THE NON-COMMUTATIVE A-POLYNOMIAL OF $(-2,3,n)$ PRETZEL KNOTS

STAVROS GAROUFALIDIS AND CHRISTOPH KOUTSCHAN

Abstract. We study q-holonomic sequences that arise as the colored Jones polynomial of knots in 3-space. The minimal-order recurrence for such a sequence is called the (non-commutative) A-polynomial of a knot. Using the *method of guessing*, we obtain this polynomial explicitly for the $K_p = (-2, 3, 3+2p)$ pretzel knots for $p = -5, \dots, 5$. This is a particularly interesting family since the pairs $(K_p, -K_{-p})$ are geometrically similar (in particular, scissors congruent) with similar character varieties. Our computation of the non-commutative A-polynomial (a) complements the computation of the A-polynomial of the pretzel knots done by the first author and Mattman, (b) supports the AJ Conjecture for knots with reducible A-polynomial and (c) numerically computes the Kashaev invariant of pretzel knots in linear time. In a later publication, we will use the numerical computation of the Kashaev invariant to numerically verify the Volume Conjecture for the above mentioned pretzel knots.

1. The colored Jones polynomial: a q-holonomic sequence of natural origin

1.1. **Introduction.** The colored Jones polynomial of a knot K in 3-space is a q-holonomic sequence of Laurent polynomials of natural origin in Quantum Topology [GL05]. As a canonical recursion relation for this sequence we choose the one with minimal order; this is the so-called non-commutative A-polynomial of a knot [Gar04]. Using the computational *method of guessing* with undetermined coefficients [Kau09a, Kau09b] combined with a carefully chosen exponent set of monomials (given by a translate of the Newton polygon of the A-polynomial) we compute very plausible candidates for the non-commutative A-polynomial of the $(-2, 3, 3+2p)$ pretzel knot family for $p = -5, \dots, 5$. Our computation of the non-commutative A-polynomial

(a) complement the computation of the A-polynomial of the pretzel knots [GM11],
(b) support the AJ Conjecture of [Gar04] (see also [Gel02]) for knots with reducible A-polynomial, and
(c) give an efficient linear time algorithm for computing numerically the Kashaev invariant of the pretzel knots (with a fixed accuracy).

In [GZ], we use the latter algorithm to numerically verify the volume conjecture of Kashaev [Kas97, MM01] for the above mentioned pretzel knots.

For an introduction to the polynomial invariants of knots that originate in Quantum Topology [Jon87, Tur88, Tur94] and the book [Jan96] where all the details of the quantum group theory can be found. For up-to-date computer calculations of several polynomial invariants of knots, see [BN05]. For an introduction to q-holonomic sequences see [Zei90, WZ92, PWZ96]. For the appearance of q-holonomic sequences in Quantum Topology, see [GL05, GS06, GS10] and also [GK11].

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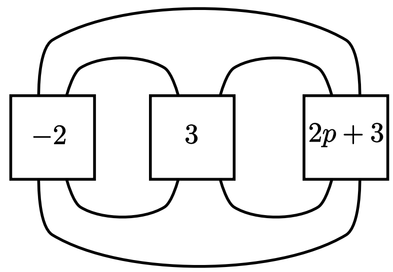
1.2. **Fusion and the colored Jones polynomial of pretzel knots.** Consider the 1-parameter family of *pretzel knots* $K_p = (-2, 3, 3+2p)$ for an integer p

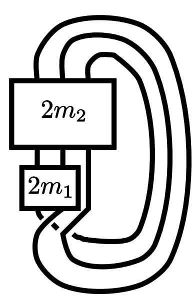


where an integer m inside a box indicates the number $|m|$ of half-twists, right-handed (if $m > 0$) or left-handed (if $m < 0$), according to the following figure:

The pretzel knots K_p are interesting from many points of view, discussed in detail in [CGLS87, GM11, Gar10a, GZ]:

- In *hyperbolic geometry*, K_p is the torus knot $5_1, 8_{19}$ and 10_{124} when $p = -1, 0, 1$, and K_p is a hyperbolic knot when $p \neq -1, 0, 1$.
- The pairs $(K_p, -K_{-p})$ (where $-K$ denotes the mirror of K) are *geometrically similar* for $p \geq 2$. In particular, their complements are scissors congruent, with equal volume, and with Chern-Simons invariants differing by torsion [Gar10a].
- The knots K_p appear in the study of *exceptional Dehn surgery* [CGLS87].
- In *Quantum Topology*, the knots K_p have different Jones polynomial and different Kashaev invariants, which numerically verify the Volume Conjecture [GZ].

Let $J_{K,n}(q)$ denote the colored *colored Jones polynomial* of a knot K colored by the n-dimensional irreducible representation of $\mathfrak{sl}_2$, framed by zero and normalized to be 1 at the unknot [Tur88, Tur94]. So, $J_{K,1}(q) = 1$ for all knots and $J_{K,2}(q)$ is the Jones polynomial of K [Jon87]. Our starting point is an explicit formula for the colored Jones polynomial $J_{p,n}(q)$ of K_p. This comes from a theorem of [Gar10b] which has two parts. The first part identifies the pretzel knots K_p with members of a 2-parameter family of 2-*fusion knots* $K(m_1, m_2)$ for integers m_1 and m_2, drawn here



and discussed in detail in [Gar10b]. The second part gives an explicit formula for the colored Jones polynomial of $K(m_1, m_2)$. To state it, we need to recall some notation. The *quantum integer* $[n]$ and the *quantum factorial* $[n]!$ of a natural number n are defined by

$$[n] = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}, \qquad [n]! = \prod_{k=1}^{n} [k]!$$

with the convention that $[0]! = 1$. Let

$$\begin{bmatrix} a \\ a_1, a_2, \dots, a_r \end{bmatrix} = \frac{[a]!}{[a_1]! \dots [a_r]!}$$

denote the q-multinomial coefficient of natural numbers a_i such that $a_1+\cdots+a_r=a$. We say that a triple (a,b,c) of natural numbers is *admissible* if $a+b+c$ is even and the triangle inequalities hold. In the formulas below, we use the following basic trivalent graphs $\mathrm{U}, \Theta, \mathrm{Tet}$ colored by one, three and six natural numbers (one in each edge of the corresponding graph) such that the colors at every vertex form an admissible triple.

