Holonomic Functions in Mathematica

In[1]:= `<< HolonomicFunctions.m`

HolonomicFunctions package by Christoph Koutschan, RISC-Linz, Version 1.7 (24.06.2013)
⟶ Type ?HolonomicFunctions for help

The package is freely available from the webpage
http://www.risc.jku.at/research/combinat/software/HolonomicFunctions/

In[2]:= `?HolonomicFunctions`

The main objective of this package is the algorithmic manipulation of ∂−finite (holonomic) functions. This includes (but is not restricted to) proving special function identities, finding recurrences, differential equations or relations of mixed type for ∂−finite functions, and computing definite sums and integrals of ∂−finite functions. Type ?DFinite to get the definition and a short introduction to ∂−finite functions.
The following commands serve the above objectives: Annihilator, CreativeTelescoping, HermiteTelescoping, FindCreativeTelescoping, FindRelation, FindSupport, Takayama, ApplyOreOperator, UnderTheStaircase, AnnihilatorDimension.
The closure properties of ∂−finite functions are implicitely executed in Annihilator. To execute them explicitly, use the commands DFinitePlus, DFiniteTimes, DFiniteSubstitute, DFiniteOreAction, DFiniteTimesHyper, DFiniteDE2RE, DFiniteRE2DE, DFiniteQSubstitute.
An important ingredient are Groebner bases in (noncommutative) Ore algebras: OreGroebnerBasis, OreReduce, GBEqual, FGLM.
A common subtask in the above algorithms is finding rational solutions of P−finite recurrences / differential equations or of coupled systems of such equations. The following commands address these purposes: RSolvePolynomial, RSolveRational, DSolvePolynomial, DSolveRational, QSolvePolynomial, QSolveRational, SolveOreSys, SolveCoupledSystem.
An element of an Ore algebra is called an Ore polynomial; the following commands explain the data type OrePolynomial that is introduced in this package and how to deal with it: OrePolynomial, ToOrePolynomial, OrePolynomialZeroQ, LeadingPowerProduct, LeadingExponent, LeadingCoefficient, LeadingTerm, OrePolynomialListCoefficients, NormalizeCoefficients, OrePlus, OreTimes, OrePower, ApplyOreOperator, ChangeOreAlgebra, ChangeMonomialOrder, OrePolynomialSubstitute, OrePolynomialDegree, Support.
In order to define own Ore algebras use the commands OreAlgebra, OreAlgebraGenerators, OreAlgebraOperators, OreAlgebraPolynomialVariables, OreOperators, OreOperatorQ, OreSigma, OreDelta, OreAction, Der, S, Delta, Euler, QS.
Some other functions that might be useful: Printlevel, RandomPolynomial.
If this package was useful in your scientific work, proper citation would be appreciated very much. Please use the following reference for this purpose:

```
@phdthesis{Koutschan09,
 author = {Christoph Koutschan},
 title = {Advanced Applications of the Holonomic Systems Approach},
 school = {RISC, Johannes Kepler University},
 address = {Linz, Austria},
 year = {2009}
}
```

Toy Example: Generating Function of Legendre Polynomials

Zeilberger: A holonomic systems approach to special functions identities (1990)

In[3]:= **TraditionalForm[**
HoldForm[Sum[LegendreP[n, x] * t ^ n, {n, 0, Infinity}] == (1 - 2 x t + t ^ 2) ^ (-1 / 2)]]

Out[3]//TraditionalForm=

$$\sum_{n=0}^{\infty} P_n(x)\, t^n = \frac{1}{\sqrt{1 - 2\, x\, t + t^2}}$$

In[4]:= **ann = Annihilator[LegendreP[n, x] * t ^ n, {S[n], Der[t], Der[x]}]**

Out[4]= $\{t\, D_t - n,\ (1+n)\, S_n + (t - t\, x^2)\, D_x + (-t\, x - n\, t\, x),\ (-1 + x^2)\, D_x^2 + 2\, x\, D_x + (-n - n^2)\}$

