Proof of the Wilf–Zeilberger Conjecture for Mixed Hypergeometric Terms

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Dedicated to Professor Peter Paule on the occasion of his 60th birthday

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Abstract

In 1992, Wilf and Zeilberger conjectured that a hypergeometric term in several discrete and continuous variables is holonomic if and only if it is proper. Strictly speaking the conjecture does not hold, but it is true when reformulated properly: Payne proved a piecewise interpretation in 1997, and independently, Abramov and Petkovšek in 2002 proved a conjugate interpretation. Both results address the pure discrete case of the conjecture. In this paper we extend their work to hypergeometric terms in several discrete and continuous variables and prove the conjugate interpretation of the Wilf–Zeilberger conjecture in this mixed setting.

Keywords: Wilf–Zeilberger Conjecture, hypergeometric term, properness, holonomic function, D-finite function, Ore-Sato Theorem

1. Introduction

The method of creative telescoping was put on an algorithmic fundament by Zeilberger [27, 28] in the early 1990's, and it has been a powerful tool in the study of special function identities since then. Zeilberger's algorithms (for binomial / hypergeometric sums and for hyperexponential integrals) terminate for *holonomic* inputs. The holonomicity of functions is defined in terms of the dimension of their annihilating ideals in algebras of operators with polynomial coefficients; in general, it is difficult to detect this property. In 1992, Wilf and

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Zeilberger [26] gave a more elegant and constructive proof that their methods are applicable to so-called *proper hypergeometric* terms, which are expressed in an explicit form. Since all examples considered in their paper are both proper and holonomic, Wilf and Zeilberger then presented the following conjecture in [26, p. 585].

Conjecture 1 (Wilf and Zeilberger, 1992)**.** *A hypergeometric term is holonomic if and only if it is proper.*

It was observed in [21, 3] that the conjecture is not true when it is taken literally, so it needs to be modified in order to be correct (see Theorem 29). For example, the term $|k_1 - k_2|$ defining a sequence over $\mathbb{N}^2$ is easily seen to be hypergeometric (since it satisfies the first-order system below) and holonomic because its generating function

$$\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} |k_1 - k_2| z_1^{k_1} z_2^{k_2} = \frac{z_1^2 z_2 + z_1 z_2^2 - 4 z_1 z_2 + z_1 + z_2}{(1-z_1)^2 (1-z_2)^2 (1-z_1 z_2)}$$

is a rational function (see Definitions 22 and 23, and Theorem 21). But $|k_1 - k_2|$ is not proper [3, p. 396]; a similar counter-example was given in [21, p. 55]. Payne in his 1997 Ph.D. dissertation [21, Chap. 4] modified and proved the conjecture in a piecewise sense; more specifically, it was shown that the domain of a holonomic hypergeometric term can be expressed as the union of a linear algebraic set and a finite number of convex polyhedral regions (the "pieces") such that the term is proper on each region. In the case of $|k_1 - k_2|$, the linear algebraic set is the line $k_1 = k_2$ and the polyhedral regions are $k_1 - k_2 > 0$ and $k_1 - k_2 < 0$ where the proper terms are $k_1 - k_2$ and $k_2 - k_1$ respectively. Unaware of [21], Abramov and Petkovšek [3] in 2002 solved the problem by showing that a holonomic hypergeometric term is conjugate to a proper term, which means roughly that both terms are solutions to a common (nontrivial) system of equations. The holonomic term $|k_1 - k_2|$ and the proper term $k_1 - k_2$ are easily seen to be solutions of the system

$$\begin{aligned}(k_1 - k_2)T(k_1 + 1, k_2) - (k_1 - k_2 + 1)T(k_1, k_2) &= 0,\\ (k_1 - k_2)T(k_1, k_2 + 1) - (k_1 - k_2 - 1)T(k_1, k_2) &= 0.\end{aligned}$$

The special case of two variables has also been shown by Hou [14, 15] and by Abramov and Petkovšek [2]. In this paper, we consider the general mixed case, but only the conjugate interpretation. For the sake of simplicity, we regard hypergeometric terms as literal functions only of the discrete variables and interpret their values as elements of a differential field. We define exponentiation of these field elements only in a formal sense, similar to what is done in symbolic integration, see Remark 12.

If Conjecture 1 above were verified, then one could algorithmically detect the holonomicity of hypergeometric terms by checking properness with the algorithms in [3, 9]. This is important because it gives a simple test for the termination of Zeilberger's algorithm. In the bivariate case, several termination criteria are developed in [1, 10, 8].

2. Mixed hypergeometric terms

Hypergeometric terms play a prominent role in combinatorics; and also a large class of special functions used in mathematics and physics can be defined in terms of them, namely as hypergeometric series. Wilf and Zeilberger in [25, 24, 26] developed an algorithmic proof theory for identities involving hypergeometric terms.

Throughout the paper, we let $\mathbb{F}$ denote an algebraically closed field of characteristic zero, and $\boldsymbol{t} = (t_1, \ldots, t_m)$ and $\boldsymbol{k} = (k_1, \ldots, k_n)$ be two sets of variables; we view $\boldsymbol{t}$ and $\boldsymbol{k}$ as continuous and discrete variables, respectively. Note that bold symbols are used for vectors and that $\boldsymbol{u} \cdot \boldsymbol{v} = u_1 v_1 + \cdots + u_n v_n$ denotes their inner product. For an element $f \in \mathbb{F}(\boldsymbol{t}, \boldsymbol{k})$, define

$$D_i(f) = \frac{\partial f}{\partial t_i} \quad \text{and} \quad S_j(f(\boldsymbol{t}, \boldsymbol{k})) = f(\boldsymbol{t}, k_1, \ldots, k_{j-1},\, k_j + 1,\, k_{j+1}, \ldots, k_n)$$

for all i, j with $1 \leq i \leq m$ and $1 \leq j \leq n$. The operators D_i and S_j are called *derivations* and *shifts*, respectively. The operators $D_1, \ldots, D_m, S_1, \ldots, S_n$ commute pairwise on $\mathbb{F}(\boldsymbol{t}, \boldsymbol{k})$.

The field $\mathbb{F}(\boldsymbol{t})$ becomes a differential field [17, p. 58] with the derivations $D_1, \ldots, D_m$. Let $\mathbb{U}$ be a universal differential extension of $\mathbb{F}(\boldsymbol{t})$, in which all consistent systems of algebraic differential equations with coefficients in $\mathbb{F}(\boldsymbol{t})$ have solutions and the extended derivations $D_1, \ldots, D_m$ commute in $\mathbb{U}$. For the existence of such a universal field, see Kolchin's book [17, p. 134, Theorem 2]. By a multivariate sequence over $\mathbb{U}$, we mean a map $H \colon \mathbb{N}^n \to \mathbb{U}$; instead of $H(\boldsymbol{k})$ we will often write $H(\boldsymbol{t}, \boldsymbol{k})$ in order to emphasize also the dependence on $\boldsymbol{t}$. Let $\mathbb{S}$ be the set of all multivariate sequences over $\mathbb{U}$. We define the addition and multiplication of two elements of $\mathbb{S}$ coordinatewise, so that the invertible elements in $\mathbb{S}$ are those sequences whose entries are all invertible in $\mathbb{U}$. The shifts S_j operate on sequences in an obvious way, and the derivations on $\mathbb{U}$ are extended to $\mathbb{S}$ coordinatewise.

A polynomial $p \in \mathbb{F}[\boldsymbol{t}, \boldsymbol{k}]$ can be viewed as an element of $\mathbb{S}$ in a natural way: for each $\boldsymbol{v} \in \mathbb{N}^n$ the entry of the sequence is given by the value $p(\boldsymbol{t}, \boldsymbol{v})$. However, in order to embed the field of rational functions $\mathbb{F}(\boldsymbol{t}, \boldsymbol{k})$ into $\mathbb{S}$, we need the following equivalence relation among multivariate sequences, used in [3].