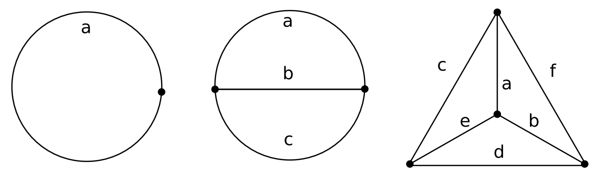


Figure 1. The U, Θ and Tet graphs colored by an admissible coloring.

If a coloring of a graph is not admissible, its evaluation vanishes. When the colorings of the graphs in Figure (1) are admissible, their evaluations can be computed by the following functions.

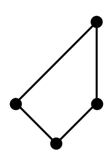
$$
\begin{aligned}
\mu(a) &= (-1)^a q^{\frac{a(a+2)}{4}} \\
\nu(c,a,b) &= (-1)^{\frac{a+b-c}{2}} q^{\frac{-a(a+2)-b(b+2)+c(c+2)}{8}} \\
\mathrm{U}(a) &= (-1)^a[a+1] \\
\Theta(a,b,c) &= (-1)^{\frac{a+b+c}{2}}[\frac{a+b+c}{2}+1]\begin{bmatrix} \frac{a+b+c}{2} \\ \frac{-a+b+c}{2}, \frac{a-b+c}{2}, \frac{a+b-c}{2} \end{bmatrix} \\
\mathrm{Tet}(a,b,c,d,e,f) &= \sum_{k=\max T_i}^{\min S_j} (-1)^k[k+1]\begin{bmatrix} k \\ S_1-k, S_2-k, S_3-k, k-T_1, k-T_2, k-T_3, k-T_4 \end{bmatrix}
\end{aligned}
$$

where

$$S_1=\frac{1}{2}(a+d+b+c) \qquad S_2=\frac{1}{2}(a+d+e+f) \qquad S_3=\frac{1}{2}(b+c+e+f) \tag{1}$$

$$T_1=\frac{1}{2}(a+b+e) \qquad T_2=\frac{1}{2}(a+c+f) \qquad T_3=\frac{1}{2}(c+d+e) \qquad T_4=\frac{1}{2}(b+d+f). \tag{2}$$

An assembly of the five building blocks can compute the colored Jones function of any knot. Consider the rational convex plane polygon P with vertices $\{(0,0),(1/2,-1/2),(1,0),(1,1)\}$ in $\mathbb{Q}^2$:



Theorem 1.1. [Gar10b] (a) For every integer p, we have $K_p=K(p,1)$.
(b) For every $m_1,m_2\in\mathbb{Z}$ and $n\in\mathbb{N}$, we have:

$$
\begin{aligned}
J_{K(m_1,m_2),n+1}(1/q) &= \frac{\mu(n)^{-w(m_1,m_2)}}{\mathrm{U}(n)} \sum_{(k_1,k_2)\in nP\cap\mathbb{Z}^2} \nu(2k_1,n,n)^{2m_1+2m_2}\nu(n+2k_2,2k_1,n)^{2m_2+1} \\
&\quad \cdot\frac{\mathrm{U}(2k_1)\mathrm{U}(n+2k_2)}{\Theta(n,n,2k_1)\Theta(n,2k_1,n+2k_2)}\mathrm{Tet}(n,2k_1,2k_1,n,n,n+2k_2)
\end{aligned} \tag{3}
$$

where P is as above and the *writhe* of $K(m_1,m_2)$ is given by $w(m_1,m_2)=2m_1+6m_2+2$.

1.3. **Our results.** Recall that a *q-holonomic sequence* $(f_n(q))$ for $n \in \mathbb{N}$ is a sequence (typically of rational functions $f_n(q) \in \mathbb{Q}(q)$ in one variable q) which satisfies a *linear recursion relation*:

$$a_d(q^n, q) f_{n+d}(q) + \cdots + a_0(q^n, q) f_n(q) = b(q^n, q) \tag{4}$$

for all $n \in \mathbb{N}$, where $a_j(u,v) \in \mathbb{Q}[u,v]$ for all $j = 0, \dots, d$ and $b(u,v) \in \mathbb{Q}[u,v]$ [Zei90]. As is custom, one can phrase Equation (4) in operator form, by considering the operators M and L that act on a sequence $(f_n(q))$ by:

$$(Lf)_n(q) = f_{n+1}(q) \qquad\qquad (Mf)_n(q) = q^n f_n(q)$$

It is easy to see that the operators M and L satisfy the q-commutation relation:

$$LM = qML$$

Thus, we can write Equation (4) in the form:

$$Pf = b, \qquad P = \sum_{j=0}^{d} a_j(M,q) L^j, \qquad b = b(q^n, q) \tag{5}$$

We will call a q-holonomic bi-infinite sequence $f_n(q)$ *palindromic* if either $f_n(q) = f_{-n}(q)$ for all integers n, or $f_n(q) = -f_{-n}(q)$ for all integers n. Given a palindromic sequence $f_n(q)$, we will call a recursion relation (4) (and the corresponding operator (P, b)) *palindromic* if Equation (4) holds for all integers n.

With our normalizations, the colored Jones polynomial $J_{K,n}(q)$ of a knot, defined for $n \geq 1$, extends to a palindromic sequence defined by $J_{K,n}(q) = J_{K,-n}(q)$ for $n < 0$ and $J_{K,0}(q) = 1$.

