Zeilberger's Holonomic Ansatz for Pfaffians

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Automatic DISCOVERY(!) and PROOF(!!)
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linear recurrences
polynomial coefficients
finitely many initial values

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where

- $a_{i,j}$ is a bivariate holonomic sequence, not depending on n,
- $b_n \neq 0$ for all $n \geq 1$.

Some Examples

$$\det_{1\le i,j\le n} \frac{1}{i+j-1} = \frac{1}{(2n-1)!}\prod_{k=1}^{n-1}\frac{(k!)^2}{(k+1)_{n-1}}$$

$$\det_{0\le i,j\le n-1} \binom{2i+2a}{j+b} = 2^{n(n-1)/2}\prod_{k=0}^{n-1}\frac{(2k+2a)!k!}{(k+b)!(2k+2a-b)!}$$

$$\det_{0\le i,j\le n-1} \sum_k \binom{i}{k}\binom{j}{k}2^k = 2^{n(n-1)/2}$$

A Prominent Example

C. K., M. Kauers, D. Zeilberger:

Proof of George Andrews's and David Robbins's
q-TSPP Conjecture

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By evaluating the q-holonomic determinant

$$\det_{1\le i,j\le n}\left(q^{i+j-1}\begin{bmatrix} i+j-2 \\ i-1 \end{bmatrix}_q + q^{i+j}\begin{bmatrix} i+j-1 \\ i \end{bmatrix}_q + (1+q^i)\delta_{i,j} - \delta_{i,j+1}\right)$$

$$= \prod_{1\le i\le j\le k\le n}\left(\frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}\right)^2$$

a long-standing combinatorial problem (first stated in 1983) was solved, the q-enumeration of totally symmetric plane partitions.

Determinant Evaluation: Proof by Induction

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Now use $c_{n,i}$ to do Laplace expansion of A_n w.r.t. the last row:

$$\det A_n = \sum_{i=1}^{n} M_{n,n} c_{n,i} a_{n,i}.$$

Showing that the sum evaluates to b_n completes the induction step.

Explanation for $c_{n,i}$

It's easy to see that $c_{n,i} = (-1)^{n+i} M_{n,i}/M_{n,n}$ is the solution of the system

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Let $A_n^{(j)}$ denote the matrix that is obtained from A_n by replacing the last row by the j-th row $(1 \leq j < n)$.

Laplace expansion of $A_n^{(j)}$ w.r.t. the last row:

$$\det A_n^{(j)} = 0 = \sum_{i=1}^{n} M_{n,n} c_{n,i} a_{j,i}.$$

This is just the j-th row in the above system.

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- Hope that $c_{n,i}$ is holonomic.
- Try to work with an implicit (recursive) definition of $c_{n,i}$.
- The values of $c_{n,i}$ can be computed for concrete $n, i \in \mathbb{N}$.
- If recurrences exist they can be guessed automatically (e.g. with M. Kauers's Mathematica package `Guess`)

Zeilberger's Holonomic Ansatz

1. Compute many values of $c_{n,i}$ (e.g. for $1 \leq i \leq n \leq 100$).
2. Guess linear recurrences for $c_{n,i}$ from that data.
3. Prove the following identities using holonomic closure properties and creative telescoping:

$$c_{n,n} = 1 \qquad (n \geq 1), \qquad \text{(D1)}$$

$$\sum_{i=1}^{n} c_{n,i} a_{j,i} = 0 \qquad (1 \leq j < n), \qquad \text{(D2)}$$

$$\sum_{i=1}^{n} c_{n,i} a_{n,i} = \frac{b_n}{b_{n-1}} \qquad (n \geq 1). \qquad \text{(D3)}$$

Note: all these steps can be executed automatically!