Definition 2 (Equality modulo an algebraic set)**.** *Two multivariate sequences $H_1(\boldsymbol{t}, \boldsymbol{k})$ and $H_2(\boldsymbol{t}, \boldsymbol{k})$ are said to be* equal modulo an algebraic set*, denoted by $H_1 \overset{\text{alg}}{=} H_2$, if there is a nonzero polynomial $p \in \mathbb{F}[k_1, \ldots, k_n]$ such that*

$$\{\boldsymbol{v} \in \mathbb{N}^n \mid H_1(\boldsymbol{t}, \boldsymbol{v}) \neq H_2(\boldsymbol{t}, \boldsymbol{v})\} \subseteq \{\boldsymbol{v} \in \mathbb{F}^n \mid p(\boldsymbol{v}) = 0\}.$$

A multivariate sequence H is nontrivial *if $H \overset{\text{alg}}{\neq} 0$.*

Equality modulo an algebraic set is not only an equivalence relation in $\mathbb{S}$, but also a congruence [3], i.e., $H_1 + H_2 \overset{\text{alg}}{=} H_1' + H_2'$ and $H_1 H_2 \overset{\text{alg}}{=} H_1' H_2'$ if $H_1 \overset{\text{alg}}{=} H_1'$ and $H_2 \overset{\text{alg}}{=} H_2'$. Now, to every rational function we can associate a sequence in $\mathbb{S}$.

Definition 3. *Let $f = p/q$ with $p, q \in \mathbb{F}[\boldsymbol{t}, \boldsymbol{k}]$ be a rational function. Then we define the* rational sequence $F(\boldsymbol{t}, \boldsymbol{k}) \in \mathbb{S}$ *corresponding to $f(\boldsymbol{t}, \boldsymbol{k})$ as follows: For each $\boldsymbol{v} \in \mathbb{N}^n$*

$$F(\boldsymbol{t}, \boldsymbol{v}) = \begin{cases} f(\boldsymbol{t}, \boldsymbol{v}), & \textit{if } q(\boldsymbol{t}, \boldsymbol{v}) \neq 0, \\ 0, & \textit{otherwise.} \end{cases}$$

Definition 4 (Mixed hypergeometric term)**.** *A multivariate sequence $H(\boldsymbol{t}, \boldsymbol{k}) \in \mathbb{S}$ is said to be a* (mixed) hypergeometric term *over $\mathbb{F}(\boldsymbol{t}, \boldsymbol{k})$ if there are polynomials $p_i, q_i \in \mathbb{F}[\boldsymbol{t}, \boldsymbol{k}]$ with $q_i \neq 0$ for $i = 1, \ldots, m$ and $u_j, v_j \in \mathbb{F}[\boldsymbol{t}, \boldsymbol{k}] \setminus \{0\}$ for $j = 1, \ldots, n$ such that*

$$q_i D_i(H) = p_i H \quad \textit{and} \quad v_j S_j(H) = u_j H.$$

Strictly speaking, such sequences should be called *hypergeometric-hyper-exponential*, but for the sake of brevity we call them just *hypergeometric*. Let $a_i(\boldsymbol{t}, \boldsymbol{k}) = p_i/q_i$ and $b_j(\boldsymbol{t}, \boldsymbol{k}) = u_j/v_j$ with p_i, q_i, u_j, v_j as in the above definition. Then we can write

$$D_i(H) \stackrel{\text{alg}}{=} a_i H \quad \text{and} \quad S_j(H) \stackrel{\text{alg}}{=} b_j H.$$

We call the rational functions a_i and b_j the *certificates* of H. The certificates of a hypergeometric term are not arbitrary rational functions. They satisfy certain compatibility conditions. The following definition is a continuous-discrete extension of the one introduced in [3].

Definition 5 (Compatible rational functions)**.** *We call the rational functions $a_1, \ldots, a_m \in \mathbb{F}(\boldsymbol{t}, \boldsymbol{k})$, $b_1, \ldots, b_n \in \mathbb{F}(\boldsymbol{t}, \boldsymbol{k}) \setminus \{0\}$* compatible *with respect to $\{D_1, \ldots, D_m, S_1, \ldots, S_n\}$ if the following three groups of conditions hold:*

$$D_i(a_j) = D_j(a_i), \qquad 1 \leq i < j \leq m, \tag{1}$$

$$\frac{S_i(b_j)}{b_j} = \frac{S_j(b_i)}{b_i}, \qquad 1 \leq i < j \leq n, \tag{2}$$

$$\frac{D_i(b_j)}{b_j} = S_j(a_i) - a_i, \quad 1 \leq i \leq m \textit{ and } 1 \leq j \leq n. \tag{3}$$

Remark 6. *Let $H \in \mathbb{S}$ be a nontrivial hypergeometric term over $\mathbb{F}(\boldsymbol{t}, \boldsymbol{k})$. By the same argument as in the proof of [3, Prop. 4], we have that the certificates of H are unique (if we take the reduced form of rational functions) and compatible with respect to $\{D_1, \ldots, D_m, S_1, \ldots, S_n\}$.*

3. Structure of compatible rational functions

The structure of rational solutions of the recurrence equation

$$F_1(k_1, k_2 + 1) F_2(k_1, k_2) = F_1(k_1, k_2) F_2(k_1 + 1, k_2)$$

has been described by Ore [20]. Note that Equation (2) is of this form. The multivariate extension of Ore's theorem was obtained by Sato [22] in the 1960s.

The proofs of the piecewise and conjugate interpretations of the discrete case of Wilf and Zeilberger's conjecture were based on the Ore–Sato theorem [21, 3]. In his thesis [7], the first-named author extended the Ore–Sato theorem to the multivariate continuous-discrete case. More recently, this result has been extended further to the case in which also q-shifted variables appear [9]. To present this extension, let us recall some notation and terminologies from [21]. For $s, t \in \mathbb{Z}$ and a sequence of expressions α_i with $i \in \mathbb{Z}$, define

$$\prod_{i}{}_{s}^{t} \alpha_i = \begin{cases} \prod_{i=s}^{t-1} \alpha_i, & \text{if } t \geq s; \\ \prod_{i=t}^{s-1} \alpha_i^{-1}, & \text{if } t < s. \end{cases} \tag{4}$$

We recall the Ore–Sato theorem, following the presentation of Payne's dissertation [21, Thm. 2.8.4]. For the proof of this theorem, one can also see [22, pp. 6–33], [3, Thm. 10] or [13, pp. 9–13].

Theorem 7 (Ore–Sato theorem)**.** *Let $b_1, \ldots, b_n \in \mathbb{F}(\boldsymbol{k})$ be nonzero compatible rational functions, i.e.,*

$$b_i S_i(b_j) = b_j S_j(b_i), \quad \text{for } 1 \leq i < j \leq n.$$

Then there exist a rational function $f \in \mathbb{F}(\boldsymbol{k})$, constants $\mu_1, \ldots, \mu_n \in \mathbb{F}$, a finite set $V \subset \mathbb{Z}^n$, and for each $\boldsymbol{v} = (v_1, \ldots, v_n) \in V$ a univariate monic rational function $r_{\boldsymbol{v}} \in \mathbb{F}(z)$ such that

$$b_j = \frac{S_j(f)}{f} \mu_j \prod_{\boldsymbol{v} \in V} \prod_{\ell}{}_{0}^{v_j} r_{\boldsymbol{v}}(\boldsymbol{v} \cdot \boldsymbol{k} + \ell) \quad \text{for } j = 1, \ldots, n.$$

The continuous analogue of the Ore–Sato theorem was first obtained by Christopher [11] for bivariate compatible rational functions and later extended by Żołądek [29] to the multivariate case using the Integration Theorem of [6, p. 5]. We offer a more algebraic proof using only some basic properties of multivariate rational functions.

Theorem 8 (Multivariate Christopher's theorem)**.** *Let $a_1, \ldots, a_m \in \mathbb{F}(\boldsymbol{t})$ be rational functions such that*

$$D_i(a_j) = D_j(a_i), \quad \text{for } 1 \leq i < j \leq m.$$

Then there exist rational functions $g_0, \ldots, g_L \in \mathbb{F}(\boldsymbol{t})$ and constants $\gamma_1, \ldots, \gamma_L \in \mathbb{F}$ such that

$$a_i = D_i(g_0) + \sum_{\ell=1}^{L} \gamma_\ell \frac{D_i(g_\ell)}{g_\ell} \quad \text{for } i = 1, \ldots, m.$$

Proof. We proceed by induction on m. To show the base case when $m = 1$, we apply the partial fraction decomposition over the algebraically closed field $\mathbb{F}$ to a_1 and get

$$a_1 = \sum_{\ell=1}^{L} \sum_{j=1}^{J} \frac{\alpha_{\ell,j}}{(t_1 - \beta_\ell)^j} \qquad \text{where } \alpha_{\ell,j}, \beta_\ell \in \mathbb{F} \text{ with } \beta_\ell \neq \beta_{\ell'} \text{ for } \ell \neq \ell'.$$

Then the theorem holds by taking

$$g_0 = \sum_{\ell=1}^{L}\sum_{j=2}^{J} \frac{(1-j)^{-1}\alpha_{\ell,j}}{(t_1-\beta_\ell)^{j-1}}, \quad \gamma_\ell = \alpha_{\ell,1}, \quad \text{and} \quad g_\ell = t_1 - \beta_\ell.$$

We now assume that $m \geq 2$ and that the theorem holds for $m-1$. Let $\mathbb{E}$ denote the field $\mathbb{F}(t_2, \ldots, t_m)$. Over the algebraic closure $\overline{\mathbb{E}}$ of $\mathbb{E}$, the base case of the theorem allows us to decompose a_1 into

$$a_1 = D_1(g_0) + \sum_{\ell=1}^{L} \gamma_\ell \frac{D_1(g_\ell)}{g_\ell} = D_1(g_0) + \sum_{\ell=1}^{L} \frac{\gamma_\ell}{t_1 - \beta_\ell}, \tag{5}$$

where $g_0 \in \overline{\mathbb{E}}(t_1)$ and, for $1 \leq \ell \leq L$, we have $g_\ell = t_1 - \beta_\ell$ and $\beta_\ell, \gamma_\ell \in \overline{\mathbb{E}}$ such that the β_ℓ are pairwise distinct.

