, z)$  are polynomials. It is well-known that such integrals satisfy a linear differential equation with respect to  $p$  having rational functions in  $p$  as coefficients, called the Picard-Fuchs equation<sup>47</sup>. The problem of determining such linear differential equations has been started by Griffiths [75] with the assumption that the variety  $D_p(y, z) = 0$  is smooth, but later techniques were developed to include examples with singular points [35,41]. The linear differential equations (Gauss-Manin systems, telescopers) occurring in integrable models [16,23,24] are of order much larger than order two<sup>48</sup> and almost never correspond to smooth varieties. Creative telescoping<sup>49</sup> and more specifically the programs [3] corresponding to a fast approach to creative telescoping [43], are a powerful way to find these linear differential operators annihilating these diagonals of rational functions in the cases emerging naturally in theoretical physics, integrable models, enumerative combinatorics, for which the order of the linear differential operators is quite large [16,23,24] and the variety  $D_p(y, z) = 0$  is (most of the time) not a smooth one. All the pedagogical (but non-trivial) examples of telescopers displayed in this paper can be viewed by an algebraic geometer as a presentation of examples of families of varieties and their Picard-Fuchs equations.

<sup>47</sup> The order of this linear differential equation is generally equal to the rank of the algebraic de Rham cohomology of  $D_p(y, z) = 0$ . For curves of genus  $g$  this rank is  $2g$ .

<sup>48</sup> Since Felix Klein it is well-known that the Picard-Fuchs equation corresponding to the (Weierstrass) elliptic curve corresponds to the hypergeometric function  ${}_2F_1(1/12, 5/12; 1; 1/J)$ .

<sup>49</sup> For a detailed introduction to creative telescoping [36] see for instance [34].

## Appendix B. Maximum number of parameters for families of planar elliptic curves. 837

We have seen, in section 3, that the previous results on diagonals of nine- or ten-parameter families of rational functions of three variables being pullbacked  ${}_2F_1$  hypergeometric functions (and in fact classical modular forms) can actually be seen as corresponding to the fact<sup>50</sup> that the most general biquadratic corresponding to elliptic curves is a nine-parameter family and that the most general ternary cubic corresponding to elliptic curves is a ten-parameter family. One can, for instance, recall page 238 of [76], which amounts to considering the collection of all cubic curves in  $\mathbb{C}P_2$  with the homogeneous equation 838  
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$$ax^3 + bx^2y + cxy^2 + dy^3 + ex^2z + fxyz^2 + gy^2z + hyz^2 + iz^3 + jxyz = 0, \quad (A7)$$

and the associated problems of passing through nine given points. One can also recall the ternary cubics in [77,78] and other problems of elliptic curves of high rank [79] (see the concept of Neron-Severi rank). 845  
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Since the rational functions of three variables we consider are essentially encoded by the denominator of these rational functions, and in the cases we have considered, the emergence of pullbacked  ${}_2F_1$  hypergeometric functions (and in fact classical modular forms) corresponds to the fact that the intersection of these denominators with the hyperbola  $p = xyz$  corresponds to elliptic curves, one sees that these rational functions are essentially classified by the possible  $n$ -parameter families  $P(x, y) = 0$  of elliptic curves. 848  
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If one considers a polynomial 854

$$P(x, y) = \sum_m \sum_n a_{m,n} \cdot x^m y^n, \quad (A8)$$

with generic coefficients  $a_{m,n} \in \mathbb{C}$ , then the genus of the algebraic curve defined by  $P$  is determined by the support  $\text{supp}(P) = \{(m, n) \in \mathbb{N}^2 : a_{m,n} \neq 0\}$ . More precisely, the genus equals the number of interior integer lattice points inside the convex hull of  $\text{supp}(P)$  [80] (see also the discussion in [81]). For example, the support of the ten-parameter family (11) consists of the following 10 points in  $\mathbb{N}^2$ :