In[5]:= **?Annihilator**

Annihilator[expr, ops] computes annihilating relations for expr w.r.t. the given operator(s). It returns the Groebner basis of an annihilating ideal (with monomial order DegreeLexicographic). If expr is ∂−finite, the result will be a ∂−finite ideal. If expr is not recognized to be ∂−finite, there is still a chance to find at least some relations (in this case the ideal is not zero−dimensional which is indicated by a warning). Annihilator[expr] automatically determines for which operators relations exist. The relations are computed by executing the ∂−finite closure properties DFinitePlus, DFiniteTimes, and DFiniteSubstitute.
The expression expr can contain hypergeometric expressions, hyperexponential expressions, and algebraic expressions.
Additionally the following functions are recognized: AiryAi, AiryAiPrime, AiryBi, AiryBiPrime, AngerJ, AppellF1, ArcCos, ArcCosh, ArcCot, ArcCoth, ArcCsc, ArcCsch, ArcSec, ArcSech, ArcSin, ArcSinh, ArcTan, ArcTanh, ArithmeticGeometricMean, BellB, BernoulliB, BesselI, BesselJ, BesselK, BesselY, Beta, BetaRegularized, Binomial, CatalanNumber, ChebyshevT, ChebyshevU, Cos, Cosh, CoshIntegral, CosIntegral, EllipticE, EllipticF, EllipticK, EllipticPi, EllipticTheta, EllipticThetaPrime, Erf, Erfc, Erfi, EulerE, Exp, ExpIntegralE, ExpIntegralEi, Factorial, Factorial2, Fibonacci, FresnelC, FresnelS, Gamma, GammaRegularized, GegenbauerC, HankelH1, HankelH2, HarmonicNumber, HermiteH, Hypergeometric0F1, Hypergeometric0F1Regularized, Hypergeometric1F1, Hypergeometric1F1Regularized, Hypergeometric2F1, Hypergeometric2F1Regularized, HypergeometricPFQ, HypergeometricPFQRegularized, HypergeometricU, JacobiP, KelvinBei, KelvinBer, KelvinKei, KelvinKer, LaguerreL, LegendreP, LegendreQ, LerchPhi, Log, LogGamma, LucasL, Multinomial, NevilleThetaC, ParabolicCylinderD, Pochhammer, PolyGamma, PolyLog, qBinomial, QBinomial, qBrackets, qFactorial, QFactorial, qPochhammer, QPochhammer, Root, Sin, Sinc, Sinh, SinhIntegral, SinIntegral, SphericalBesselJ, SphericalBesselY, SphericalHankelH1, SphericalHankelH2, Sqrt, StirlingS1, StirlingS2, StruveH, StruveL, Subfactorial, WeberE, WhittakerM, WhittakerW, Zeta.
If expr contains the commands D and ApplyOreOperator then the closure property DFiniteOreAction is performed: Note the difference between Annihilator[D[LegendreP[n, x], x], {S[n], Der[x]}] and expr = D[LegendreP[n, x], x]; Annihilator[expr, {S[n], Der[x]}].
Similarly, if expr contains Sum or Integrate then not Mathematica is asked to simplify the expression, but CreativeTelescoping is executed automatically on the summand (resp. integrand). For evaluating the delta part, Mathematica's FullSimplify is used; if it fails (or if you don't trust it), you can use the option Inhomogeneous −> True, in order to obtain an inhomogeneous recurrence (resp. differential equation).

In[6]:= **{ts, cs} = CreativeTelescoping[ann, S[n] - 1]**

Out[6]= $\{\{(-1 - t^2 + 2\, t\, x)\, D_x + t,\ (1 + t^2 - 2\, t\, x)\, D_t + (t - x)\},\ \{(-1 + t\, x)\, D_x - n\, t,\ (-1 + x)\, (1 + x)\, D_x + \frac{n - n\, t\, x}{t}\}\}$

In[7]:= **ct1 = ts[[1]] + (S[n] - 1) ** cs[[1]]**

Out[7]= $(-1 + t\, x)\, S_n\, D_x - (1+n)\, t\, S_n + (-t^2 + t\, x)\, D_x + (t + n\, t)$

In[8]:= **OreReduce[ct1, ann]**

Out[8]= 0

In[9]:= **Annihilator[(1 - 2 x t + t ^ 2) ^ (-1 / 2)]**

Out[9]= $\{(1 + t^2 - 2\, t\, x)\, D_x - t,\ (1 + t^2 - 2\, t\, x)\, D_t + (t - x)\}$

In[10]:=
```
OreAlgebra[%]
```
Out[10]= $\mathbb{K}(t, x)[D_t; 1, D_t][D_x; 1, D_x]$