First, we claim that all γ_ℓ's are actually constants in $\mathbb{F}$. For any $u \in \overline{\mathbb{F}(\boldsymbol{t})}$ and $1 \leq i < j \leq m$ we have the commutation formulas

$$D_i(D_j(u)) = D_j(D_i(u)),$$
$$D_i\left(\frac{D_j(u)}{u}\right) = D_j\left(\frac{D_i(u)}{u}\right)$$

which, together with $\gamma_\ell \in \overline{\mathbb{E}}$, imply that for $2 \leq i \leq m$

$$D_i(a_1) = D_1(D_i(g_0)) + D_1\left(\sum_{\ell=1}^{L} \gamma_\ell \frac{D_i(g_\ell)}{g_\ell}\right) + \sum_{\ell=1}^{L} D_i(\gamma_\ell)\frac{D_1(g_\ell)}{g_\ell}.$$

Now it follows from the compatibility condition $D_1(a_i) = D_i(a_1)$ and after remembering $g_\ell = t_\ell - \beta_\ell$ that

$$D_1\left(a_i - D_i(g_0) - \sum_{\ell=1}^{L} \gamma_\ell \frac{D_i(g_\ell)}{g_\ell}\right) = \sum_{\ell=1}^{L} \frac{D_i(\gamma_\ell)}{t_1 - \beta_\ell}. \tag{6}$$

We now take, for some fixed $1 \leq \ell \leq L$, the residue at $t_1 = \beta_\ell$ on both sides of (6): the left-hand side vanishes as it is the derivative (with respect to t_1) of a rational function, and on the right-hand side we obtain precisely $D_i(\gamma_\ell)$ since $\beta_\ell \neq \beta_{\ell'}$ for $\ell \neq \ell'$. We get that $D_i(\gamma_\ell) = 0$ for $2 \leq i \leq m$ and $1 \leq \ell \leq L$, and therefore the γ_ℓ's are constants in $\mathbb{F}$.

Next, we claim that there always exist $\tilde{g}_0 \in \mathbb{E}(t_1)$, $\tilde{\gamma}_\ell \in \mathbb{F}$, and $\tilde{g}_\ell \in \mathbb{E}[t_1] \setminus \mathbb{E}$ with $\gcd(\tilde{g}_\ell, \tilde{g}_{\ell'}) = 1$ for $\ell \neq \ell'$ such that

$$a_1 = D_1(\tilde{g}_0) + \sum_{\ell=1}^{L} \tilde{\gamma}_\ell \frac{D_1(\tilde{g}_\ell)}{\tilde{g}_\ell}. \tag{7}$$

Let $\mathbb{K}$ be a finite normal extension of $\mathbb{E}$ containing the coefficients of both g_0 and the g_ℓ's from (5) and let G be the Galois group of $\mathbb{K}$ over $\mathbb{E}$. Since t_1

is transcendental over $\mathbb{K}$, we have that G is also the Galois group of $\mathbb{K}(t_1)$ over $\mathbb{E}(t_1)$. Let $d = |G|$. Then Equation (5) leads to

$$a_1 = D_1 \left(\frac{1}{d} \sum_{\sigma \in G} \sigma(g_0) \right) + \sum_{\ell=1}^{L} \frac{\gamma_\ell}{d} \frac{D_1(\prod_{\sigma \in G} \sigma(g_\ell))}{\prod_{\sigma \in G} \sigma(g_\ell)}$$

and the claim follows by taking

$$\tilde{g}_0 = \frac{1}{d} \sum_{\sigma \in G} \sigma(g_0) \in \mathbb{E}(t_1), \quad \tilde{\gamma}_\ell = \frac{\gamma_\ell}{d} \in \mathbb{F} \quad \text{and} \quad \tilde{g}_\ell = \prod_{\sigma \in G} \sigma(g_\ell) \in \mathbb{E}[t_1].$$

We have already shown that $\tilde{\gamma}_\ell \in \mathbb{F}$, and therefore the right-hand side of Equation (6) vanishes. Thus, for $2 \leq i \leq m$, we obtain

$$a_i = D_i(\tilde{g}_0) + \sum_{\ell=1}^{L} \tilde{\gamma}_\ell \frac{D_i(\tilde{g}_\ell)}{\tilde{g}_\ell} + \bar{a}_i, \quad \text{for some } \bar{a}_i \in \mathbb{E}.$$

The compatibility conditions $D_i(a_j) = D_j(a_i)$ imply that $D_i(\bar{a}_j) = D_j(\bar{a}_i)$ for all i, j with $2 \leq i < j \leq m$. By the induction hypothesis, for the $m-1$ compatible rational functions $\bar{a}_i$, there exist $\bar{g}_0 \in \mathbb{E}$, nonzero elements $\bar{\gamma}_\ell \in \mathbb{F}$ and $\bar{g}_\ell \in \mathbb{E} \setminus \mathbb{F}$ for $\ell = 1, \ldots, \bar{L}$ such that

$$\bar{a}_i = D_i(\bar{g}_0) + \sum_{\ell=1}^{\bar{L}} \bar{\gamma}_\ell \frac{D_i(\bar{g}_\ell)}{\bar{g}_\ell}, \quad \text{for all } i \text{ with } 2 \leq i \leq m.$$

Since $\bar{g}_0$ and the $\bar{g}_\ell$'s are free of t_1, we get

$$a_i = D_i(\tilde{g}_0 + \bar{g}_0) + \sum_{\ell=1}^{L} \tilde{\gamma}_\ell \frac{D_i(\tilde{g}_\ell)}{\tilde{g}_\ell} + \sum_{\ell=1}^{\bar{L}} \bar{\gamma}_\ell \frac{D_i(\bar{g}_\ell)}{\bar{g}_\ell}, \quad \text{for all } i \text{ with } 1 \leq i \leq m.$$

This completes the proof. □

The next theorem describes the full structure of compatible rational functions in the general continuous-discrete setting.

Theorem 9. *Assume that $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbb{F}(\boldsymbol{t}, \boldsymbol{k})$ are compatible rational functions with respect to $\{D_1, \ldots, D_m, S_1, \ldots, S_n\}$. Then there exist a rational function $f \in \mathbb{F}(\boldsymbol{t}, \boldsymbol{k}) \setminus \{0\}$, rational functions $g_0, \ldots, g_L, h_1, \ldots, h_n \in \mathbb{F}(\boldsymbol{t}) \setminus \{0\}$, univariate rational functions $r_{\boldsymbol{v}} \in \mathbb{F}(z)$ for each $\boldsymbol{v}$ in a finite set $V \subset \mathbb{Z}^n$, and constants $\gamma_1, \ldots, \gamma_L, \mu_1, \ldots, \mu_n \in \mathbb{F}$ such that*

$$\begin{aligned} a_i &= D_i(g_0) + \frac{D_i(f)}{f} + \sum_{\ell=1}^{L} \gamma_\ell \frac{D_i(g_\ell)}{g_\ell} + \sum_{j=1}^{n} k_j \frac{D_i(h_j)}{h_j}, & 1 \leq i \leq m, \\ b_j &= \frac{S_j(f)}{f} \mu_j h_j \prod_{v \in V} \prod_{\ell}{}_0^{v_j} r_{\boldsymbol{v}}(\boldsymbol{v} \cdot \boldsymbol{k} + \ell), & 1 \leq j \leq n. \end{aligned}$$