$$(0, 0), (0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)$$

which form a right triangle of side length 3. Only one of these points is an interior point, namely  $(1, 2)$ , hence the genus is 1. 855  
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Therefore we may ask: which integer lattice polytopes exist which have exactly one interior point and what is the largest such polytope? Not surprisingly, the answer is known: there are (up to transformations like translation, rotation, shearing) exactly 16 different polytopes with a single interior point [82] (see also Figure 5, page 548 in [83]), the above-mentioned right triangle being the one with the highest total number of lattice points. 857  
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This shows that there cannot be a family of elliptic curves with more than ten parameters. 863  
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## Appendix C. Monomial transformations preserving pullbacked hypergeometric results 865 866

More generally, recalling subsection 4.2 in [2] and subsection 4.2 page 17 in [1], let us consider the monomial transformation 867  
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$$(x, y, z) \longrightarrow M(x, y, z) = (x_M, y_M, z_M) = (x^{A_1} \cdot y^{A_2} \cdot z^{A_3}, x^{B_1} \cdot y^{B_2} \cdot z^{B_3}, x^{C_1} \cdot y^{C_2} \cdot z^{C_3}), \quad (A9)$$

<sup>50</sup> Well-known in integrable models and integrable mappings.

where the  $A_i$ 's,  $B_i$ 's and  $C_i$ 's are positive integers such that  $A_1 = A_2 = A_3$  is excluded (as well as  $B_1 = B_2 = B_3$  and  $C_1 = C_2 = C_3$ ), and that the determinant<sup>51</sup> of the  $3 \times 3$  matrix [1,2]

$$\begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}, \quad (\text{A10})$$

is not equal to zero<sup>52</sup>, and that:

$$A_1 + B_1 + C_1 = A_2 + B_2 + C_2 = A_3 + B_3 + C_3. \quad (\text{A11})$$

We will denote by  $n = A_i + B_i + C_i$  the integer in these three equal sums (A11). Condition (A11) is introduced in order to impose that the product<sup>53</sup> of  $x_M y_M z_M$  is an integer power of the product of  $x y z$ :  $x_M y_M z_M = (x y z)^n$ .

If we take a rational function  $\mathcal{R}(x, y, z)$  in three variables and perform such a monomial transformation (A9)  $(x, y, z) \rightarrow M(x, y, z)$ , on this rational function  $\mathcal{R}(x, y, z)$ , we get another rational function that we denote by  $\tilde{\mathcal{R}} = \mathcal{R}(M(x, y, z))$ . Now the diagonal of  $\tilde{\mathcal{R}}$  is the diagonal of  $\mathcal{R}(x, y, z)$  where we have changed  $x$  into  $x^n$ :

$$\Phi(x) = \text{Diag}\left(\mathcal{R}(x, y, z)\right), \quad \text{Diag}\left(\tilde{\mathcal{R}}(x, y, z)\right) = \Phi(x^n). \quad (\text{A12})$$

#### Appendix D. Weierstrass and Legendre forms

The telescope of the rational function in three variables

$$\frac{xy}{(1+y)^2 - x \cdot (1-x) \cdot (x - xyz)}, \quad (\text{A13})$$

associated<sup>54</sup> with the elliptic curve in a *Weierstrass form*:

$$(1+y)^2 - x \cdot (1-x) \cdot (x-p) = 0, \quad (\text{A14})$$

is the order-two linear differential operator

$$L_2 = -1 + 4 \cdot (1-2x) \cdot D_x + 4 \cdot x \cdot (1-x) \cdot D_x^2, \quad (\text{A15})$$

which has the hypergeometric solution:

$$\begin{aligned} & {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right) \\ &= (1-x+x^2)^{-1/4} \cdot {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; \frac{27}{4} \cdot \frac{x^2 \cdot (1-x)^2}{(1-x+x^2)^3}\right). \end{aligned} \quad (\text{A16})$$

<sup>51</sup> Note a typo in the footnote 28 page 17 of [1] as well as in the second footnote page 18 in [2]. The sentence has been truncated. One should read: For  $n = 1$ , the  $3 \times 3$  matrix (A10) is stochastic and transformation (A9) is a birational transformation if the determinant of the matrix (A10) is  $\pm 1$ .