Let $A_p(M,L) \in \mathbb{Q}[M^2, L]$ denote the *A-polynomial* of the pretzel knot K_p, given in [GM11]. Let $\epsilon_p(M) \in \mathbb{Q}[M]$ denote the M-factors given in Appendix B. Let $\Delta_p(t) \in \mathbb{Z}[t^{\pm 1}]$ denote the *Alexander polynomial* of K_p; [Kau87]. Δ_p satisfies a linear recursion relation:

$$\Delta_{p+2} - (t + t^{-1})\Delta_{p+1} + \Delta_p = 0 \tag{6}$$

for all $p \in \mathbb{Z}$, with initial conditions

$$\Delta_0 = \frac{1}{t^3} - \frac{1}{t^2} + 1 - t^2 + t^3, \qquad \Delta_1 = \frac{1}{t^4} - \frac{1}{t^3} + \frac{1}{t} - 1 + t - t^3 + t^4.$$

Theorem 1.2. (a) Consider the operators $(A_p(M,L,q), b_p(M,q))$ of the appendix for $p = -5, \dots, 5$. Then, we have:

$$A_p(M,L,1) = A_p(M^{1/2}, L)\epsilon_p(M)$$

in accordance with the AJ Conjecture [Gar04].
(b) $(A_p(M,L,q), b_p(M,q))$ is palindromic.
(c) We also have:

$$\frac{A_p(M,1,1)}{b_p(M,1)} = \Delta_p(M) \tag{7}$$

for $p \neq -3$. When $p = -3$ we have $A_{-3}(M,1,1) = b_{-3}(M,1) = 0$ and

$$\frac{A_{-3,q}(M,1,1)}{\Delta_{-3}(M)} - A_{-3,L}(M,1,1) M \frac{\Delta'_{-3}(M)}{\Delta_{-3}(M)^2} = b_{-3,q}(M,1) \tag{8}$$

where primes indicate partial derivatives. This is in accordance with the loop expansion of the colored Jones polynomial [Gar08]. For a definition of the loop expansion [Roz98].

Conjecture 1.1. We conjecture that:

$$A_p(M,L,q) J_{p,n}(q) = b_p(q^n, q) \tag{9}$$

for $p = -5, \dots, 5$ and all $n \in \mathbb{Z}$.

2. Consistency checks

Three consistency checks of Conjecture 1.1 were already mentioned in Theorem 1.2. In this section we discuss four independent consistency checks regarding Conjecture 1.1.

2.1. **Consistency with the height.** In this section we discuss the height of Equation (9). For fixed integers p and natural numbers n, both sides of Equation (9) are Laurent polynomials in q with integer coefficients.

In general, the minimum and maximum degree of a q-holonomic sequence of Laurent polynomials is a quadratic quasi-polynomial [Gar11a]. In [Gar10b] the first author studied the minimum and maximum degree of the colored Jones polynomial of the 2-fusion knots $K(m_1, m_2)$. For the case of pretzel knots K_p, the maximum degree of $J_{p,n}(q)$ is a quadratic quasi-polynomial of n, and the minimum degree is a linear function of n. It follows that the terms in the left hand side of Equation (9) are polynomials of q of minimum degree a linear function of n and maximum degree a quasi-polynomial quadratic function of n. Explicitly, for the case of $K_2 = K(2,1)$, it was shown in [Gar10b, Gar11b] that $J_{2,n}(q)$ is a polynomial of q of minimum degree $\delta^*(n)$ and maximum degree $\delta(n)$ given by:

$$\begin{aligned} \delta(n) &= \left[\frac{37}{8}n^2 + \frac{3}{4}n - \frac{31}{8}\right] = \frac{37}{8}n^2 + \frac{3}{4}n - \frac{31}{8} + \epsilon(n), \\ \delta^*(n) &= 5(n-1) \end{aligned}$$

where $\epsilon(n)$ is a periodic sequence of period 4 given by $1/8, 0, 1/8, 1/2$ if $n \equiv 0, 1, 2, 3 \bmod 4$ respectively. Keep in mind that $J_{K,n}(q)$ denotes the colored Jones polynomial of K colored by the n-dimensional irreducible representation of $\mathfrak{sl}_2$. It follows that for $p = 2$, the left hand side of Equation (9) is a polynomial in q of minimum degree $60n + O(1)$ and maximum degree $37/8n^2 + 9/4n + O(1)$. We computed $J_{2,n}(q)$ explicitly for $1 \leq n \leq 70$ using Theorem 1.1. For instance, $J_{2,70}(q)$ is a polynomial of

- minimum (resp. maximum) exponent 345 (resp. 22606),
- maximum (in absolute value) coefficient 14287764770955 and
- sum of absolute values of its coefficients 28587411833908277.

It follows that that Equation (9) for $n = 70$ (which actually holds, by an explicit computation) involves the matching of about 22500 many powers of q with coefficients 14 digit integers. One can compare this to the modest size of $(A_2(M,L,q), b_2(M,q))$ given in Appendix A.

Of course, one can come up with operators that satisfy parts (a), (b) and (c) of Theorem 1.2 and Equation (9) for a fixed natural number n (such as $n = 70$). This is simply a problem of linear algebra with more unknowns than coefficients which in fact has infinitely many solutions. On the other hand, the operators given in the appendix are of small height, given the height of the input, as was illustrated above. Note also that Theorem 1.2 can be proven rigorously if we knew a priori bounds for the M, L degrees of the operators involved, and if we were able to compute enough values of the colored Jones polynomials, using the formula of Theorem 1.1.

2.2. **Consistency with the loop expansion of the colored Jones polynomial.** This section concerns the consistency of Conjecture 1.1 with the loop expansion of the colored Jones polynomial of K_p. The latter was introduced by Rozansky in [Roz98], and has the form:

$$J_{K,n}(q) = \sum_{k=0}^{\infty} \frac{P_{K,k}(q^n)}{\Delta_K(q^n)^{2k+1}}(q-1)^k \in \mathbb{Q}[[q-1]] \tag{10}$$

where $P_k(t) \in \mathbb{Z}[t^{\pm 1}]$ are Laurent polynomials with $P_{K,0} = 1$ and $\Delta_K(t) \in \mathbb{Z}[t^{\pm 1}]$ is the Alexander polynomial of K. With some effort, one can compute the k-loop polynomials $P_{K,k}(M)$ of a knot for various small values of k. The loop expansion given by Equation (10) *highly constrains* the coefficients of $(A_K(M,L,q), b_K(M,q))$, as was discussed in detail in [Gar08]. The simplest constraint for the knots K_p is given in Equation (7), which in fact is the first of a hierarchy of constraints. Each such constraint gives a consistency check for Conjecture 1.1, and offers a practical way to compute the loop expansion of the knots K_p. Consistency with the higher loop constraints have also been checked. We plan to discuss their details in a forthcoming publication.