Proof. By Proposition 5.1 in [9] or Theorem 4.4.6 in [7], there exist $f \in \mathbb{F}(\boldsymbol{t}, \boldsymbol{k})$, $\bar{a}_1, \ldots, \bar{a}_m, h_1, \ldots, h_n \in \mathbb{F}(\boldsymbol{t})$, and $\bar{b}_1, \ldots, \bar{b}_n \in \mathbb{F}(\boldsymbol{k})$ such that

$$a_i = \frac{D_i(f)}{f} + \sum_{j=1}^{n} k_j \frac{D_i(h_j)}{h_j} + \bar{a}_i \quad \text{for all } i \text{ with } 1 \leq i \leq m, \tag{8}$$

and

$$b_j = \frac{S_j(f)}{f} h_j \bar{b}_j \quad \text{for all } j \text{ with } 1 \leq j \leq n, \tag{9}$$

and where $\bar{a}_1, \ldots, \bar{a}_m$ are compatible with respect to $\{D_1, \ldots, D_m\}$, and $\bar{b}_1, \ldots, \bar{b}_n$ are compatible with respect to $\{S_1, \ldots, S_n\}$. Now the full structure of the a_i's and b_j's follows from applying Theorems 8 and 7 respectively to the $\bar{a}_i$'s and $\bar{b}_j$'s. □

4. Structure of mixed hypergeometric terms

In this section, we derive the structure of hypergeometric terms from that of their associated certificates, which are compatible rational functions. To this end, let us recall some terminologies from [3].

Definition 10. *Two hypergeometric terms H_1, H_2 are said to be* conjugate *if they have the same certificates.*

Note that if $H_1 \overset{\text{alg}}{=} H_2$ then H_1 and H_2 are also conjugate to each other; however, the converse is not true. The first reason to introduce the notion of conjugacy is that it is the main tool to "correct" Conjecture 1 (see Theorem 29). As it was mentioned in the introduction the hypergeometric term $|k_1 - k_2|$ is holonomic (see Definition 23), but not proper. On the other hand, $|k_1 - k_2|$ is conjugate to the proper term $k_1 - k_2$, but they are not equal modulo an algebraic set.

The second reason for introducing the notion of conjugacy is related to the inversion of sequences. Recall that a multivariate sequence in $\mathbb{S}$ is invertible if all its entries are nonzero. The concept of nonvanishing rising factorials will allow us to construct invertible hypergeometric terms which are conjugate to those given in classical notation (rising factorials, binomial coefficients, etc.). Recall that the classical *rising factorial* $(\alpha)_k$ for $\alpha \in \mathbb{F}$ and $k \in \mathbb{Z}$ is defined by

$$(\alpha)_k = \begin{cases} \prod_{i=0}^{k-1}(\alpha + i), & \text{if } k \geq 0; \\ \prod_{i=1}^{-k}(\alpha - i)^{-1}, & \text{if } k < 0 \text{ and } \alpha \neq 1, 2, \ldots, -k; \\ 0, & \text{otherwise.} \end{cases}$$

As a companion notion of rising factorials, Abramov and Petkovšek introduced the *nonvanishing rising factorial* $(\alpha)_k^*$ for $\alpha \in \mathbb{F}$ and $k \in \mathbb{Z}$ as follows:

$$(\alpha)_k^* = \begin{cases} (\alpha)_k, & \text{if } (\alpha)_k \neq 0; \\ (\alpha)_{1-\alpha}(0)_{\alpha+k}, & \text{if } \alpha \in \mathbb{Z} \text{ and } \alpha > 0 \text{ and } \alpha + k \leq 0; \\ (\alpha)_{-\alpha}(1)_{\alpha+k-1}, & \text{if } \alpha \in \mathbb{Z} \text{ and } \alpha \leq 0 \text{ and } \alpha + k > 0. \end{cases}$$

It is easy to verify that $(\alpha)_k$ and $(\alpha)_k^*$ are conjugate, since they have the same certificate. Indeed, they both satisfy the recurrence

$$(k+\alpha)f(k+1) - (k+\alpha)^2 f(k) = 0.$$

Similarly, we can consider factorials with an integer-linear combination of several variables in the argument: for $(\alpha)_{\boldsymbol{v}\cdot\boldsymbol{k}}$ with some fixed $\boldsymbol{v} \in \mathbb{Z}^n$, a direct calculation yields the certificate

$$S_i\big((\alpha)_{\boldsymbol{v}\cdot\boldsymbol{k}}\big) \stackrel{\text{alg}}{=} \left(\prod\nolimits_{\ell}{}_0^{v_i}(\alpha + \boldsymbol{v}\cdot\boldsymbol{k} + \ell)\right)(\alpha)_{\boldsymbol{v}\cdot\boldsymbol{k}},$$

where we use the notation introduced in (4).

Definition 11 (Factorial term)**.** *A hypergeometric term $T(\boldsymbol{k}) \in \mathbb{S}$ over $\mathbb{F}(\boldsymbol{k})$ is called a* factorial term *if it has the form*

$$T(\boldsymbol{k}) = \mu_1^{k_1}\cdots\mu_n^{k_n}\left(\prod_{i=1}^{I}(\alpha_i)^*_{\boldsymbol{v}_i\cdot\boldsymbol{k}}\right)\left(\prod_{j=1}^{J}(\beta_j)^*_{\boldsymbol{w}_j\cdot\boldsymbol{k}}\right)^{-1}$$

where $\mu_1,\ldots,\mu_n \in \mathbb{F}$, $\alpha_i, \beta_j \in \mathbb{F}$ and $\boldsymbol{v}_i, \boldsymbol{w}_j \in \mathbb{Z}^n$ for $1 \le i \le I$ and $1 \le j \le J$.

The dictionary in Table 1 below enables us to translate the structure of compatible rational functions in Theorem 9 to that of their corresponding hypergeometric terms.

For a rational function $g \in \mathbb{F}[\boldsymbol{t}]$ and a constant $\gamma \in \mathbb{F}$, by g^γ we mean a term with certificate $\gamma\frac{D_i(g)}{g}$ for each $1 \le i \le m$, as indicated in row (4) of Table 1, where $L = 1$ is taken. In other words, g^γ is a solution of the system $D_i f - \gamma\frac{D_i(g)}{g} f = 0$ $(1 \le i \le m)$ in the unknown f. Likewise, $\exp(g)$ is a solution of the system $D_i f - D_i(g) f = 0$ $(1 \le i \le m)$.

Remark 12. *It is important to note that $g(\boldsymbol{t})^\gamma$ and $\exp(g(\boldsymbol{t}))$ are determined only up to a constant, because they are defined only as solutions of differential equations without boundary conditions. Consequently we have $g(\boldsymbol{t})^{\gamma_1} g(\boldsymbol{t})^{\gamma_2} = cg(\boldsymbol{t})^{(\gamma_1+\gamma_2)}$ for some nonzero constant $c \in \mathbb{F}$, but we do not have that $c = 1$ as we would like. Similarly, $(g(\boldsymbol{t})^{\gamma_1})^{\gamma_2} = cg(\boldsymbol{t})^{(\gamma_1\gamma_2)}$ where c is not necessarily 1, even when γ_2 is an integer. Analogously, the power laws for $\exp(g(\boldsymbol{t}))$ are different from the usual ones. In our context, however, these differences turn out to be irrelevant. This is similar to the way how exponentials and logarithms are introduced in the context of symbolic integration [5]. A version of the Wilf-Zeilberger conjecture without this kind of pseudo-exponentiation remains open and might be achieved with a more careful construction of the extension field or by considering the case where the continuous variables are complex.*

Theorem 13. *Any hypergeometric term over $\mathbb{F}(\boldsymbol{t},\boldsymbol{k})$ is conjugate to a multivariate sequence of the form*

$$F(\boldsymbol{t},\boldsymbol{k})\exp\big(g_0(\boldsymbol{t})\big)\left(\prod_{\ell=1}^{L} g_\ell(\boldsymbol{t})^{\gamma_\ell}\right)\left(\prod_{j=1}^{n} h_j(\boldsymbol{t})^{k_j}\right)T(\boldsymbol{k}) \tag{10}$$