<sup>52</sup> We want the rational function  $\tilde{\mathcal{R}} = \mathcal{R}(M(x, y, z))$  deduced from the monomial transformation (A9) to remain a rational function of three variables and not of two, or one, variables.

<sup>53</sup> Recall that taking the diagonal of a rational function of three variables extracts, in the multi-Taylor expansion, only the terms that are  $n$ -th powers of the product  $xyz$ .

<sup>54</sup> The diagonal extracts the terms function of the product  $p = xyz$  in the multi-Taylor series.

The elliptic curve (A14) has the Hauptmodul 885

$$\mathcal{H} = \frac{27}{4} \cdot \frac{p^2 \cdot (1-p)^2}{(1-p+p^2)^3}. \quad (\text{A17})$$

in agreement with the pullback in (A16). 886

*Appendix D.1. K3 surfaces as products or foliations of two elliptic curves.* 887

The examples of section 4 correspond to denominators which are algebraic varieties that can be seen as Weierstrass elliptic curves for fixed values of all the variables except two. Let us show other simple telescopers for rational functions with denominators which are algebraic varieties with some foliation in elliptic curves<sup>55</sup>. 888  
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The telescoper of the rational function in four variables 892

$$\frac{xyz}{(1+z)^2 - x \cdot (1-x) \cdot y \cdot (x-y) \cdot (y-xyzw)}, \quad (\text{A18})$$

associated with the  $K_3$  surface written in a Legendre form<sup>56</sup> 893

$$(1+z)^2 - x \cdot (1-x) \cdot y \cdot (x-y) \cdot (y-p) = 0, \quad (\text{A19})$$

is an order-three *self-adjoint*<sup>57</sup> linear differential operator  $L_3$  894

$$L_3 = x \cdot (2\theta + 1)^3 - 8 \cdot \theta^3, \quad (\text{A20})$$

which has the following  ${}_3F_2$  solution (which is also, because of Clausen's formula, the square of a  ${}_2F_1$  function): 895  
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$${}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1; x\right) = {}_2F_1\left(\frac{1}{4}, \frac{1}{4}; 1; x\right)^2. \quad (\text{A21})$$

The  $K_3$  surface (A19) can be seen as associated with the product of two Weierstrass elliptic curves<sup>58</sup> of Hauptmoduls respectively: 897  
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$$\mathcal{H}_x = \frac{27}{4} \cdot \frac{p^2 \cdot (1-p)^2}{(1-p+p^2)^3}, \quad \mathcal{H}_y = \frac{27}{4} \cdot \frac{y^2 \cdot (1-y)^2}{(1-y+y^2)^3}. \quad (\text{A22})$$

This order-three linear differential operator  $L_3$  is the symmetric square of the order-two linear differential operator 899  
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$$M_2 = -1 + 8 \cdot (2-3x) \cdot D_x + 16 \cdot x \cdot (1-x) \cdot D_x^2, \quad (\text{A23})$$

which has the hypergeometric solutions: 901

$${}_2F_1\left(\frac{1}{4}, \frac{1}{4}; 1; x\right) = \left(1 - \frac{x}{4}\right)^{-1/4} \cdot {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; -\frac{27 \cdot x^2}{(x-4)^3}\right). \quad (\text{A24})$$

<sup>55</sup> Like K3 surfaces, or Calabi-Yau threefolds.

<sup>56</sup> Along this line see the first equation page 19 of [84].

<sup>57</sup> The order-three linear differential operator is thus the symmetric square of an order-two linear differential operator.

<sup>58</sup>  $K_3$  surfaces are not abelian varieties, but they are "close" to abelian varieties: from a creative telescoping viewpoint they essentially can be seen as products of two elliptic curves.

