Effective algebraic analysis approach to linear systems over Ore algebras

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Software for the Symbolic Study of Functional Equations

Part I: HolonomicFunctions

◇ The Mathematica package HolonomicFunctions provides:

1. Algorithms for multivariate holonomic functions/sequences
   - computation of annihilating ideals
   - closure properties (addition, multiplication, substitutions)
   - summation and integration (via creative telescoping)
2. Computations in Ore algebras
   - noncommutative polynomial arithmetic with mixed difference-differential operators
   - Gröbner bases (Buchberger, FGLM, modules)
3. Solving coupled linear systems of differential-difference equations

◇ HolonomicFunctions is freely available at

`http://www.risc.jku.at/research/combinat/software/HolonomicFunctions/`

Ore algebras

◇ Let $\mathbb{A}$ be a ring and let $\sigma\colon \mathbb{A} \to \mathbb{A}$ be an endomorphism of $\mathbb{A}$.
◇ Let $\delta\colon \mathbb{A} \to \mathbb{A}$ be a σ-derivation, i.e.,

$$\forall\, a, b \in \mathbb{A}\colon\ \delta(a+b) = \delta(a) + \delta(b),\ \delta(ab) = \sigma(a)\delta(b) + \delta(a)b.$$

◇ An Ore extension $\mathbb{A}[\partial; \sigma, \delta]$ of $\mathbb{A}$ is the noncommutative ring of polynomials in ∂ with coefficients in $\mathbb{A}$ subject to

$$\forall\, a \in \mathbb{A}\colon\ \partial \cdot a = \sigma(a) \cdot \partial + \delta(a).$$

$\rightsquigarrow$ The multiplication corresponds to composition of operators.

◇ Examples:

- $\mathbb{K}[x][\partial; \sigma, \delta]$ with $\sigma = \mathrm{id}$ and $\delta = \frac{d}{dx}$: ring of linear ordinary differential operators with polynomial coefficients
- $\mathbb{K}(n)[\partial; \sigma, \delta]$ with $\sigma(n) = n + 1$ and $\delta = 0$: linear recurrence operators with rational function coefficients

Holonomic functions

$\diamond$ A function f is called holonomic if it satisfies "sufficiently many" linear differential/(q-)difference/mixed equations.

$\diamond$ The annihilator of f consists of the (usually infinite) set of such equations satisfied by f (it has the structure of a left ideal):

$$\mathrm{Ann}_D(f) = \{P \in D : P(f) = 0\}.$$

$\rightsquigarrow$ Its (finite) left Gröbner basis yields a data structure for encoding holonomic functions.

$\diamond$ Example: a holonomic function identity:

$$\int_0^\infty e^{-t} t^{\frac{a}{2}+n} J_a\big(2\sqrt{tx}\big)\,\mathrm{d}t = e^{-x} x^{a/2} n!\, L_n^a(x).$$

$\rightsquigarrow$ The HolonomicFunctions package can assist in proving it.

Functionality: Ore operators

◇ Built-in operators in the HolonomicFunctions package:

- partial derivatives $D_x = \frac{d}{dx}$, $D_y = \frac{d}{dy}$, etc.
- Euler operator $\theta_x = xD_x$
- forward/backward shifts ("time-delay operators"): S_n, S_n^{-1}
- difference operator $\Delta_n = S_n - 1$
- q-shift operator $S_{q,x}f(x) = f(qx)$

◇ Example: Weyl algebra in two variables:

```
> OreAlgebra[x, y, Der[x], Der[y]]
```

$$\mathbb{K}[x, y][D_x; 1, D_x][D_y; 1, D_y]$$

◇ Example: mixed multivariate rational Ore algebra:

```
> OreAlgebra[Der[x], Euler[z], S[n], QS[M,q^m]]
```

$$\mathbb{K}(k, M, n, q, x, z)[D_x; 1, D_x][\theta_z; 1, \theta_z][S_n; S_n, 0][S_{M,q}; S_{M,q}, 0]$$

Functionality: Ore polynomials

◇ Elements in an Ore algebra are called Ore polynomials.

$$D = \mathbb{F}[\partial_1; \sigma_1, \delta_1] \cdots [\partial_s; \sigma_s, \delta_s]$$

◇ An Ore polynomial $P \in D$ is a (finite) sum of the form

$$\sum_{i_1 \geq 0} \cdots \sum_{i_s \geq 0} c_{i_1,\dots,i_s} \partial_1^{i_1} \cdots \partial_s^{i_s}, \qquad c_{i_1,\dots,i_s} \in \mathbb{F}.$$