	Hypergeometric terms	t_i-certificate	k_j-certificate
(1)	$H_1 \cdot H_2$	$\dfrac{D_i(H_1)}{H_1} + \dfrac{D_i(H_2)}{H_2}$	$\dfrac{S_j(H_1)}{H_1} \cdot \dfrac{S_j(H_2)}{H_2}$
(2)	$f(\boldsymbol{t},\boldsymbol{k}) \in \mathbb{F}(\boldsymbol{t},\boldsymbol{k}) \setminus \{0\}$	$\dfrac{D_i(f)}{f}$	$\dfrac{S_j(f)}{f}$
(3)	$\exp(g_0(\boldsymbol{t}))$	$D_i(g_0)$	1
(4)	$\displaystyle\prod_{\ell=1}^{L} g_\ell(\boldsymbol{t})^{\gamma_\ell}$	$\displaystyle\sum_{\ell=1}^{L} \gamma_\ell \frac{D_i(g_\ell)}{g_\ell}$	1
(5)	$\displaystyle\prod_{j=1}^{n} h_j(\boldsymbol{t})^{k_j}$	$\displaystyle\sum_{j=1}^{n} k_j \frac{D_i(h_j)}{h_j}$	$h_j(\boldsymbol{t})$
(6)	$(\alpha)^*_{\boldsymbol{v}\cdot\boldsymbol{k}}$	0	$\prod_{\ell}{}_0^{v_j} (\alpha + \boldsymbol{v}\cdot\boldsymbol{k} + \ell)$

Table 1: Dictionary between hypergeometric terms and their certificates.

where $F(\boldsymbol{t},\boldsymbol{k}) \in \mathbb{S}$ is the rational sequence corresponding to some rational function $f \in \mathbb{F}(\boldsymbol{t},\boldsymbol{k})$, and where $g_0, \ldots, g_L, h_1, \ldots, h_n \in \mathbb{F}(\boldsymbol{t})$, $\gamma_1, \ldots, \gamma_L \in \mathbb{F}$, and $T(\boldsymbol{k})$ is a factorial term that is nontrivial, i.e., that is not equal to the zero sequence modulo an algebraic set, in symbols: $T \overset{\text{alg}}{\neq} 0$.

Proof. This follows from Theorem 9, Corollary 4 in [3], and the dictionary in Table 1. □

Definition 14. *We call the form in* (10) *a* standard form *if the denominator of the rational sequence F, i.e., the denominator of the underlying rational function f, contains no factors in $\mathbb{F}[\boldsymbol{t}]$ and no* integer-linear *factors of the form $\alpha + \boldsymbol{v}\cdot\boldsymbol{k}$ with $\boldsymbol{v} \in \mathbb{Z}^n$ and $\alpha \in \mathbb{F}$.*

Remark 15. *We can always turn* (10) *into a standard form by moving all factors in $\mathbb{F}[\boldsymbol{t}]$ from the denominator of f into the part $\prod_{\ell=1}^{L} g_\ell(\boldsymbol{t})^{\gamma_\ell}$, and moving all* integer-linear *factors into the factorial term via the formula $\alpha + \boldsymbol{v}\cdot\boldsymbol{k} = (\alpha)^*_{\boldsymbol{v}\cdot\boldsymbol{k}+1}/(\alpha)^*_{\boldsymbol{v}\cdot\boldsymbol{k}}$.*

According to the definition by Wilf and Zeilberger [26], Theorem 13 distinguishes an arbitrary hypergeometric term from a proper one as follows.

Definition 16 (Properness)**.** *A hypergeometric term over $\mathbb{F}(\boldsymbol{t},\boldsymbol{k})$ is said to be* proper *if it of the form*

$$p(\boldsymbol{t},\boldsymbol{k})\exp\bigl(g_0(\boldsymbol{t})\bigr)\left(\prod_{\ell=1}^{L} g_\ell(\boldsymbol{t})^{\gamma_\ell}\right)\left(\prod_{j=1}^{n} h_j(\boldsymbol{t})^{k_j}\right)T(\boldsymbol{k}) \tag{11}$$

where p is a sequence corresponding to some polynomial in $\mathbb{F}[\boldsymbol{t},\boldsymbol{k}]$, and where $g_0,\dots,g_L,h_1,\dots,h_n \in \mathbb{F}(\boldsymbol{t})$, $\gamma_1,\dots,\gamma_L \in \mathbb{F}$, and where $T(\boldsymbol{k}) \overset{\text{alg}}{\neq} 0$ is a nontrivial factorial term.

Definition 17 (Conjugate-Properness)**.** *A hypergeometric term over $\mathbb{F}(\boldsymbol{t},\boldsymbol{k})$ is said to be* conjugate-proper *if it is conjugate to a proper term.*

Proposition 18. *Let $H(\boldsymbol{t},\boldsymbol{k})$ be a hypergeometric term such that $S_j(H) \overset{\text{alg}}{=} H$ for all j with $1 \le j \le n$. Then H is conjugate-proper.*

Proof. Let $a_1,\dots,a_m,b_1,\dots,b_n \in \mathbb{F}(\boldsymbol{t},\boldsymbol{k})$ be the certificates of $H(\boldsymbol{t},\boldsymbol{k})$ with respect to $t_1,\dots,t_m,k_1,\dots,k_n$, respectively. If $S_j(H) \overset{\text{alg}}{=} H$ for all j with $1 \le j \le n$, then $b_j = 1$ for all j with $1 \le j \le n$ by [3, Proposition 1]. The compatibility conditions $D_i(b_j)/b_j = S_j(a_i) - a_i$ for all i,j with $1 \le i \le m$ and $1 \le j \le n$ imply that $a_i \in \mathbb{F}(\boldsymbol{t})$ for all i with $1 \le i \le m$. By Theorem 13, H is conjugate to a hypergeometric term of the form $F(\boldsymbol{t})\exp\bigl(g_0(\boldsymbol{t})\bigr)\prod_{\ell=1}^{L} g_\ell(\boldsymbol{t})^{\gamma_\ell}$, where $F,g_0,g_1,\dots,g_L \in \mathbb{F}(\boldsymbol{t})$ and $\gamma_\ell \in \mathbb{F}$. By setting g_{L+1} as the denominator of $F(\boldsymbol{t})$ and γ_{L+1} as -1, we conclude that H is conjugate to a proper hypergeometric term. $\square$

5. Holonomic Functions

In this section, we recall some results concerning holonomic functions and D-finite functions from [12, 19]. The *Weyl algebra* $\mathbb{W}_{\boldsymbol{t}} := \mathbb{F}[\boldsymbol{t}]\langle \boldsymbol{D_t}\rangle$ is the noncommutative polynomial ring in the variables $\boldsymbol{t} = t_1,\dots,t_m$ and $\boldsymbol{D_t} = D_1,\dots,D_m$, in which the following multiplication rules hold:

$$\begin{aligned} D_iD_j &= D_jD_i, && 1 \le i,j \le m,\\ D_i\,p &= p\,D_i + \frac{\partial p}{\partial t_i}, && 1 \le i \le m,\ p \in \mathbb{F}[\boldsymbol{t}]. \end{aligned}$$

The Weyl algebra is the ring of linear partial differential operators with polynomial coefficients. Analogously, we define the *Ore algebra* $\mathbb{O}_{\boldsymbol{t}}$ as the ring $\mathbb{F}(\boldsymbol{t})\langle \boldsymbol{D_t}\rangle$ of linear partial differential operators with rational function coefficients.

Definition 19 (Holonomicity)**.** *A finitely generated left $\mathbb{W}_{\boldsymbol{t}}$-module is* holonomic *if it is zero, or if it has Bernstein dimension m (see for example [12, Chap. 9]). Let $H(\boldsymbol{t})$ be a function in a left $\mathbb{W}_{\boldsymbol{t}}$-module of functions. We define the annihilator of H in $\mathbb{W}_{\boldsymbol{t}}$ as*

$$\operatorname{ann}_{\mathbb{W}_{\boldsymbol{t}}}(H) := \{P \in \mathbb{W}_{\boldsymbol{t}} \mid P \cdot H = 0\},$$

which is a left ideal in $\mathbb{W}_{\boldsymbol{t}}$*. Then* $H(\boldsymbol{t})$ *is said to be* holonomic *with respect to* $\mathbb{W}_{\boldsymbol{t}}$ *if the left* $\mathbb{W}_{\boldsymbol{t}}$*-module* $\mathbb{W}_{\boldsymbol{t}}/\operatorname{ann}_{\mathbb{W}_{\boldsymbol{t}}}(H)$ *is holonomic. Differently stated, this means that the left ideal* $\operatorname{ann}_{\mathbb{W}_{\boldsymbol{t}}}(H)$ *has dimension* m*.*

By Bernstein's inequality [4, Thm. 1.3], any finitely generated nonzero left $\mathbb{W}_{\boldsymbol{t}}$-module has dimension at least m. So holonomicity indicates the minimality of dimension for nonzero $\mathbb{W}_{\boldsymbol{t}}$-modules, and in terms of functions this means: holonomic functions are solutions of maximally overdetermined systems of linear partial differential equations.