In[11]:=
```
FullForm[%%]
```
Out[11]//FullForm=
```
List[OrePolynomial[
  List[List[Plus[1, Power[t, 2]], Times[-2, t, x]], List[0, 1]], List[Times[-1, t], List[0, 0]]],
  OreAlgebraObject[List[Der[t], Der[x]], Expand, Function[Plus[Slot[1], Slot[2]]],
   Function[Expand[Times[Slot[1], Slot[2]]]], None], DegreeLexicographic],
 OrePolynomial[List[List[Plus[1, Power[t, 2]], Times[-2, t, x]], List[1, 0]],
   List[Plus[t, Times[-1, x]], List[0, 0]]],
  OreAlgebraObject[List[Der[t], Der[x]], Expand, Function[Plus[Slot[1], Slot[2]]],
   Function[Expand[Times[Slot[1], Slot[2]]]], None], DegreeLexicographic]]
```

A Monthly Problem

Proposed by L. Glasser (2009)

In[12]:=
```
TraditionalForm[HoldForm[Integrate[arccos[x / Sqrt[(a + b) * x - a * b]], {x, a, b}] == "?"]]
```
Out[12]//TraditionalForm=

$$\int_a^b \arccos\left(\frac{x}{\sqrt{(a+b)\,x - a\,b}}\right) dx = ?$$

In[13]:=
```
Annihilator[Integrate[ArcCos[x / Sqrt[(a + b) * x - a * b]], {x, a, b}],
 {Der[a], Der[b]}, Assumptions → a ≥ 0 && b > a]
```
Out[13]= $\left\{\left(-a^2+b^2\right) D_b + (-3\,a-b),\ \left(a^2-b^2\right) D_a + (-a-3\,b)\right\}$

In[14]:=
```
ApplyOreOperator[First[%], f[b]]
```
Out[14]= $(-3\,a-b)\,f[b] + \left(-a^2+b^2\right) f'[b]$

In[15]:=
```
DSolve[% == 0, f[b], b]
```
Out[15]= $\left\{\left\{f[b] \to \frac{(-a+b)^2\, C[1]}{a+b}\right\}\right\}$

q-Summation from Knot Theory

In[16]:=
```
TraditionalForm[
 HoldForm[Sum[(-1)^k * q^(-k * n - k * (k + 3) / 2) * QPochhammer[q^(n - 1), 1 / q, k] *
    QPochhammer[q^(n + 1), q, k] * c[k]^2, {k, 0, n - 1}]]]
```
Out[16]//TraditionalForm=

$$\sum_{k=0}^{n-1} (-1)^k\, q^{-k\,n-\frac{1}{2}\,k\,(k+3)} \left(q^{n-1}; \frac{1}{q}\right)_k \left(q^{n+1}; q\right)_k\, c(k)^2$$

where $c(k+2) = -q^{k+3}\left(q^{2\,k+4} - q^{k+2} + q + 1\right) c(k+1) + q^{2\,k+6}\left(q^{k+1}-1\right) c(k)$.

In[17]:=
```
{QK, QN} = {QS[qk, q^k], QS[qn, q^n]};
```
In[18]:=
```
annc = ToOrePolynomial[{QN - 1, QK^2 + (q^(k + 3) * (1 + q - q^(k + 2) + q^(2 k + 4))) ** QK +
    q^(2 k + 6) * (1 - q^(k + 1))}, OreAlgebra[QK, QN]]
```
Out[18]= $\left\{S_{qn,q} - 1,\ S_{qk,q}^2 + \left(q^3\,qk + q^4\,qk - q^5\,qk^2 + q^7\,qk^3\right) S_{qk,q} + \left(q^6\,qk^2 - q^7\,qk^3\right)\right\}$

In[19]:=
```
annf = Annihilator[(-1)^k*q^(-k*n - k*(k + 3)/2)*
  QPochhammer[q^(n - 1), 1/q, k]*QPochhammer[q^(n + 1), q, k], {QK, QN}]
```

Out[19]=

$$\{(qk - qn - q\,qk\,qn + q\,qn^2)\,S_{qn,q} + (-1 + qn + q\,qk\,qn - q\,qk\,qn^2),$$
$$q^3\,qk^2\,qn\,S_{qk,q} + (q\,qk - qn - q^2\,qk^2\,qn + q\,qk\,qn^2)\}$$