◇ Data structure for Ore polynomials and pretty printing

◇ Arithmetic with Ore polynomials in HolonomicFunctions:

```
> p = ToOrePolynomial[S[n]^2 - ((2n+2)/x)**S[n] + 1]
```

$$S_n^2 + \left(-\frac{2n}{x} - \frac{2}{x}\right) S_n + 1$$

```
> p ** p - 1 + n^9
```

$$S_n^4 - \left(\frac{4n}{x} + \frac{8}{x}\right) S_n^3 + \left(\frac{4n^2}{x^2} + \frac{12n}{x^2} + \frac{8}{x^2} + 2\right) S_n^2 - \left(\frac{4n}{x} + \frac{4}{x}\right) S_n + n^9$$

Functionality: Gröbner bases

◇ Gröbner bases in Ore algebras

- Buchberger's algorithm works in the Ore setting almost exactly as in commutative polynomial rings.
- Various term orders, e.g., lexicographic, graded-lexicographic, graded-reverse-lexicographic, grlex with weights, elimination order, order defined by a matrix.
- FGLM algorithm

◇ Example: Gröbner basis for the annihilator of $f(x, y) = \sin(xy)$:

```
> OreGroebnerBasis[{Der[x]^2 + y^2, Der[y]^2 + x^2},
    OreAlgebra[Der[x], Der[y]]]
```

$$\left\{-xD_x + yD_y, D_y^2 + x^2\right\}$$

Functionality: module Gröbner bases

◇ Let D be an Ore algebra and M be a left module over D.

◇ If M is finitely generated, then there exists a finite set $\{m_1, \ldots, m_r\}$ every $m \in M$ can be represented as

$$m = \sum_{i=1}^{r} d_i m_i, \quad d_i \in D.$$

◇ Gröbner bases for ideals can be easily extended to modules by

- using indicator variables: either $m_1, \ldots, m_r$ or $p, p^2, \ldots, p^r$,
- using an appropriate term order: position over term or term over position.

Special Features

◇ Possibility to define own Ore operators:

```
> OreSigma[T] = σ; OreDelta[T] = δ;
> ToOrePolynomial[T^2 ** a]
```

$$\sigma(\sigma(a))T^2 + (\delta(\sigma(a)) + \sigma(\delta(a)))T + \delta(\delta(a))$$

◇ Work with arbitrary coefficient domains, e.g., containing trigonometric functions or special functions:

```
> p1 = x^n ** p + Gamma[n] ** S[n]
```

$$x^n S_n^2 + \left(-2nx^{n-1} - 2x^{n-1} + \Gamma(n)\right) S_n + x^n$$

```
> S[n]^2 ** p1
```

$$x^{n+2} S_n^4 + \left(-2nx^{n+1} - 6x^{n+1} + \Gamma(n+2)\right) S_n^3 + x^{n+2} S_n^2$$

Special Features

◇ Redefine procedures for adding, multiplying, and normalizing coefficients of Ore polynomials:

- By default, coefficients are kept in expanded form (problematic for rational functions!).
- This information is stored in each `OreAlgebra` object.
- Use `Factor[#1+#2]&` and `#1*#2&` for factored form.
- More advanced application of this feature: automatic simplification in general coefficient domains:
  - by using Mathematica commands like `Simplify`,
  - by performing some reduction (e.g., $s^2 + c^2 - 1$),
  - by applying some replacement rules.
- The latter is realized in the `OreAlgebraWithRelations` command of the OreAlgebraicAnalysis package.

Special Features

```
> A = OreAlgebraWithRelations[Der[x],
        {D[f[x], {x, k_}] :> a*D[f[x], {x, k-1}]}];
```

$$\mathbb{K}(x)[D_x; 1, D_x]$$

```
> ToOrePolynomial[Der[x]^2 ** f[x]]
```

$$f[x]D_x^2 + 2f'[x]D_x + f''[x]$$

```
> ToOrePolynomial[Der[x]^2 ** f[x], A]
```

$$f[x]D_x^2 + 2a\,f[x]D_x + a^2 f[x]$$

Part II: OreAlgebraicAnalysis

◇ Starting point: HolonomicFunctions provides:

1. A rather general implementation of Ore algebras
2. Gröbner bases for ideals over such Ore algebras
   ⇝ Gröbner bases for modules over such Ore algebras

◇ Gröbner bases techniques can then be used to compute left kernels (syzygy modules), left factorizations, left inverses, . . .
⇝ We can perform constructive homological algebra computations

⇒ Study linear systems over Ore algebras via algebraic analysis

⇝ OreAlgebraicAnalysis: dedicated MATHEMATICA package freely available with a list of examples at

http://www.unilim.fr/pages_perso/thomas.cluzeau

Problems in control theory

$\diamond$ Given a linear system of functional equations, typical questions in control theory are:

1. Is the system controllable or does-it admit autonomous elements? $\rightarrow$ Compute autonomous elements
2. Is the system parametrizable? $\rightarrow$ Compute a parametrization
3. Is the system flat? $\rightarrow$ Compute flat outputs

$\diamond$ Algebraic analysis approach: unified mathematical framework to systematically answer to these questions

$\diamond$ Main tools for computations: constructive homological algebra, non-commutative Gröbner bases

The left D-module M

$\diamond$ D Ore algebra of functional operators, $R \in D^{q \times p}$ and a left D-module $\mathcal{F}$ (the functional space).