Appendix D.2. Calabi-Yau threefolds as foliation in three elliptic curves. 902

The telescope of the rational function in five variables  $x, y, z, v$  and  $w$  903

$$\frac{xyzv}{(1+w)^2 - x \cdot (1-x) \cdot y \cdot (x-y) \cdot z \cdot (y-z) \cdot (z-x) y z v w}, \quad (\text{A25})$$

associated<sup>59</sup> with the Calabi-Yau threefold written in a Legendre form 904

$$(1+w)^2 - x \cdot (1-x) \cdot y \cdot (x-y) \cdot z \cdot (y-z) \cdot (z-p) = 0, \quad (\text{A26})$$

is an order-four (self-adjoint) linear differential operator  $L_4$  905

$$L_4 = 16 \cdot \theta^4 - x \cdot (2\theta + 1)^4, \quad (\text{A27})$$

which is a Calabi-Yau operator<sup>60</sup> with the  ${}_4F_3$  solution: 906

$${}_4F_3\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1, 1; x\right). \quad (\text{A28})$$

For  $y$  and  $z$  fixed, the Calabi-Yau threefold (A26) is foliated in genus-one curves 907

$$(1+w)^2 - \lambda \cdot x \cdot (1-x) \cdot (x-y) = 0, \quad (\text{A29})$$

where  $\lambda$  is the constant expression ( $p$  is fixed): 908

$$\lambda = y \cdot z \cdot (y-z) \cdot (z-p). \quad (\text{A30})$$

The Hauptmodul of these genus-one curves is independent of  $p$  and  $z$ , reading: 909

$$\mathcal{H}_{y,z} = \frac{27}{4} \cdot \frac{y^2 \cdot (1-y)^2}{(1-y+y^2)^3}. \quad (\text{A31})$$

Similarly for  $x$  and  $z$  fixed, the Calabi-Yau threefold (A26) is foliated in genus-one curves 910

$$(1+w)^2 - \mu \cdot y \cdot (x-y) \cdot (y-z) = 0, \quad (\text{A32})$$

where  $\mu$  is the constant expression ( $p$  is fixed): 911

$$\mu = x \cdot z \cdot (1-x) \cdot (z-p). \quad (\text{A33})$$

The genus-one curves (A32) can be written in a simpler Weierstrass form: 912

$$(1+w)^2 - \rho \cdot Y \cdot \left(1-Y\right) \cdot \left(Y - \frac{z}{x}\right) = 0, \quad (\text{A34})$$

where the constant  $\rho$  reads  $\rho = \mu \cdot x^3$ , and the variable  $y$  has been rescaled into  $Y = y/x$ . 913  
 The Hauptmodul of these genus-one curves (A32) is the same as the Hauptmodul of the 914  
 genus-one curves (A29), and corresponds to expression (A31) where  $y$  has been changed 915  
 into  $z/x$  (see the canonical form (A34)), namely: 916

$$\mathcal{H}_{x,z} = \frac{27}{4} \cdot \frac{x^2 \cdot z^2 \cdot (x-z)^2}{(x^2 - xz + z^2)^3}. \quad (\text{A35})$$

<sup>59</sup> The diagonal extracts the terms function of the product  $p = xyzvw$  in the multi-Taylor series.

<sup>60</sup> This linear differential operator is self-adjoint, its exterior square is of order five, and it is MUM (maximum unipotent monodromy [27,66,67]).