In[20]:=
```
annSmnd = DFiniteTimes[annf, annc, annc]
```

Out[20]=

$$\{(-qk + qn + q\,qk\,qn - q\,qn^2)\,S_{qn,q} + (1 - qn - q\,qk\,qn + q\,qk\,qn^2),$$
$$(qn^3 + q\,qn^3 - q^2\,qk\,qn^3 + q^4\,qk^2\,qn^3)\,S^3_{qk,q} +$$
$$(q^4\,qk\,qn^2 + 2\,q^5\,qk\,qn^2 + 2\,q^6\,qk\,qn^2 + q^7\,qk\,qn^2 - q^6\,qk^2\,qn^2 - 3\,q^7\,qk^2\,qn^2 - 3\,q^8\,qk^2\,qn^2 -$$
$$2\,q^9\,qk^2\,qn^2 + q^8\,qk^3\,qn^2 + 4\,q^9\,qk^3\,qn^2 + 5\,q^{10}\,qk^3\,qn^2 + 4\,q^{11}\,qk^3\,qn^2 + 2\,q^{12}\,qk^3\,qn^2 -$$
$$2\,q^{11}\,qk^4\,qn^2 - 5\,q^{12}\,qk^4\,qn^2 - 3\,q^{13}\,qk^4\,qn^2 - 2\,q^{14}\,qk^4\,qn^2 + 3\,q^{14}\,qk^5\,qn^2 + 4\,q^{15}\,qk^5\,qn^2 +$$
$$q^{16}\,qk^5\,qn^2 + q^{17}\,qk^5\,qn^2 - 2\,q^{17}\,qk^6\,qn^2 - q^{18}\,qk^6\,qn^2 + q^{20}\,qk^7\,qn^2 - q\,qn^3 - 2\,q^2\,qn^3 - 2\,q^3\,qn^3 -$$
$$q^4\,qn^3 + q^3\,qk\,qn^3 + 3\,q^4\,qk\,qn^3 + 3\,q^5\,qk\,qn^3 + 2\,q^6\,qk\,qn^3 - q^5\,qk^2\,qn^3 - 4\,q^6\,qk^2\,qn^3 -$$
$$6\,q^7\,qk^2\,qn^3 - 6\,q^8\,qk^2\,qn^3 - 4\,q^9\,qk^2\,qn^3 - q^{10}\,qk^2\,qn^3 + 2\,q^8\,qk^3\,qn^3 + 6\,q^9\,qk^3\,qn^3 +$$
$$6\,q^{10}\,qk^3\,qn^3 + 5\,q^{11}\,qk^3\,qn^3 + 2\,q^{12}\,qk^3\,qn^3 - 4\,q^{11}\,qk^4\,qn^3 - 8\,q^{12}\,qk^4\,qn^3 - 6\,q^{13}\,qk^4\,qn^3 -$$
$$5\,q^{14}\,qk^4\,qn^3 - 2\,q^{15}\,qk^4\,qn^3 + 4\,q^{14}\,qk^5\,qn^3 + 6\,q^{15}\,qk^5\,qn^3 + 3\,q^{16}\,qk^5\,qn^3 + 2\,q^{17}\,qk^5\,qn^3 -$$
$$4\,q^{17}\,qk^6\,qn^3 - 4\,q^{18}\,qk^6\,qn^3 - q^{19}\,qk^6\,qn^3 - q^{20}\,qk^6\,qn^3 + 2\,q^{20}\,qk^7\,qn^3 + q^{21}\,qk^7\,qn^3 -$$
$$q^{23}\,qk^8\,qn^3 + q^4\,qk\,qn^4 + 2\,q^5\,qk\,qn^4 + 2\,q^6\,qk\,qn^4 + q^7\,qk\,qn^4 - q^6\,qk^2\,qn^4 - 3\,q^7\,qk^2\,qn^4 -$$
$$3\,q^8\,qk^2\,qn^4 - 2\,q^9\,qk^2\,qn^4 + q^8\,qk^3\,qn^4 + 4\,q^9\,qk^3\,qn^4 + 5\,q^{10}\,qk^3\,qn^4 + 4\,q^{11}\,qk^3\,qn^4 +$$
$$2\,q^{12}\,qk^3\,qn^4 - 2\,q^{11}\,qk^4\,qn^4 - 5\,q^{12}\,qk^4\,qn^4 - 3\,q^{13}\,qk^4\,qn^4 - 2\,q^{14}\,qk^4\,qn^4 + 3\,q^{14}\,qk^5\,qn^4 +$$