2.3. **Consistency with the AJ Conjecture.** This section concerns the AJ Conjecture of [Gar04] for the knots $K_{\pm 3}$ with reducible A-polynomial. The A-polynomial $A_p(M,L)$ of the pretzel knots K_p was computed in [GM11]. In [Mat02] it was shown that $A_p(M,L)$ is irreducible if 3 does not divide p and otherwise it is the product of two irreducible factors when 3 divides p. Explicitly, we have:

$A_{-3}(M,L) = -(-1+L)(L^3 - L^4 - 5L^3M^2 + L^4M^2 - 2L^2M^4 - 2L^4M^4 - LM^6 - 4L^2M^6 + 3L^3M^6 + 2L^4M^6 - M^8 - 5LM^8 - 3L^3M^8 + L^4M^8 + LM^{10} - 3L^2M^{10} - 5L^4M^{10} - L^5M^{10} + 2LM^{12} + 3L^2M^{12} - 4L^3M^{12} - L^4M^{12} - 2LM^{14} - 2L^3M^{14} + LM^{16} - 5L^2M^{16} - LM^{18} + L^2M^{18})$

$A_3(M,L) = (-1+LM^{24})(-1+LM^{16} - LM^{18} + 2LM^{20} - 5LM^{22} + LM^{24} + 5L^2M^{40} - 4L^2M^{42} + L^2M^{46} + L^3M^{62} + 3L^3M^{66} + 2L^3M^{68} - 2L^4M^{84} - 3L^4M^{86} + 3L^4M^{88} + 2L^4M^{90} - 2L^5M^{106} - 3L^5M^{108} - L^5M^{112} - L^6M^{128} + 4L^6M^{132} - 5L^6M^{134} - L^7M^{150} + 5L^7M^{152} - 2L^7M^{154} + L^7M^{156} - L^7M^{158} + L^8M^{174})$

Theorem 1.2 matches exactly with the above values of the A-polynomials. The case of the knot K_{-3} is particularly interesting, since its $\mathrm{SL}_2(\mathbb{C})$ character variety has *three* components: the geometric one, the abelian $L-1$ component, and an additional $L-1$ component of nonabelian representations. Theorem 1.2 and Conjecture 1.1 support the idea that the AJ Conjecture captures the multiplicity of the various components of the character variety.

2.4. **Consistency with the Volume Conjecture.** This section concerns the consistency of Conjecture 1.1 with the computation of the Kashaev invariant of the K_p knots. The N-th *Kashaev invariant* $\langle K\rangle_N$ of a knot K is defined by [Kas97, MM01]:

$$\langle K\rangle_N = J_{K,N}(e^{2\pi i/N}) \tag{11}$$

The Volume Conjecture of Kashaev states that if K is a hyperbolic knot, then

$$\lim_{N\to\infty} \frac{|\langle K\rangle_N|}{N} = \frac{\mathrm{vol}(K)}{2\pi} \tag{12}$$

where $\mathrm{vol}(K)$ is the volume of the hyperbolic knot K. Since we are specializing to a root of unity, we might as well consider the remainder $\tau_{K,N}(q)$ of $J_{K,N-1}(q)$ by the N-th cyclotomic polynomial $\Phi_N(q)$. In [GZ], it was shown that given a recursion relation for $J_{K,N}(q)$, there is a linear time algorithm to numerically compute $\langle K\rangle_N$. Using the guessed recursion relation for K_2, we compute $\tau_{K_2,N}(q)$ for $N = 1,\dots,1000$. Here is a sample computation.

$$\begin{aligned}
&\tau_{K_2,100}(q) = \\
&- 1420771679897311607360 - 1402034476570732425908q - 1377764083694494707679q^2 \\
&- 1348056285420017550322q^3 - 1313028324854995190830q^4 - 1272818441358081463973q^5 \\
&- 1227585324968178744317q^6 - 1177507490130630983388q^7 - 1122782571182284245313q^8 \\
&- 1063626542375688303231q^9 + 420498814366636734411q^{10} + 469062907903390306537q^{11} \\
&+ 515775824438145014436q^{12} + 560453209429428890901q^{13} + 602918741648741441924q^{14} \\
&+ 643004829043136905736q^{15} + 680553270138355921566q^{16} + 715415878390451489264q^{17} \\
&+ 747455067013913965248q^{18} + 776544391967778302155q^{19} - 618202628922511743188q^{20} \\
&- 576608139973286430388q^{21} - 532738042123286363977q^{22} - 486765470606610517117q^{23} \\
&- 438871858158259827294q^{24} - 389246218987652812332q^{25} - 338084402821172432280q^{26} \\
&- 285588321971646221647q^{27} - 231965154488540570326q^{28} - 177426526516296620808q^{29} \\
&+ 1298584002796105745794q^{30} + 1335567867823634101034q^{31} + 1367280856639633305993q^{32} \\
&+ 1393597812566394292363q^{33} + 1414414874600710903331q^{34} + 1429649887309469255114q^{35} \\
&+ 1439242725058651352936q^{36} + 1443155529298983637839q^{37} + 1441372857979981026638q^{38} \\
&+ 1433901746491878528487q^{39}
\end{aligned}$$

Let

$$a_N = 2\pi \frac{\log|\langle K_2\rangle_N|}{N}$$

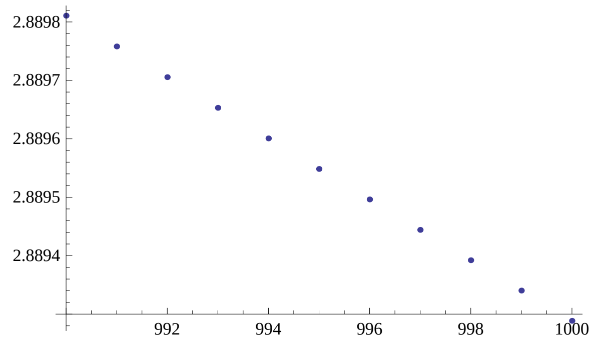
Since $\langle K_2\rangle_N = \tau_{K_2,N}(e^{2\pi i/N})$, the above expression gives the numerical value:

$$a_{100} = 3.22309\dots$$

which is a rather poor approximation of the volume $\mathrm{vol}(K_2) = 2.8281220883307827\dots$ of K_2. On the other hand, for $N = 990,\dots,1000$ we have:

N	990	991	992	993	994	995	996	997	998	999	1000
a_N	2.88981	2.88976	2.88971	2.88965	2.8896	2.88955	2.8895	2.88944	2.88939	2.88934	2.88929

The above data plots as follows:



and numerically fits the following curve:

$$2.82813 + 9.41764\frac{\log(n)}{n} - 3.89193\frac{1}{n}$$

which is a 4-digit approximation to the volume. In [GZ] a more precise approximation to the volume and its correction is given.