Pfaffians

Consider a skew-symmetric matrix A, i.e., $A = -A^T$:

$$A = \begin{pmatrix} 0 & a_{1,2} & a_{1,3} & \cdots \\ -a_{1,2} & 0 & a_{2,3} & \cdots \\ -a_{1,3} & -a_{2,3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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(it is easy to see that $\det A = 0$ if A has odd dimensions).

Now let A be a skew-symmetric matrix of size $2n \times 2n$.
Then the Pfaffian of A is defined as

$$\mathrm{Pf}\, A := \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(2i-1),\sigma(2i)}.$$

Note that $(\mathrm{Pf}\, A)^2 = \det A$.

Try to Apply Determinant Techniques

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In the paper

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- This double step method works in theory,
- but is complicated in practice and
- leads to very large computations.

Laplace Expansion for Pfaffians

- Let $A = (a_{i,j})_{1 \le i,j \le 2n}$ be a skew-symmetric matrix.
- Denote by $A(i,j)$ the $(2n-2) \times (2n-2)$ matrix which is obtained by deleting the rows and columns i and j from A.
- Define the cofactors $\Gamma_{i,j} := \begin{cases} (-1)^{j-i-1} \operatorname{Pf} A(i,j) & \text{if } i < j, \\ (-1)^{i-j} \operatorname{Pf} A(j,i) & \text{if } j < i, \\ 0 & \text{if } i = j. \end{cases}$

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Then there exists a Laplace-type expansion for the Pfaffian of A:

$$\delta_{j,k} \operatorname{Pf} A = \sum_{i=1}^{2n} a_{j,i} \Gamma_{k,i} = \sum_{i=1}^{2n} a_{i,j} \Gamma_{i,k}.$$

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Setting $j = k = 2n$ leads to

$$\operatorname{Pf} A = \sum_{i=1}^{2n} a_{2n,i} \Gamma_{2n,i} = \sum_{i=1}^{2n} a_{i,2n} \Gamma_{i,2n}.$$

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Now use $c_{2n,i}$ in the expansion formula for the Pfaffian of A_{2n}:

$$\operatorname{Pf} A_{2n} = \sum_{i=1}^{2n-1} b_{n-1} c_{2n,i} a_{i,2n}.$$

Showing that the sum evaluates to b_n completes the induction step.

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Now the holonomic ansatz can be formulated for Pfaffians:

1. Compute many values of $c_{2n,i}$ (e.g. for $1 \leq i \leq 2n \leq 100$).
2. Guess linear recurrences for $c_{2n,i}$ from that data.
3. Prove the following identities using holonomic closure properties and creative telescoping*:

$$c_{2n,2n-1} = 1 \qquad (n \geq 1), \tag{P1}$$

$$\sum_{i=1}^{2n-1} c_{2n,i} a_{i,j} = 0 \qquad (1 \leq j < 2n), \tag{P2}$$

$$\sum_{i=1}^{2n-1} c_{2n,i} a_{i,2n} = \frac{b_n}{b_{n-1}} \qquad (n \geq 1). \tag{P3}$$

*implementations are available in F. Chyzak's Maple package `Mgfun` and C.K.'s Mathematica package `HolonomicFunctions`; here we will use the latter one.

A Worked Example

Consider the Pfaffian $\operatorname{Pf}\big((j-i)M_{i+j-3}\big)_{1\le i,j\le 2n}$ where

$$M_n = \sum_{k=0}^{\lfloor n/2\rfloor} \frac{1}{k+1}\binom{n}{2k}\binom{2k}{k}$$

denotes the n-th Motzkin number: $1, 2, 4, 9, 21, 51, 127, 323, \ldots$

Clearly, $a_{i,j} = (j-i)M_{i+j-3}$ is a holonomic sequence:

$$(i-j-2)a_{i,j} = (i-j)a_{i-1,j+1}$$

$$\begin{aligned}&(i-j+1)(i-j+2)(i+j-1)a_{i,j} = \\ &\quad (i-j)(i-j+2)(2i+2j-5)a_{i,j-1} + \\ &\quad 3(i-j)(i-j+1)(i+j-4)a_{i,j-2}\end{aligned}$$