Definition 20 (D-finiteness [18])**.** *A left ideal* $\mathcal{I}$ *of* $\mathbb{O}_{\boldsymbol{t}}$ *is said to be D-finite if* $\dim_{\mathbb{F}(\boldsymbol{t})}(\mathbb{O}_{\boldsymbol{t}}/I) < \infty$*. Assume that a function* $H(\boldsymbol{t})$ *can be viewed as an element of a left* $\mathbb{O}_{\boldsymbol{t}}$*-module. Then* $H(\boldsymbol{t})$ *is said to be* D-finite *with respect to* $\mathbb{O}_{\boldsymbol{t}}$ *if the left ideal* $\operatorname{ann}_{\mathbb{O}_{\boldsymbol{t}}}(H) := \{P \in \mathbb{O}_{\boldsymbol{t}} \mid P \cdot H = 0\}$ *is D-finite. Equivalently, the vector space generated by all derivatives* $D_1^{i_1} \cdots D_m^{i_m}(H)$*,* $i_1, \ldots, i_m \geq 0$*, is finite-dimensional over* $\mathbb{F}(\boldsymbol{t})$*.*

The next theorem shows that the notions of holonomicity and D-finiteness coincide, which follows from two deep results of Bernstein [4] and Kashiwara [16]. An elementary proof of the direction "$\Longrightarrow$" has been given by Takayama [23, Thm. 2.4]. The other direction follows from the elimination property [27, Lemma 4.1], see also the paragraph before Proposition 26 below.

Theorem 21 (Bernstein–Kashiwara equivalence)**.** *Let* $\mathcal{I}$ *be a left ideal of* $\mathbb{O}_{\boldsymbol{t}}$*. Then* $\mathcal{I}$ *is D-finite if and only if* $\mathbb{W}_{\boldsymbol{t}}/(\mathcal{I} \cap \mathbb{W}_{\boldsymbol{t}})$ *is a holonomic* $\mathbb{W}_{\boldsymbol{t}}$*-module.*

In order to define holonomicity in the case of several continuous and discrete variables, the concept of generating functions is employed. The reason is that Definition 19 cannot be literally translated to $\mathbb{F}[\boldsymbol{k}]\langle \boldsymbol{S_k} \rangle$, the shift analog of the Weyl algebra, since there Bernstein's inequality does not hold.

Definition 22. *For* $H(\boldsymbol{t}, \boldsymbol{k}) \in \mathbb{S}$ *we call the formal power series*

$$G(\boldsymbol{t}, \boldsymbol{z}) = \sum_{k_1, k_2, \ldots, k_n \geq 0} H(\boldsymbol{t}, \boldsymbol{k}) z_1^{k_1} \cdots z_n^{k_n}$$

the generating function *of* H*.*

The definition requires evaluating H at integer points $\boldsymbol{k} \in \mathbb{N}^n$; note that this is always possible by the construction of $\mathbb{S}$ and the way in which the rational functions are embedded into it, see Definition 3.

Definition 23. *An element* $H(\boldsymbol{t}, \boldsymbol{k}) \in \mathbb{S}$ *is said to be* holonomic *with respect to* $\boldsymbol{t}$ *and* $\boldsymbol{k}$ *if its generating function* $G(\boldsymbol{t}, \boldsymbol{z})$ *is holonomic with respect to* $\mathbb{W}_{\boldsymbol{t},\boldsymbol{z}} = \mathbb{F}[\boldsymbol{t}, \boldsymbol{z}]\langle \boldsymbol{D_t}, \boldsymbol{D_z} \rangle$*.*

We recall the notion of *diagonals* of formal power series, which will be useful for proving the following results about closure properties. For a formal power series

$$G(\boldsymbol{z}) = \sum_{i_1, \ldots, i_n \geq 0} g_{i_1, \ldots, i_n} z_1^{i_1} \cdots z_n^{i_n},$$

the primitive diagonal $I_{z_1,z_2}(G)$ is defined as

$$I_{z_1,z_2}(G) := \sum_{i_1,i_3,\ldots,i_n\geq 0} g_{i_1,i_1,\ldots,i_n} z_1^{i_1} z_3^{i_3}\cdots z_n^{i_n}.$$

Similarly, one can define the other primitive diagonals I_{z_i,z_j} for $i<j$. By a diagonal we mean any composition of the I_{z_i,z_j}. The following theorem states that D-finiteness is closed under the diagonal operation for formal power series.

Theorem 24 (Lipshitz [18], 1988)**.** *If $G(\boldsymbol{z})\in\mathbb{F}[[z_1,\ldots,z_n]]$ is D-finite, then any diagonal of G is D-finite.*

Zeilberger [27, Props. 3.1 and 3.2] proved that the class of holonomic functions satisfies certain closure properties, based on certain D-module constructions. Here, we give an alternative proof, based on Lipshitz' work on D-finite functions.

Proposition 25. *Let $H_1(\boldsymbol{t},\boldsymbol{k}), H_2(\boldsymbol{t},\boldsymbol{k})\in\mathbb{S}$ be holonomic. Then both H_1+H_2 and H_1H_2 are also holonomic.*

Proof. Let $G_1(\boldsymbol{t},\boldsymbol{y})=\sum_{\boldsymbol{k}\geq\boldsymbol{0}} H_1(\boldsymbol{t},\boldsymbol{k})\,\boldsymbol{y}^{\boldsymbol{k}}$ and $G_2(\boldsymbol{t},\boldsymbol{z})=\sum_{\boldsymbol{k}\geq\boldsymbol{0}} H_2(\boldsymbol{t},\boldsymbol{k})\,\boldsymbol{z}^{\boldsymbol{k}}$. By Definition 23, G_1 and G_2 are holonomic with respect to $\mathbb{F}[\boldsymbol{t},\boldsymbol{y},\boldsymbol{z}]\langle \boldsymbol{D_t},\boldsymbol{D_y},\boldsymbol{D_z}\rangle$, and therefore also D-finite since they only involve continuous variables. The class of D-finite functions forms an algebra over $\mathbb{F}(\boldsymbol{t},\boldsymbol{y},\boldsymbol{z})$ [19, Prop. 2.3], i.e., it is closed under addition and multiplication. It follows that $G_1(\boldsymbol{t},\boldsymbol{z})+G_2(\boldsymbol{t},\boldsymbol{z})$ is also D-finite, and therefore H_1+H_2 is holonomic. Similarly, $G_1(\boldsymbol{t},\boldsymbol{y})G_2(\boldsymbol{t},\boldsymbol{z})$ is D-finite; now note that the generating function of H_1H_2 is equal to the diagonal of G_1G_2:

$$\begin{aligned}\sum_{\boldsymbol{k}\geq\boldsymbol{0}} H_1(\boldsymbol{t},\boldsymbol{k})\,H_2(\boldsymbol{t},\boldsymbol{k})\,\boldsymbol{y}^{\boldsymbol{k}} &= I_{\boldsymbol{y},\boldsymbol{z}}(G_1(\boldsymbol{t},\boldsymbol{y})\,G_2(\boldsymbol{t},\boldsymbol{z}))\\ &= I_{y_1,z_1}(\cdots I_{y_n,z_n}(G_1(\boldsymbol{t},\boldsymbol{y})\,G_2(\boldsymbol{t},\boldsymbol{z}))\cdots).\end{aligned}$$

By Lipshitz's theorem and Definition 23, we conclude that H_1H_2 is holonomic with respect to $\boldsymbol{t}$ and $\boldsymbol{k}$. □

In the continuous case, Zeilberger [27, Lemma 4.1] shows that a holonomic ideal $\mathcal{I}$ in $\mathbb{W}_{\boldsymbol{t}}$, $\boldsymbol{t}=t_1,\ldots,t_m$, possesses the *elimination property*, i.e., for any subset of $m+1$ elements among the $2m$ generators of $\mathbb{W}_{\boldsymbol{t}}$ there exists a nonzero operator in $\mathcal{I}$ that involves only these $m+1$ generators and is free of the remaining $m-1$ generators. The proof is based on a simple counting argument that employs the Bernstein dimension. For later use, we show a similar elimination property in the algebra $\mathbb{F}[\boldsymbol{t},\boldsymbol{k}]\langle\boldsymbol{D_t},\boldsymbol{S_k}\rangle$.