3. The computation

We use the computer to guess the recurrences for the colored Jones polynomials $J_{p,n}(q)$ [Kau09a, Kou09, Kou10, GK11]. The term *guessing* here refers to the method of making an ansatz with undetermined coefficients. The first values of the sequence $J_{p,n}(q)$ can be computed explicitly using Equation (3) (see Table 1 for an example). These values are then plugged into an ansatz with undetermined coefficients in order to produce an overdetermined linear system of equations. The more equations are used (note that at least as many equations as unknowns are needed to obtain a reliable result), the higher is the certainty that the result is the recurrence for the sequence. Alternatively, we can verify that the recurrence is satisfied for some values of the sequence that were not used for guessing, gaining further confidence into the result.

n	$J_{0,n}(q)$
1	1
2	$-q^8+q^5+q^3$
3	$q^{23}-q^{22}+q^{20}-q^{19}-q^{16}-q^{13}+q^{12}+q^9+q^6$
4	$-q^{43}+q^{41}+q^{40}-q^{39}+q^{37}-q^{35}+q^{33}-q^{31}+q^{29}-q^{27}-q^{26}+q^{25}-q^{23}-q^{22}+q^{21}-q^{19}+q^{17}+q^{13}+q^9$
5	$q^{70}-q^{69}+q^{65}-2q^{64}+q^{60}-q^{59}+q^{57}+q^{55}-q^{54}+q^{52}-q^{49}+q^{47}-q^{44}+q^{42}-q^{39}+q^{37}-q^{35}-q^{34}+q^{32}-q^{30}-q^{29}+q^{27}-q^{25}+q^{22}+q^{17}+q^{12}$

Table 1. The first elements of the colored Jones polynomial of the (-2,3,3) pretzel knot K_0

For guessing q-difference equations, there are two different choices for the ansatz and the nature of its coefficients.

The first ansatz (we consider it the more classical one) is of the form

$$\sum_{(\alpha,\beta)\in S} c_{\alpha,\beta} M^\alpha L^\beta \tag{13}$$

where the unknown coefficients $c_{\alpha,\beta}$ have to be determined in $\mathbb{Q}(q)$. We will refer to the finite set $S \subseteq \mathbb{N}^2$ as the *structure set* of the ansatz. It is easy to see that with ansatz (13) at least the first $(o+1)(d+2)-1$ (resp. $(o+2)(d+2)-2$) values of a sequence are needed in order to guess a homogeneous (resp. inhomogeneous) recursion of order o and coefficient degree d, i.e., when $0 \le \alpha \le d$ and $0 \le \beta \le o$. Table 2 illustrates how fast the entries of the colored Jones polynomials grow, and it becomes obvious that we cannot go very far with this ansatz.

p	$d(J_{p,10}(q))$	$d(J_{p,20}(q))$	$d(J_{p,30}(q))$
-5	453	1919	4400
-4	363	1546	3549
-3	282	1197	2735
-2	225	950	2175
-1	225	950	2175
0	265	1130	2595
1	330	1410	3240
2	406	1736	3991
3	491	2098	4821
4	579	2469	5671
5	667	2843	6529

Table 2. Size of the colored Jones polynomial at $n = 10, 20, 30$ for the pretzel knot family, where $d(p) = d_1 + d_2$ for a Laurent polynomial $p = \sum_{i=-d_1}^{d_2} c_i q^i$ with $c_{-d_1} \neq 0$ and $c_{d_2} \neq 0$

The second ansatz is of the form

$$\sum_{(\alpha,\beta,\gamma)\in S} c_{\alpha,\beta,\gamma} q^\gamma M^\alpha L^\beta \tag{14}$$

where the unknowns $c_{\alpha,\beta,\gamma}$ are elements of $\mathbb{Q}$ and $S \subseteq \mathbb{N}^3$ is again a finite structure set. This alternative is particularly promising when the q-degrees of the sequence grow very fast as it is the case with the colored Jones polynomials. On the one hand, we have many more unknowns than in (13), but already the first 30 values of $J_{p,n}(q)$ suffice to generate thousands of equations. However, this method has its limits, too, as it can be seen from Table 3. Guessing the last entry ($p = 5$) would require to solve a linear system over $\mathbb{Q}$ with $17 \cdot 289 \cdot 2175$ (about 10 million) unknowns.

The small instances in our family of problems can be done with either technique and without further previous knowledge. But for finding the larger recurrences presented in this paper, some optimizations are necessary. What helped considerably in reducing the size of the computations, is the fact that certain properties of the structure sets in (13) or (14) can be deduced a priori. In particular, the AJ Conjecture of [Gar04], and the Newton polygon of the A-polynomial of the pretzel knots K_p from [Mat02] allow us to make guess of the exponents (α, β) that appear in Equation (14), up to an overall translation in the M-direction. The only missing link to obtain the structure set is how far we have to translate the Newton polygon in M-direction (this has to be found out by trial and error), see Figure 2. Obviously this translated Newton polygon has much fewer lattice points than the rectangular box, and this helps a lot in the computations.

The second trick that allows us to compute the large recurrences listed in Table 3 is to use ansatz (13) in connection with modular computations. This means that we compute the sequence $J_{p,n}(q)$ only for specific integers q and modulo some prime number m. Then the monstrous Laurent polynomials (see Tables 1 and 2) shrink to a natural number between 0 and $m - 1$. Thus we can compute the first few hundred values of the colored Jones sequence "easily" (i.e., in a few hours). The guessing then is done efficiently, since only a single nullspace computation modulo m of a moderately-sized matrix (less than 1000 columns) is required. This procedure has to be performed for many specific values of q to be able to interpolate and reconstruct the polynomial coefficients $c_{\alpha,\beta}$. In general, we also would have to use several prime numbers in order to recover the integer coefficients via chinese remaindering and rational reconstruction. However, the integer coefficients in the present problems are so small (see Table 3) that a single prime suffices to recover the coefficients in $\mathbb{Z}$. Note that this strategy is perfectly suited for parallel computations.