Similarly for  $x$  and  $y$  fixed, the Calabi-Yau threefold (A26) is foliated in genus-one curves, 917

$$(1 + w)^2 - v \cdot z \cdot (y - z) \cdot (z - p) = 0, \quad (\text{A36})$$

where  $v$  reads: 918

$$v = x \cdot (1 - x) \cdot y \cdot (x - y). \quad (\text{A37})$$

A reduction to a canonical Weierstrass form similar to (A34) gives immediately the Hauptmodul of the genus-one curve (A36) which reads: 919 920

$$\mathcal{H}_{x,y} = \frac{27}{4} \cdot \frac{y^2 \cdot p^2 \cdot (y - p)^2}{(y^2 - y p + p^2)^3}. \quad (\text{A38})$$

The Calabi-Yau threefold (A26) thus has a foliation in a triple of elliptic curves  $\mathcal{E}_1$ ,  $\mathcal{E}_2$ , and  $\mathcal{E}_3$ . 921 922

## Appendix E. Rational functions with tri-quadratic and $N$ -quadratic denominators. 923

### Appendix E.1. Rational functions with tri-quadratic denominators. 924

Let us consider the most general tri-quadratic surface 925

$$\sum_{m=0,1,2} \sum_{n=0,1,2} \sum_{l=0,1,2} a_{m,n,l} \cdot x^m y^n z^l = 0, \quad (\text{A39})$$

depending on  $27 = 3^3$  parameters  $a_{m,n,l}$ . It can be rewritten as: 926

$$A(x, y) \cdot z^2 + B(x, y) \cdot z + C(x, y) = 0. \quad (\text{A40})$$

It is straightforward to see that condition (A40) is preserved by the birational involution  $I_z$  927

$$I_z : \quad \left( x, y, z \right) \longrightarrow \left( x, y, \frac{C(x, y)}{A(x, y)} \cdot \frac{1}{z} \right), \quad (\text{A41})$$

and we have of course two other similar birational involutions  $I_x$  and  $I_y$  that single out  $x$  and  $y$ , respectively. The (generically) infinite-order birational transformations  $K_x = I_y \cdot I_z$ ,  $K_y = I_z \cdot I_x$  and  $K_z = I_x \cdot I_y$  are birational symmetries of the surface (A39) or (A40). They are related by  $K_x \cdot K_y \cdot K_z = \text{identity}$ . Note that the birational transformation  $K_x$  preserves  $x$ . The iteration of the (generically) infinite-order birational transformation  $K_x$  gives elliptic curves. Since equation (A39) or (A40) is preserved by  $K_x$ , which also preserves  $x$ , the equation of the elliptic curves corresponding to the iteration<sup>61</sup> of  $K_x$  is actually (A39) for fixed values of  $x$ . Equation (A39), for fixed values of  $x$ , is a (general) biquadratic curve in  $y$  and  $z$  and is thus an elliptic curve depending on  $x$ . Therefore one has a canonical foliation of the algebraic surface (A39) in elliptic curves. Of course the iteration of  $K_y$  (resp.  $K_z$ ) also yields elliptic curves, and similarly yields two other foliations in elliptic curves. 928 929 930 931 932 933 934 935 936 937 938

We have a foliation in two families of elliptic curves  $\mathcal{E}$  and  $\mathcal{E}'$  of the surface. Consequently, this tri-quadratic surface (A39), having an infinite set of birational automorphisms, an infinite set of birational symmetries, cannot be of the "general type" (it has Kodaira dimension less than 2). 939 940 941 942

<sup>61</sup> The birational transformation  $K_x$  maps the elliptic curve onto itself (self-map). One can use the iteration of the birational transformation  $K_x$  to actually visualize the elliptic curve [45,85].

*Appendix E.2. Rational functions with N-quadratic denominators.*

The calculations of Appendix E.1 can straightforwardly be generalized to N-quadratic equations, writing the N-quadratic (72) as

$$A(x_1, x_2, \dots, x_{N-1}) \cdot x_N^2 + B(x_1, x_2, \dots, x_{N-1}) \cdot x_N + C(x_1, x_2, \dots, x_{N-1}) = 0, \tag{A42}$$

and introducing the birational involution  $I_N$

$$I_N : \begin{aligned} & (x_1, x_2, \dots, x_N) \\ & \longrightarrow \left( x_1, x_2, \dots, x_{N-1}, \frac{C(x_1, x_2, \dots, x_{N-1})}{A(x_1, x_2, \dots, x_{N-1})} \cdot \frac{1}{x_N} \right). \end{aligned} \tag{A43}$$

Similarly to Appendix E.1, we can introduce N involutive birational transformations  $I_m$  and consider the products of two such involutive birational transformations  $K_{m,n} = I_m \cdot I_n$ . These  $K_{m,n}$ 's are (generically) infinite-order birational transformations preserving the  $N - 2$  variables that are not  $x_m$  and  $x_n$ .