Guessed Recurrences for $c_{2n,i}$

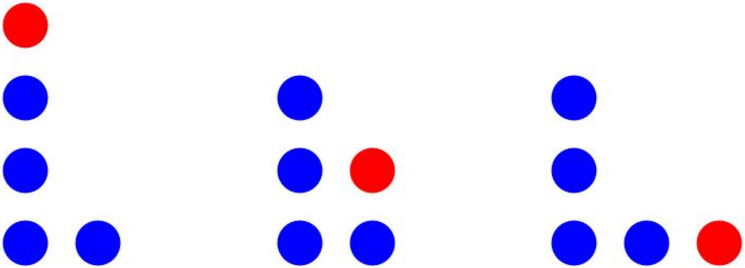
$$(i-1)(2n-3)(4n-7)c_{2n,i} =$$
$$-(2n+i-4)(8in-8i-8n^2+6n+3)c_{2(n-1),i-1} +$$
$$(i-1)(16in-16i+8n^2-34n+27)c_{2(n-1),i} +$$
$$24i(i-1)(n-1)c_{2(n-1),i+1} - (2n-3)(4n-7)(2n-i)c_{2n,i-1}$$

$$(n-2)(2n-5)(4n-11)(4n-7)(2n-i-2)(2n-i-1)c_{2n,i} =$$
$$(2n-5)(4n-11)(8i^2n^2-24i^2n+17i^2-16in^2+48in-$$
$$33i-16n^4+108n^3-258n^2+258n-92)c_{2(n-1),i} -$$
$$(n-1)(4n-7)(2n+i-5)(32in^2-122in+$$
$$117i-32n^3+168n^2-280n+144)c_{2(n-2),i} +$$
$$6i(4i+1)(n-2)(n-1)(2n-3)(4n-7)c_{2(n-2),i+1} +$$
$$36i(i+1)(n-2)(n-1)(2n-3)(4n-7)c_{2(n-2),i+2}$$

$$18n(i-3)(i-2)(i-1)c_{2n,i} =$$
$$(2n+i-4)(10i^2n-24in^2-63in+i+16n^3+76n^2+97n-3)c_{2n,i-3}$$
$$2(i-3)n(7i^2-12in-46i+33n+73)c_{2n,i-2} -$$
$$3(i-3)(i-2)n(14i-12n-39)c_{2n,i-1} -$$
$$(2n-1)(4n-3)(2n-i+4)(2n-i+3)c_{2(n+1),i-3}$$

Guessed Recurrences for $c_{2n,i}$

The support of these recurrences looks as follows:



The annihilating ideal they generate has rank 4.

Identity (P1)

$$c_{2n,2n-1} = 1 \qquad (n \geq 1), \tag{P1}$$

Apply the holonomic closure property "integer-linear substitution":

```
DFiniteSubstitute[c2ni, {i -> 2 n - 1}]
```

Identity (P1)

$$c_{2n,2n-1} = 1 \qquad (n \geq 1), \tag{P1}$$

Apply the holonomic closure property "integer-linear substitution":

```
DFiniteSubstitute[c2ni, {i -> 2 n - 1}]
```

The result is a recurrence of order 4 for $c_{2n,2n-1}$:

$$\begin{aligned}
&648(n+2)(n+3)(2n+3)(2n+5)(4n+9)\\
&(1003520n^7 + 6117888n^6 + 12424768n^5 + 9388056n^4\\
&+318598n^3 - 2766651n^2 - 1249360n - 163269)c_{2(n+4),2(n+4)-1}\\
&\qquad -9(n+2)(2n+3)(2247884800n^{10} + \dots)c_{2(n+3),2(n+3)-1}\\
&\qquad\qquad +2(4n+7)(6470696960n^{11} + \dots)c_{2(n+2),2(n+2)-1}\\
&\qquad -(4n+3)(4n+7)(1485209600n^{10} + \dots)c_{2(n+1),2(n+1)-1}\\
&+2(4n+1)(4n+3)(4n+7)(4n-1)^2(1003520n^7 + \dots)c_{2n,2n-1} = 0
\end{aligned}$$

which has $S_n - 1$ as a right factor and initial values $1, 1, 1, 1$.