We have seen that the number of sequence entries that we need to compute is determined by the number of unkowns in the ansatz, and the number of interpolation points (specific values for q) by the q-degree of the coefficients in the recurrence. We mention some further tricks for reducing these two parameters. The q-degree can be decreased by considering a slight variation of the original sequence $f(n)$, namely $g(n) := f(n+s)$ for some $s \in \mathbb{Z}$. Even more, a similar trick can be used to halve the number of unknowns, by exploiting the

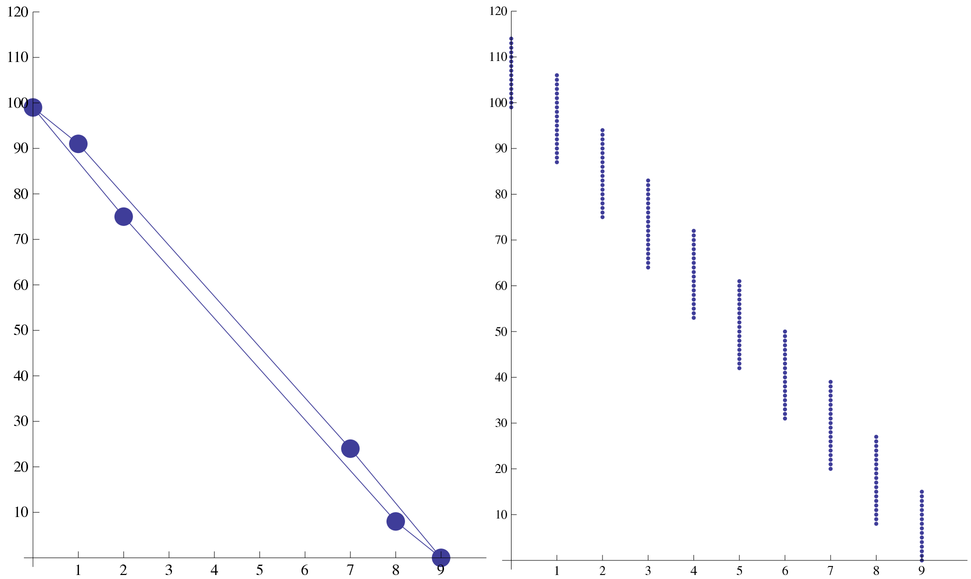


Figure 2. The Newton polygon for the (-3,2,9) pretzel knot K_3 (left) and the structure set of the recurrence for its colored Jones polynomial (right)

fact the there is an $s = t/2$ for $t \in \mathbb{Z}$ such that the substitution $n \to n+s$ in the coefficients of the recurrence reveals the following symmetry:

$$c_{\alpha,\beta} = c_{m-\alpha,l-\beta} \text{ for } t \text{ even}, \qquad \text{and } c_{\alpha,\beta} = -c_{m-\alpha,l-\beta} \text{ for } t \text{ odd},$$

where m is the M-degree and l is the L-degree of the recurrence. The above symmetry is equivalent to the palindromic property of the sought recursion.

We want to remark that most of the computation time was used to compute the data which then later served as input for the guessing. In the most difficult example presented here, the recurrence for K_5, it took 1607 CPU days to produce the first 744 entries of $J_{5,n}(q)$ for 700 different values of q. Since we ran this computation on a cluster with several hundred processors, it finished within a few days.

p	-5	-4	-3	-2	-1	0	1	2	3	4	5
ByteCount	5.7×10^7	1.1×10^7	1.1×10^6	32032	1192	1616	1616	47016	2.3×10^6	1.9×10^7	8.6×10^7
L-degree	12	9	6	3	1	2	2	6	9	12	15
M-degree	125	66	27	12	6	13	16	58	114	191	288
q-degree	946	392	85	19	3	13	16	233	514	1151	2174
largest cf.	3.0×10^8	12345	33	4	1	2	2	6	118	386444	2.2×10^{11}
translation	68	39	18	5	1	1	1	3	15	36	65

Table 3. Some data concerning the recursion relations for the colored Jones polynomial of the pretzel knots K_p, $p = -5, \ldots, 5$: the size of the recurrence (in bytes, using the Mathematica command `ByteCount`), the order (or L-degree), the coefficient degree (or M-degree), the degree of q in the coefficients, the largest integer coefficient, and how far the Newton polygon had to be translated in order to find this recurrence.

Appendix A. The recursion for K_{-2} and K_2

The operators $A_{\pm 2}(M,L,q)$ and $b_{\pm 2}(M,q)$ are given as follows.

$A_{-2}(M,L,q) = -(q^2M-1)(q^2M+1)(q^4M-1)(q^3M^2-1)L^3 - q(q^3M-1)^2(q^3M+1)(q^3M^2-1)(q^{14}M^5 - q^{11}M^4 - (q^{10}-q^9-q^8+q^7)M^3 + (q^7+q^4)M^2 + 2q^3M - 1)L^2 + q^7M^2(q^2M-1)^2(q^2M+1)(q^7M^2-1)(q^{11}M^5 - 2q^9M^4 - (q^8+q^5)M^3 + (q^6-q^5-q^4+q^3)M^2 + q^2M - 1)L - q^{16}M^7(qM-1)(q^3M-1)(q^3M+1)(q^7M^2-1)$

$b_{-2}(M,q) = q^6M^2(q^2M+1)(q^3M+1)(q^3M^2-1)(q^5M^2-1)(q^7M^2-1)$

$A_2(M,L,q) = q^{59}(q^2M-1)(q^3M-1)(q^7M-1)L^6 - q^{112}M^8(q^2M-1)(q^3M-1)(q^6M-1)^3L^5 - q^{167}M^{18}(q^2M-1)(q^5M-1)^2(q^5M+q+1)L^4 + (q-1)(q+1)q^{207}M^{27}(q^2M-1)(q^4M-1)^2(q^6M-1)L^3 + q^{240}M^{36}(q^3M-1)^2(q^4M+q^3M+1)(q^6M-1)L^2 + q^{263}M^{45}(q^2M-1)^3(q^5M-1)(q^6M-1)L - q^{279}M^{55}(qM-1)(q^5M-1)(q^6M-1)$