Using such remarkable N-variable algebraic varieties, with an infinite set of birational automorphisms, one can build rational functions of  $N + 1$  variables, any of the parameter of the algebraic variety, becoming an arbitrary rational<sup>62</sup> function of the product  $p = x_1 x_2 \dots x_N$  in order to build the denominator of the rational function. The telescopers of such rational functions are seen (experimentally using creative telescoping) to be of substantially smaller order than the ones for rational functions whose denominators are, after reduction by  $p = x_1 x_2 \dots x_N$ , associated with algebraic varieties of "general type".

**Appendix F. Telescopers of rational functions of several variables: some examples**

Let us consider here the following family of rational functions in four variables

$$R(x, y, z, u) = \frac{1}{1 + 3y + z + 9yz + 11z^2y + 3ux + x \cdot P(x, y, z)}, \tag{A44}$$

where  $P(x, y, z)$  is an arbitrary polynomial of the three variables  $x, y$  and  $z$ .

*Appendix F.1. Telescopers of rational functions of several variables: a second example with four variables*

Let us now consider the rational function in four variables  $x, y, z, u$ :

$$R(x, y, z, u) = \frac{1}{1 + 3y + z + 9yz + 11z^2y + 3ux + 9x + 2xy + 5xz + 7x^2y}, \tag{A45}$$

which corresponds to  $P(x, y, z) = 9 + 2y + 5z + 7xy$ . The telescoper of this rational function of four variables is the same order-two linear differential operator  $L_2$  as for the telescoper of (91). It has the same pullbacked hypergeometric solution (92). The diagonal of the rational function (A45) is the expansion of (92), namely (93).

Performing the intersection of the codimension-one algebraic variety

$$1 + 3y + z + 9yz + 11z^2y + 3ux + 9x + 2xy + 5xz + 7x^2y = 0,$$

<sup>62</sup> Or even an arbitrary algebraic function of the product  $p = x_1 x_2 \dots x_N$ , with a Taylor series expansion at  $p = 0$ , the diagonal of rational functions becoming diagonal of algebraic functions.



corresponding to the denominator of (A45), with the hyperbola  $p = xyz u$  amounts to eliminating, for instance,  $u$  (writing  $u = \frac{p}{xyz}$ ). This gives  $P_u = 0$  where  $P_u$  reads:

$$P_u = 7x^2y^2z + 2xy^2z + 5xyz^2 + 9xyz + 11y^2z^3 + 9y^2z^2 + 3y^2z + yz^2 + yz + 3p. \quad (\text{A46})$$

Assuming  $x$  to be constant, the previous condition  $P_u(y, z) = 0$  is an algebraic curve. Calculating its genus, one finds immediately that it has genus one. Calculating its  $j$ -invariant, one deduces the expression of the Hauptmodul  $\mathcal{H}_{p,x} = \frac{1728}{J}$  as a rational expression of  $p$  and  $x$ :

$$\mathcal{H}_{p,x} = \frac{1728}{J} = -\frac{46656 p^3 \cdot (7x^2 + 2x + 3)^2 \cdot N}{D^3}, \quad (\text{A47})$$

where  $N$  is a polynomial expression of degree eight in  $w$  and three in  $p$ , and  $D$  is a polynomial expression of degree four in  $w$  and two in  $p$ . In the  $x \rightarrow 0$  limit of the Hauptmodul  $\mathcal{H}_{p,x} = \frac{1728}{J}$ , one finds:

$$\mathcal{H}_p = -\frac{419904 \cdot p^3 \cdot (5 - 12p - 19440p^2 + 2665872p^3)}{(1 - 2592p^2)^3}, \quad (\text{A48})$$

which is actually the Hauptmodul in (92). In other words, the exact expression of the diagonal of the rational function (A45), which is (92), and is essentially encapsulated in the Hauptmodul in (92), could have been obtained from the  $x = 0$  selection of the Hauptmoduls  $\mathcal{H}_{p,x}$ .