$\diamond$ Consider the linear system (behavior)

$$\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}.$$

$\diamond$ To $\ker_{\mathcal{F}}(R.)$ we associate the left D-module:

$$M = D^{1\times p}/(D^{1\times q}\,R)$$

given by the finite presentation

$$\begin{array}{ccccccc} D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0, \\ \lambda = (\lambda_1, \ldots, \lambda_q) & \longmapsto & \lambda\, R. & & & & \end{array}$$

Remark (*Malgrange'62*):

$$\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F}) := \{f : M \to \mathcal{F},\ f \text{ is left } D\text{-linear}\}.$$

Module theory

⋄ Classification of finitely generated left D-modules

⊙ M free if $\exists\, r \in \mathbb{Z}_+$ such that $M \cong D^r$

⊙ M stably free if $\exists\, r, s \in \mathbb{Z}_+$ such that $M \oplus D^s \cong D^r$

⊙ M projective if $\exists\, r \in \mathbb{Z}_+$ and a D-module P s.t. $M \oplus P \cong D^r$

⊙ M reflexive if $\varepsilon : M \longrightarrow \hom_D(\hom_D(M, D), D)$ is an isomorphism, where $\varepsilon(m)(f) = f(m)$, $\forall\, m \in M$, $f \in \hom_D(M, D)$

⊙ M torsion-free if $t(M) = \{m \in M \,|\, \exists\, 0 \neq d \in D : d\, m = 0\} = 0$

⊙ M torsion if $t(M) = M$

⋄ This classification can be effectively checked via the computation of $\mathrm{ext}^i_D(., D)$'s (constructive homological algebra)

⋄ Existing implementations in MAPLE, SINGULAR/PLURAL, COCOA, GAP4/HOMALG, . . .

Dictionary system / module properties

◇ Linear system $\ker_{\mathcal{F}}(R.) := \{\eta := (\eta_1 \; \ldots \; \eta_p)^T \in \mathcal{F}^p \mid R\,\eta = 0\}$

1. Autonomous element: linear comb. of the η_i satisfying a D-linear relation (controllable system if no autonomous elmt)
2. Parametrizable system: $\exists Q \in D^{p \times m}$ s.t. $\ker_{\mathcal{F}}(R.) = Q\,\mathcal{F}^m$
3. Flat system: there exists a parametrization $Q \in D^{p \times m}$ which admits a left inverse $T \in D^{m \times p}$, i.e, $T\,Q = I_p$

◇ Associated left D-module $M = D^{1 \times p}/(D^{1 \times q}\,R)$

1. $\ker_{\mathcal{F}}(R.)$ controllable iff M torsion-free
   (Autonomous elements $\leftrightarrow$ Torsion elements)
2. $\ker_{\mathcal{F}}(R.)$ parametrizable iff $\exists Q \in D^{p \times m}$ s.t. $M \cong D^{1 \times p}\,Q$
   (iff M torsion-free and Q parametrization, $\ker_{\mathcal{F}}(R.) = Q\,\mathcal{F}^m$)
3. $\ker_{\mathcal{F}}(R.)$ flat iff M free
   (Flat outputs $\leftrightarrow$ Elements of the bases of M)

Content of the package

- D Ore algebra handled by the package HolonomicFunctions
- Main functions:
  1. Functions for the study of matrices with entries in D
     (`LeftKernel`, `LeftFactorize`, `LeftInverse`,. . .)
  2. Functions for the study of systems/modules over D
     (`AutonomousElements`, `Parametrization`, `IsTorsion`,. . .)
  3. Functions for homological algebra over D
     (`FreeResolution`, `Ext1`, `Ext`,. . .)
  4. Functions for morphisms and decomposition problems over D
     (`Morphisms`, `IdempotentMorphisms`, `Decomposition`,. . .)

Concluding remarks

◇ OreAlgebraicAnalysis includes the main procedures implemented in the maple packages OreModules (Chyzak-Q.-Robertz) and OreMorphisms (C.-Q.)

◇ Advantages of OreAlgebraicAnalysis:

1. Study larger classes of linear functions systems since HolonomicFunctions handles more Ore algebras than Ore_algebra
2. Internal design of mathematica allows us to consider classes of systems which could not easily be considered in maple:
   - Generic linearizations of nonlinear functional systems defined by explicit equations
   - Systems containing transcendental functions (trigonometric functions, special functions, . . .)

OREALGEBRAICANALYSIS is freely available with a library of examples at

http://www.unilim.fr/pages_perso/thomas.cluzeau

Reference: *Effective algebraic analysis approach to linear systems over Ore algebras*, in collaboration with C. Koutschan, A. Quadrat and M. Tõnso, To appear, 2016.

Thank you for your attention!