$b_2(M,q) = q^{89}M^5(q^{192}M^{48} - q^{186}(q+1)M^{47} + q^{181}M^{46} - q^{187}M^{45} + q^{181}(q+1)M^{44} - q^{176}(q^7+1)M^{43} + q^{177}(q^4+q+1)M^{42} - q^{172}(q^7+q^4+q^3+1)M^{41} + q^{170}(q^4+q^3+1)M^{40} + q^{168}(q^6-1)M^{39} - q^{168}(-q^3+q+1)M^{38} - q^{163}(q^6+q^3+q^2-1)M^{37} + q^{160}(q^4+q^3+1)M^{36} - q^{158}(q^5+1)M^{35} + q^{157}(q^4-q^3+q+1)M^{34} + q^{152}(q^6-q^4+q^2-1)M^{33} - q^{149}(q^3-q+1)M^{32} - q^{148}(q^3+1)M^{31} + q^{142}(q^3+1)M^{30} + q^{142}(q^2-1)M^{29} + q^{136}(q^4-q^2+1)M^{28} + q^{134}(q^2-1)M^{27} - q^{134}M^{26} - q^{124}M^{25} + 2q^{122}M^{24} - q^{116}M^{23} - q^{118}M^{22} + q^{110}(q^2-1)M^{21} + q^{104}(q^4-q^2+1)M^{20} + q^{102}(q^2-1)M^{19} + q^{94}(q^3+1)M^{18} - q^{92}(q^3+1)M^{17} - q^{85}(q^3-q+1)M^{16} + q^{80}(q^6-q^4+q^2-1)M^{15} + q^{77}(q^4-q^3+q+1)M^{14} - q^{70}(q^5+1)M^{13} + q^{64}(q^4+q^3+1)M^{12} - q^{59}(q^6+q^3+q^2-1)M^{11} - q^{56}(-q^3+q+1)M^{10} + q^{48}(q^6-1)M^9 + q^{42}(q^4+q^3+1)M^8 - q^{36}(q^7+q^4+q^3+1)M^7 + q^{33}(q^4+q+1)M^6 - q^{24}(q^7+1)M^5 + q^{21}(q+1)M^4 - q^{19}M^3 + q^5M^2 - q^2(q+1)M + 1)$

The values of $(A_p(M,L,q), b_p(M,q))$ for $p = -5, \dots, 5$ are available from

http://www.risc.jku.at/people/ckoutsch/pretzel/

or

http://www.math.gatech.edu/~stavros/publications/pretzel.data/

Appendix B. The M-factors

The M-factors $\epsilon_p(M) \in \mathbb{Q}[M]$ given as follows for $p = -5, \dots, 5$.

$\epsilon_{-5}(M) = -(-1+M)^9(1+M)^5(1+M+M^2)(1-2M+5M^2+6M^3-14M^4+20M^5+17M^6-48M^7+43M^8+40M^9-67M^{10}+40M^{11}+43M^{12}-48M^{13}+17M^{14}+20M^{15}-14M^{16}+6M^{17}+5M^{18}-2M^{19}+M^{20})(12-32M+34M^2+18M^3-171M^4+462M^5-680M^6+240M^7+1054M^8-2126M^9+1332M^{10}+1080M^{11}-3016M^{12}+2558M^{13}-227M^{14}-2256M^{15}+3187M^{16}-2256M^{17}-227M^{18}+2558M^{19}-3016M^{20}+1080M^{21}+1332M^{22}-2126M^{23}+1054M^{24}+240M^{25}-680M^{26}+462M^{27}-171M^{28}+18M^{29}+34M^{30}-32M^{31}+12M^{32})$

$\epsilon_{-4}(M) = (-1+M)^9(1+M)^6(1+M^4)^2(4-2M+7M^2+10M^3-14M^4+34M^5-7M^6+6M^7+24M^8+6M^9-7M^{10}+34M^{11}-14M^{12}+10M^{13}+7M^{14}-2M^{15}+4M^{16})$

$\epsilon_{-3}(M) = 2(-1+M)^5(1+M)^5(1-M+M^2)^3(1+M+M^2)$

$\epsilon_{-2}(M) = -(-1+M)^3(1+M)^2$

$\epsilon_{-1}(M) = -1+M$

$\epsilon_0(M) = -1+M$

$\epsilon_1(M) = 1-M$

$\epsilon_2(M) = (-1+M)^3$

$\epsilon_3(M) = 2(-1+M)^5(1+M)^2(1+M+M^2)^2(1-6M+13M^2-6M^3+M^4)$

$\epsilon_4(M) = -(-1+M)^7(1+M)^3(1+M^2)(1-4M+4M^2+4M^3-8M^4+4M^5+4M^6-4M^7+M^8)(4-14M+33M^2+12M^3-330M^4+328M^5-9M^6+226M^7+650M^8+226M^9-9M^{10}+328M^{11}-330M^{12}+12M^{13}+33M^{14}-14M^{15}+4M^{16})$

$\epsilon_5(M) = (-1+M)^9(1+M)^2(1+M+M^2)(1-2M+4M^2-2M^3-M^4-109M^6-78M^7+406M^8+162M^9-417M^{10}+162M^{11}+406M^{12}-78M^{13}-109M^{14}-M^{16}-2M^{17}+4M^{18}-2M^{19}+M^{20})(12-88M+318M^2-698M^3+381M^4+4924M^5-20623M^6+30440M^7+4694M^8-61074M^9+47268M^{10}+15096M^{11}-11350M^{12}-42702M^{13}+37078M^{14}+55502M^{15}-112131M^{16}+55502M^{17}+37078M^{18}-42702M^{19}-11350M^{20}+15096M^{21}+47268M^{22}-61074M^{23}+4694M^{24}+30440M^{25}-20623M^{26}+4924M^{27}+381M^{28}-698M^{29}+318M^{30}-88M^{31}+12M^{32})$

An explicit calculation shows that $\epsilon_p(M)$ are palindromic polynomials, i.e., $\epsilon_p(M)/e_p(1/M) = -M^{\delta_p}$ where δ_p is given by

$$\{68, 39, 18, 5, 1, 1, 1, 3, 15, 36, 65\}$$

for $p=-5,\dots,5$. One factor of $\epsilon_p(M)$ is easy to spot, namely it is the Alexander polynomial $\Delta_p(M)$, in accordance with the loop expansion; see Equation (7). The other factors of $\epsilon_p(M)$ are more mysterious with no clear geometric definition. Note finally that δ_p given above agree with the translation factor of the Newton polygon of K_p given in the last row of Table 3. This agreement is consistent with the fact that our computed recursion relations $(A_p(M,L,q), b_p(M,q))$ are palindromic for $p=-5,\dots,5$.