Proposition 26. *Let $H(\boldsymbol{t},\boldsymbol{k})\in\mathbb{S}$ be holonomic with respect to $\boldsymbol{t}$ and $\boldsymbol{k}$. Then for any $i\in\{1,2,\ldots,m\}$ and $j\in\{1,2,\ldots,n\}$, there exists a nonzero operator $P(\boldsymbol{t},k_1,\ldots,k_{j-1},k_{j+1},\ldots,k_n,D_i,S_j)\in\mathbb{F}[\boldsymbol{t},\boldsymbol{k}]\langle\boldsymbol{D_t},\boldsymbol{S_k}\rangle$ such that $P(H)=0$.*

Proof. Without loss of generality, we may assume that $i = 1$ and $j = 1$. Let $G(\boldsymbol{t}, \boldsymbol{z})$ be the generating function of $H(\boldsymbol{t}, \boldsymbol{k})$ and let Θ_ℓ denote the Euler derivation $z_\ell \frac{\partial}{\partial z_\ell}$ for $1 \leq \ell \leq n$. By [19, Lemma 2.4], there exists a nonzero operator

$$Q(\boldsymbol{t}, z_1, D_1, \Theta_2, \ldots, \Theta_n) \in \mathbb{F}[\boldsymbol{t}, \boldsymbol{z}]\langle \boldsymbol{D_t}, \boldsymbol{D_z} \rangle$$

such that $Q(G) = 0$. Write

$$Q = \sum_{\boldsymbol{w} \in W} q_{\boldsymbol{w}}(\boldsymbol{t}, D_1) z_1^{w_1} \Theta_2^{w_2} \cdots \Theta_n^{w_n}, \quad \text{where } W \subset \mathbb{N}^n \text{ and } |W| < +\infty.$$

Set $u_1 = \deg_{z_1}(Q) = \max\{w_1 \mid (w_1, \ldots, w_m) \in W\}$, and let $W' = \{(u_1 - w_1, w_2, \ldots, w_m) \mid \boldsymbol{w} \in W\}$. By a straightforward calculation, we have

$$\begin{aligned}
Q(G) &= \sum_{\boldsymbol{w} \in W} q_{\boldsymbol{w}}(\boldsymbol{t}, D_1) z_1^{w_1} \Theta_2^{w_2} \cdots \Theta_n^{w_n} \left(\sum_{\boldsymbol{k} \geq \boldsymbol{0}} H(\boldsymbol{t}, \boldsymbol{k})\, \boldsymbol{z}^{\boldsymbol{k}} \right) \\
&= \sum_{\boldsymbol{w} \in W'} q_{\boldsymbol{w}}(\boldsymbol{t}, D_1) z_1^{u_1 - w_1} \Theta_2^{w_2} \cdots \Theta_n^{w_n} \left(\sum_{\boldsymbol{k} \geq \boldsymbol{0}} H(\boldsymbol{t}, \boldsymbol{k})\, \boldsymbol{z}^{\boldsymbol{k}} \right) \\
&= \sum_{\boldsymbol{w} \in W'} \sum_{\boldsymbol{k} \geq \boldsymbol{0}} q_{\boldsymbol{w}}(\boldsymbol{t}, D_1)\, H(\boldsymbol{t}, \boldsymbol{k})\, k_2^{w_2} \cdots k_n^{w_n}\, z_1^{k_1 + u_1 - w_1} z_2^{k_2} \cdots z_n^{k_n} \\
&= z_1^{u_1} \sum_{\boldsymbol{w} \in W'} \sum_{\substack{k_1 \geq -w_1 \\ k_2, \ldots, k_n \geq 0}} q_{\boldsymbol{w}}(\boldsymbol{t}, D_1)\, H(\boldsymbol{t}, k_1 + w_1, k_2, \ldots, k_n)\, k_2^{w_2} \cdots k_n^{w_n}\, z_1^{k_1} \cdots z_n^{k_n} \\
&= z_1^{u_1} \sum_{\boldsymbol{w} \in W'} \sum_{\substack{k_1 \geq 0 \\ k_2, \ldots, k_n \geq 0}} q_{\boldsymbol{w}}(\boldsymbol{t}, D_1)\, (S_1^{w_1} H)(\boldsymbol{k})\, k_2^{w_2} \cdots k_n^{w_n}\, z_1^{k_1} \cdots z_n^{k_n} \\
&\qquad + z_1^{u_1} \sum_{\boldsymbol{w} \in W'} \sum_{\substack{-w_1 \leq k_1 < 0 \\ k_2, \ldots, k_n \geq 0}} q_{\boldsymbol{w}}(\boldsymbol{t}, D_1)\, (S_1^{w_1} H)(\boldsymbol{k})\, k_2^{w_2} \cdots k_n^{w_n}\, z_1^{k_1} \cdots z_n^{k_n} \\
&= z_1^{u_1} \left(\sum_{\substack{k_1 \geq 0 \\ k_2, \ldots, k_n \geq 0}} (PH)(\boldsymbol{k})\, z_1^{k_1} \cdots z_n^{k_n} \right) + r(\boldsymbol{z}) = 0
\end{aligned}$$

where P is the desired operator

$$P = \sum_{\boldsymbol{w} \in W'} q_{\boldsymbol{w}}(\boldsymbol{t}, D_1) k_2^{w_2} \cdots k_n^{w_n} S_1^{w_1}$$

and $r(\boldsymbol{z})$ is a polynomial in z_1 of degree less than u_1 with coefficients being power series in $z_2, \ldots, z_n$. Recalling that the extreme left member $Q(G)$ of the equality above is 0 and noting that r and the sum in the extreme right member of the equality have no powers of z_1 in common and hence, no monomials $z_1^{k_1} \cdots z_n^{k_n}$ in common, coefficient comparison with respect to $z_1^{k_1} \cdots z_n^{k_n}$ reveals that $P(H) = 0$ and $r = 0$. $\square$

6. Proof of the Conjecture

In the case of several discrete variables, piecewise and conjugate interpretations of Conjecture 1 were proved by Payne [21] and by Abramov and Petkovšek [3], respectively. In the continuous case, any multivariate hypergeometric term is D-finite, and therefore holonomic by the Bernstein–Kashiwara equivalence. By Proposition 18, it is also conjugate-proper. Thus, Wilf and Zeilberger's conjecture holds naturally in this case. It remains to prove that the conjecture also holds in a mixed setting with several continuous and discrete variables; this is done in the rest of this section. We start by proving one direction of the equivalence in Wilf and Zeilberger's conjecture, namely that properness implies holonomicity.

Proposition 27. *Any proper hypergeometric term over* $\mathbb{F}(\boldsymbol{t},\boldsymbol{k})$ *is holonomic. Any conjugate-proper hypergeometric term over* $\mathbb{F}(\boldsymbol{t},\boldsymbol{k})$ *is conjugate to a holonomic one.*

Proof. By Definition 16 and Proposition 25, it suffices to show that all factors in the multiplicative form (11) are holonomic with respect to $\boldsymbol{t}$ and $\boldsymbol{k}$. First, we see that

$$\sum_{k\in\mathbb{N}} k(k-1)\cdots(k-i+1)z^{k-i} = D_z^i\left(\frac{1}{1-z}\right),$$

which is a rational function in z for each fixed $i\in\mathbb{N}$, obtained by taking the ith derivative on both sides of $\sum_{k\in\mathbb{N}} z^k = 1/(1-z)$. This fact implies that the generating function of any polynomial in $\mathbb{F}[\boldsymbol{t},\boldsymbol{k}]$ is a rational function in $\mathbb{F}(\boldsymbol{t},\boldsymbol{z})$ and therefore is holonomic.

Second, any hypergeometric term $H(\boldsymbol{t})$ that depends only on the continuous variables $\boldsymbol{t}$ is holonomic. According to Definition 23, its generating function is $G(\boldsymbol{t},\boldsymbol{z}) = H(\boldsymbol{t})\prod_{i=1}^{n}(1/(1-z_i))$. Clearly $G(\boldsymbol{t},\boldsymbol{z})$ satisfies a system of first-order linear differential equations and therefore is D-finite with respect to $\mathbb{O}_{\boldsymbol{t},\boldsymbol{z}} = \mathbb{F}(\boldsymbol{t},\boldsymbol{z})\langle \boldsymbol{D_t},\boldsymbol{D_z}\rangle$. By Theorem 21, the generating function G is holonomic with respect to $\mathbb{W}_{\boldsymbol{t},\boldsymbol{z}}$, and thus H is holonomic with respect to $\boldsymbol{t}$ and $\boldsymbol{k}$. In particular, the factor $\exp(g_0(\boldsymbol{t}))\prod_{\ell=1}^{L} g_\ell(\boldsymbol{t})^{\gamma_\ell}$ in (11) is holonomic.