#### Appendix F.2. Telescopers of rational functions of several variables: a third example with four variables

Let us consider the rational function in four variables  $x, y, z, u$ :

$$R(x, y, z, u) = \frac{1}{1 + 3y + z + 9yz + 11z^2y + 3ux + x \cdot (y^2z^2 + xy^3)}, \quad (\text{A49})$$

which corresponds to  $P(x, y, z) = y^2z^2 + xy^3$  in the family (A44). Again, the telescoper of this rational function of four variables is the same order-two linear differential operator  $L_2$  as for the telescoper of (91). It has the same pullbacked hypergeometric solution (92). Actually the diagonal of the rational function (91) is the expansion (93) of the pullbacked hypergeometric function (92). In this case (A49), the elimination of  $u = \frac{p}{xyz}$  in the vanishing condition of the denominator (A49) gives the algebraic curve:

$$x^2y^4z + xy^3z^3 + 11y^2z^3 + 9y^2z^2 + 3y^2z + yz^2 + yz + 3p = 0. \quad (\text{A50})$$

For  $x$  fixed (and of course  $p$  fixed) this algebraic curve (A50) is a genus-five curve, but, of course, in the  $x = 0$  case it reduces to the same genus-one curve as for the first example (91), namely:

$$11y^2z^3 + 9y^2z^2 + 3y^2z + yz^2 + yz + 3p = 0. \quad (\text{A51})$$

which corresponds to the Hauptmodul (A48).

The generalization of this result is straightforward. Let us consider the rational function in four variables  $x, y, z$  and  $u$

$$R(x, y, z, u) = \frac{1}{1 + 3y + z + 9yz + 11z^2y + 3ux + x \cdot P(x, y, z)}, \quad (\text{A52})$$

where  $P(x, y, z)$  is an arbitrary polynomial of the three variables  $x, y$  and  $z$ . On a large set of examples one verifies that the diagonal of (A52) is actually the expansion (93) of the pullbacked hypergeometric function (92):

$$1 + 648x^2 - 72900x^3 + 1224720x^4 - 330674400x^5 + 23370413220x^6 - 1276733858400x^7 + 180019474034400x^8 - 12013427240614800x^9 + \dots \quad (\text{A53})$$

However, as far as creative telescoping calculations are concerned<sup>63</sup>, the telescoper corresponding to different polynomials  $P(x, y, z)$  becomes quickly a quite large *non-minimal* linear differential operator. For instance, even for the simple polynomial  $P(x, y, z) = x + y$ , one obtains a quite large order-ten telescoper. Of course, since this telescoper has the pullbacked hypergeometric function (92) as a solution, it is not minimal, it is right-divisible by the order-two linear differential operator having (92) as a solution. It is straightforward to see that the previous elimination of  $u = \frac{p}{xyz}$  in the vanishing condition of the denominator (A52) gives an algebraic curve<sup>64</sup>

$$11y^2z^3 + 9y^2z^2 + 3y^2z + yz^2 + yz + 3p + yz \cdot P(x, y, z) = 0. \quad (\text{A54})$$

which reduces again, in the  $x = 0$  case, to the same genus-one curve (A51).

With that general example (A52) we see that there is an infinite set of rational functions depending on an arbitrary polynomial  $P(x, y, z)$  of three variables whose diagonals are actually a pullbacked  ${}_2F_1$  hypergeometric solution, namely (92).

<sup>63</sup> Using the HolonomicFunctions package [3].

<sup>64</sup> Of arbitrary large genus for increasing degrees of the polynomial  $P(x, y, z)$ .