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Contents

References

[BN05] Dror Bar-Natan, *Knotatlas*, 2005, http://katlas.org.

[CGLS87] Marc Culler, C. McA. Gordon, J. Luecke, and Peter B. Shalen, *Dehn surgery on knots*, Ann. of Math. (2) **125** (1987), no. 2, 237–300.

[Gar04] Stavros Garoufalidis, *On the characteristic and deformation varieties of a knot*, Proceedings of the Casson Fest, Geom. Topol. Monogr., vol. 7, Geom. Topol. Publ., Coventry, 2004, pp. 291–309 (electronic).

[Gar08] ———, *Difference and differential equations for the colored Jones function*, J. Knot Theory Ramifications **17** (2008), no. 4, 495–510.

[Gar10a] ———, *The geometry of the $(-2,3,n)$ pretzel knots*, 2010, Preprint.

[Gar10b] ———, *Tropicalization and the Jones slopes of 2-fusion knots*, 2010, Preprint.

[Gar11a] ———, *The degree of a q-holonomic sequence is a quadratic quasi-polynomial*, Electron. J. Combin. **18** (2011), no. 2, Research Paper P4, 23.

[Gar11b] ———, *The Jones slopes of a knot*, Quantum Topol. **2** (2011), no. 1, 43–69.

[Gel02] Răzvan Gelca, *On the relation between the A-polynomial and the Jones polynomial*, Proc. Amer. Math. Soc. **130** (2002), no. 4, 1235–1241 (electronic).

[GK11] Stavros Garoufalidis and Christoph Koutschan, *The $\mathfrak{sl}_3$ jones polynomial of the trefoil: a case study of q-holonomic recursions*, Advances in Applied Mathematics **47** (2011), no. 4, 829–839.

[GL05] Stavros Garoufalidis and Thang T. Q. Lê, *The colored Jones function is q-holonomic*, Geom. Topol. **9** (2005), 1253–1293 (electronic).

[GM11] Stavros Garoufalidis and Thomas W. Mattman, *The A-polynomial of the $(-2,3,3+2n)$ pretzel knots*, New York J. Math. **17** (2011), 269–279.

[GS06] Stavros Garoufalidis and Xinyu Sun, *The C-polynomial of a knot*, Algebr. Geom. Topol. **6** (2006), 1623–1653 (electronic).

[GS10] ———, *The non-commutative A-polynomial of twist knots*, J. Knot Theory Ramifications **19** (2010), no. 12, 1571–1595.

[GZ] Stavros Garoufalidis and Don Zagier, *The kashaev invariant of* $(-2,3,n)$ *pretzel knots*, in preparation.

[Jan96] Jens Carsten Jantzen, *Lectures on quantum groups*, Graduate Studies in Mathematics, vol. 6, American Mathematical Society, Providence, RI, 1996.

[Jon87] V. F. R. Jones, *Hecke algebra representations of braid groups and link polynomials*, Ann. of Math. (2) **126** (1987), no. 2, 335–388.

[Kas97] R. M. Kashaev, *The hyperbolic volume of knots from the quantum dilogarithm*, Lett. Math. Phys. **39** (1997), no. 3, 269–275.

[Kau87] Louis H. Kauffman, *On knots*, Annals of Mathematics Studies, vol. 115, Princeton University Press, Princeton, NJ, 1987.

[Kau09a] Manuel Kauers, *Guessing handbook*, Tech. Report 09-07, RISC Report Series, Johannes Kepler University Linz, Austria, 2009.

[Kau09b] ———, *Mathematica software*, 2009, http://www.risc.uni-linz.ac.at/research/combinat/software/Guess/.

[Kou09] Christoph Koutschan, *Advanced applications of the holonomic systems approach*, Ph.D. thesis, RISC, Johannes Kepler University Linz, Austria, 2009.

[Kou10] ———, *HolonomicFunctions (user's guide)*, Tech. Report 10-01, RISC Report Series, Johannes Kepler University Linz, 2010.

[Mat02] Thomas W. Mattman, *The Culler-Shalen seminorms of the* $(-2,3,n)$ *pretzel knot*, J. Knot Theory Ramifications **11** (2002), no. 8, 1251–1289.

[MM01] Hitoshi Murakami and Jun Murakami, *The colored Jones polynomials and the simplicial volume of a knot*, Acta Math. **186** (2001), no. 1, 85–104.

[PWZ96] Marko Petkovšek, Herbert S. Wilf, and Doron Zeilberger, $A = B$, A K Peters Ltd., Wellesley, MA, 1996, With a foreword by Donald E. Knuth, With a separately available computer disk.

[Roz98] L. Rozansky, *The universal R-matrix, Burau representation, and the Melvin-Morton expansion of the colored Jones polynomial*, Adv. Math. **134** (1998), no. 1, 1–31.

[Tur88] V. G. Turaev, *The Yang-Baxter equation and invariants of links*, Invent. Math. **92** (1988), no. 3, 527–553.

[Tur94] ———, *Quantum invariants of knots and 3-manifolds*, de Gruyter Studies in Mathematics, vol. 18, Walter de Gruyter & Co., Berlin, 1994.

[WZ92] Herbert S. Wilf and Doron Zeilberger, *An algorithmic proof theory for hypergeometric (ordinary and "q") multi-sum/integral identities*, Invent. Math. **108** (1992), no. 3, 575–633.

[Zei90] Doron Zeilberger, *A holonomic systems approach to special functions identities*, J. Comput. Appl. Math. **32** (1990), no. 3, 321–368.

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