Identity (P2)

$$\sum_{i=1}^{2n-1} c_{2n,i} a_{i,j} = 0 \qquad (1 \le j < 2n), \qquad \text{(P2)}$$

Apply closure property "times" and use creative telescoping:

```
smnd = DFiniteTimes[c2ni, aij]
FindCreativeTelescoping[smnd, S[i] - 1]
```

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```
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```

The result is a system of recurrences which is satisfied by the sum

$$j(4n-3)(j+2n)s_{n+1,j} - n(4n+1)(-j+2n-1)s_{n,j+1} \\ -j(4n+1)(j-n)s_{n,j} = 0,$$

$$(j-2n+2)(j+2n)s_{n,j+2} - (j+1)(2j+1)s_{n,j+1} - 3j(j+1)s_{n,j} = 0,$$

and whose initial values are all equal to 0.

Identity (P3)

$$\sum_{i=1}^{2n-1} c_{2n,i} a_{i,2n} = \frac{b_n}{b_{n-1}} \qquad (n \geq 1). \tag{P3}$$

This is done by closure property "times" and creative telescoping.

Identity (P3)

$$\sum_{i=1}^{2n-1} c_{2n,i} a_{i,2n} = \frac{b_n}{b_{n-1}} \qquad (n \geq 1). \tag{P3}$$

This is done by closure property "times" and creative telescoping.

The result is a recurrence for the ratio $r_n := \det A_n / \det A_{n-1}$:

$$\begin{aligned} &2(4n-11)(4n-7)(4n-5)(7n-13)r_n \\ -(4n-11)&(350n^3 - 1413n^2 + 1798n - 714)r_{n-1} \\ &+9(n-2)(2n-3)(4n-7)(7n-6)r_{n-2} = 0 \end{aligned}$$

with initial values $r_1 = 1$ and $r_2 = 5$.

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with initial values $r_1 = 1$ and $r_2 = 5$.

Its closed-form solution is $r_n = 4n-3$ and therefore

$$\mathrm{Pf}\left((j-i)M_{i+j-3}\right)_{1\leq i,j\leq 2n} = \prod_{k=1}^{n}(4k-3).$$

Turn Conjectures into Theorems

THEOREM 2. Let $M_n = \sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{2k}\binom{2k}{k}$ denote the n-th Motzkin number. Then for $n \in \mathbb{N}$ we have

$$\mathrm{Pf}\left((j-i)M_{i+j-3}\right)_{1\le i,j\le 2n} = \prod_{k=0}^{n-1}(4k+1).$$

THEOREM 3. Let $D_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}$ denote the n-th central Delannoy number. Then for $n \in \mathbb{N}$ we have

$$\mathrm{Pf}\left((j-i)D_{i+j-3}\right)_{1\le i,j\le 2n} = 2^{(n+1)(n-1)}(2n-1)\prod_{k=1}^{n-1}(4k-1).$$

THEOREM 4. Let $N_n(x)$ denote the n-th Narayana polynomial defined by $N_0(x) = 1$ and $N_n(x) = \sum_{k=0}^{n} \frac{1}{n}\binom{n}{k}\binom{n}{k-1}x^k$, $n \ge 1$. Then for $n \in \mathbb{N}$ we have

$$\mathrm{Pf}\left((j-i)N_{i+j-2}(x)\right)_{1\le i,j\le 2n} = x^{n^2}\prod_{k=0}^{n-1}(4k+1).$$