Third, a direct calculation implies that the generating function of the factor $\prod_{j=1}^{n} h_j(\boldsymbol{t})^{k_j}$ is equal to $\prod_{j=1}^{n} 1/(1-h_j(\boldsymbol{t})z_j)$, which is holonomic by a similar reasoning.

Finally, we have to show that the factorial term $T(\boldsymbol{k})$ is holonomic. For this, we point to [3, Def. 6 and Thm. 3]: there, a *proper term* is defined as the product of a polynomial in $\mathbb{F}[\boldsymbol{k}]$ and a factorial term $T(\boldsymbol{k})$, and subsequently, it is shown that every proper term is holonomic.

The second assertion now follows from the symmetry of the conjugacy relation. □

The following proposition characterizes those rational functions in continuous and discrete variables which are holonomic.

Proposition 28. *Let $f(\boldsymbol{t},\boldsymbol{k}) \in \mathbb{F}(\boldsymbol{t},\boldsymbol{k})$ be a rational function and $F(\boldsymbol{t},\boldsymbol{k}) \in \mathbb{S}$ be the corresponding rational sequence. The following statements are equivalent:*

(i) *F is conjugate to a holonomic term.*

(ii) *F is conjugate-proper.*

(iii) *The denominator of f splits into the form*

$$g(\boldsymbol{t}) \prod_{i=1}^{I} (\alpha_i + \boldsymbol{v}_i \cdot \boldsymbol{k})$$

where $g \in \mathbb{F}[\boldsymbol{t}]$, and $\alpha_i \in \mathbb{F}$ and $\boldsymbol{v}_i \in \mathbb{Z}^n$ for $1 \leq i \leq I$.

Proof. We first prove that (iii) implies (ii) and that (ii) implies (i). Assume that the denominator of f has the form prescribed in (iii). According to Remark 15 there is a proper term of the form (11) that is conjugate to F. Hence we have shown (ii). Now by Proposition 27 we get that F is conjugate to a holonomic term, which is part (i) of the assertion.

It remains to show that (i) implies (iii). For this purpose, assume that F is conjugate to a holonomic term. The rest of the proof is divided into two parts: first it is proved that the denominator of f splits into $g(\boldsymbol{t})h(\boldsymbol{k})$ and then it is argued that $h(\boldsymbol{k})$ factors into integer-linear factors.

We may assume that f is not a polynomial, otherwise the statement is trivially true. Let $p, d, s \in \mathbb{F}[\boldsymbol{t},\boldsymbol{k}]$ such that $f = p/(ds)$, $\gcd(p, ds) = 1$, and d is irreducible. We will show that d is either free of $\boldsymbol{t}$ or free of $\boldsymbol{k}$. Suppose to the contrary that d depends on both continuous and discrete variables; without loss of generality, assume that d is neither free of t_1 nor of k_1. Let $\overline{\boldsymbol{k}}$ denote $(k_2, k_3, \ldots, k_n)$. Performing a pseudo-division of p by d with respect to k_1, one obtains $e \in \mathbb{F}[\boldsymbol{t},\overline{\boldsymbol{k}}]$ and $q, r \in \mathbb{F}[\boldsymbol{t},\boldsymbol{k}]$ with $\deg_{k_1}(r) < \deg_{k_1}(d)$ such that $ep = qd + r$; note that $r \neq 0$ since p and d are coprime.

By Proposition 27, the embeddings of the polynomials e and s into $\mathbb{S}$ are holonomic. Then the rational sequence R that corresponds to efs is conjugate to some holonomic term H by the product closure property (Proposition 25), since F is conjugate to a holonomic term by assumption.

Proposition 26 states that there exists a nonzero operator P in $\mathbb{F}[\boldsymbol{t},\overline{\boldsymbol{k}}]\langle D_1, S_1\rangle$ such that $P(H) = 0$. It follows that $P(R) \stackrel{\text{alg}}{=} 0$, by the same argument as in the proof of [3, Thm. 13]. Then we have that the operator P also annihilates the rational function $efs = ep/d = q + r/d$ in $\mathbb{F}(\boldsymbol{t},\boldsymbol{k})$, since a rational function that is zero on a non-algebraic set (i.e., "almost everywhere"), must be identically zero. Write

$$P = \sum_{i \geq 0} \sum_{j \geq 0} c_{i,j}(\boldsymbol{t},\overline{\boldsymbol{k}}) D_1^i S_1^j$$

where only finitely many $c_{i,j}$ are nonzero. Since d is irreducible and not free of k_1, it is easy to see that $S_1^j(d)$ is also irreducible for all $j \in \mathbb{N}$ and that $\gcd\big(S_1^i(d), S_1^j(d)\big) = 1$ when $i \neq j$. By induction on i and noting that d is not free of t_1, we have

$$D_1^i S_1^j \left(\frac{r}{d}\right) = \frac{r_{i,j}}{\big(S_1^j(d)\big)^{i+1}}$$

for some polynomials $r_{i,j} \in \mathbb{F}[\boldsymbol{t}, \boldsymbol{k}]$ for which $\gcd\big(r_{i,j}, S_1^j(d)\big) = 1$ over $\mathbb{F}(\boldsymbol{t}, \overline{\boldsymbol{k}})$. Now let j' be such that not all $c_{i,j'}$ are zero and choose i' to be the largest integer such that $c_{i',j'} \neq 0$. Then, in the expression

$$P(efs) = P(q) + \sum_{i \geq 0} \sum_{j \geq 0} \frac{c_{i,j} r_{i,j}}{\big(S_1^j(d)\big)^{i+1}}$$

we have a pole at $S_1^{j'}(d)$ of order $i' + 1$, but this pole cannot be canceled with any other term of $P(efs)$. This contradicts the assumption that P annihilates efs. Thus any irreducible factor in the denominator of f is free of $\boldsymbol{t}$ or free of $\boldsymbol{k}$. It follows that the denominator of f can be written as $g(\boldsymbol{t})h(\boldsymbol{k})$ with $g \in \mathbb{F}[\boldsymbol{t}]$ and $h \in \mathbb{F}[\boldsymbol{k}]$.

It remains to show that $h(\boldsymbol{k})$ is a product of integer-linear factors of the form $\alpha + \boldsymbol{v} \cdot \boldsymbol{k}$. Multiplying f by g and noting that g is holonomic, we get that $fg = p/h$ is holonomic. We remark that a holonomic term in $\boldsymbol{t}$ and $\boldsymbol{k}$ is also holonomic when viewed as a term in $\boldsymbol{k}$ alone with parameters $\boldsymbol{t}$. Then, Theorem 13 in [3] or Lemma 4.1.6. in [21] implies that h factors into integer-linear factors by regarding p/h as a holonomic term of $\boldsymbol{k}$ alone. $\square$

We are now ready to state the main result of this paper.

Theorem 29. *A hypergeometric term is conjugate-proper if and only if it is conjugate to a holonomic one.*

Proof. In Proposition 27 it was proved that any conjugate-proper hypergeometric term is conjugate to a holonomic one. For the other direction, recall that Theorem 13 implies that any hypergeometric term is conjugate to a product of a rational sequence, an exponential function, a factorial term, and several power functions. Also in Proposition 27 it was proved that all factors in the multiplicative form (10) are holonomic, except possibly the first one, the rational sequence $F(\boldsymbol{t}, \boldsymbol{k})$. Similarly, the reciprocals of these factors are holonomic as well. By the product closure property given in Proposition 25, we are reduced to show that any rational sequence $F(\boldsymbol{t}, \boldsymbol{k})$ that is conjugate to a holonomic one, is conjugate-proper. The proof is concluded by invoking Proposition 28. $\square$

7. Conclusion

In this paper we have formulated and proved Wilf and Zeilberger's conjecture in the mixed continuous-discrete setting, using the notion of conjugate hypergeometric terms. In the discrete setting, an alternative formulation by Payne [21] using piecewise hypergeometric terms was employed to prove the following version of the conjecture: a hypergeometric term is holonomic if and only if it is piecewise-proper. It would be interesting to investigate the mixed setting in this framework and compare it with the previous one.

As there exists a notion of q-proper-hypergeometric terms [26, p. 589], it would be natural to formulate and prove a more general version of Theorem 29 that also includes the q-case. One important ingredient, namely the structure theorem for compatible rational functions in all three types of variables (continuous, discrete, and q-discrete) is already available [9].

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