$$4\,q^{15}\,qk^5\,qn^4 + q^{16}\,qk^5\,qn^4 + q^{17}\,qk^5\,qn^4 - 2\,q^{17}\,qk^6\,qn^4 - q^{18}\,qk^6\,qn^4 + q^{20}\,qk^7\,qn^4)\,S^2_{qk,q} +$$
$$(q^8\,qk^2\,qn + 2\,q^9\,qk^2\,qn + 2\,q^{10}\,qk^2\,qn + q^{11}\,qk^2\,qn - 3\,q^{10}\,qk^3\,qn - 5\,q^{11}\,qk^3\,qn -$$
$$5\,q^{12}\,qk^3\,qn - 2\,q^{13}\,qk^3\,qn + 5\,q^{12}\,qk^4\,qn + 8\,q^{13}\,qk^4\,qn + 8\,q^{14}\,qk^4\,qn + 3\,q^{15}\,qk^4\,qn +$$
$$q^{16}\,qk^4\,qn - 5\,q^{14}\,qk^5\,qn - 9\,q^{15}\,qk^5\,qn - 9\,q^{16}\,qk^5\,qn - 4\,q^{17}\,qk^5\,qn - q^{18}\,qk^5\,qn +$$
$$3\,q^{16}\,qk^6\,qn + 7\,q^{17}\,qk^6\,qn + 7\,q^{18}\,qk^6\,qn + 4\,q^{19}\,qk^6\,qn - q^{18}\,qk^7\,qn - 4\,q^{19}\,qk^7\,qn -$$
$$5\,q^{20}\,qk^7\,qn - 2\,q^{21}\,qk^7\,qn + q^{21}\,qk^8\,qn + 3\,q^{22}\,qk^8\,qn - q^{24}\,qk^9\,qn - q^5\,qk\,qn^2 - 3\,q^6\,qk\,qn^2 -$$
$$4\,q^7\,qk\,qn^2 - 3\,q^8\,qk\,qn^2 - q^9\,qk\,qn^2 + 3\,q^7\,qk^2\,qn^2 + 8\,q^8\,qk^2\,qn^2 + 10\,q^9\,qk^2\,qn^2 +$$
$$7\,q^{10}\,qk^2\,qn^2 + 2\,q^{11}\,qk^2\,qn^2 - 5\,q^9\,qk^3\,qn^2 - 14\,q^{10}\,qk^3\,qn^2 - 19\,q^{11}\,qk^3\,qn^2 - 15\,q^{12}\,qk^3\,qn^2 -$$
$$7\,q^{13}\,qk^3\,qn^2 - 2\,q^{14}\,qk^3\,qn^2 + 5\,q^{11}\,qk^4\,qn^2 + 17\,q^{12}\,qk^4\,qn^2 + 26\,q^{13}\,qk^4\,qn^2 + 23\,q^{14}\,qk^4\,qn^2 +$$
$$12\,q^{15}\,qk^4\,qn^2 + 3\,q^{16}\,qk^4\,qn^2 - 3\,q^{13}\,qk^5\,qn^2 - 15\,q^{14}\,qk^5\,qn^2 - 27\,q^{15}\,qk^5\,qn^2 - 27\,q^{16}\,qk^5\,qn^2 -$$
$$15\,q^{17}\,qk^5\,qn^2 - 4\,q^{18}\,qk^5\,qn^2 - q^{19}\,qk^5\,qn^2 + q^{15}\,qk^6\,qn^2 + 10\,q^{16}\,qk^6\,qn^2 + 23\,q^{17}\,qk^6\,qn^2 +$$
$$25\,q^{18}\,qk^6\,qn^2 + 15\,q^{19}\,qk^6\,qn^2 + 5\,q^{20}\,qk^6\,qn^2 + q^{21}\,qk^6\,qn^2 - 4\,q^{18}\,qk^7\,qn^2 - 14\,q^{19}\,qk^7\,qn^2 -$$
$$17\,q^{20}\,qk^7\,qn^2 - 11\,q^{21}\,qk^7\,qn^2 - 4\,q^{22}\,qk^7\,qn^2 + q^{20}\,qk^8\,qn^2 + 6\,q^{21}\,qk^8\,qn^2 + 10\,q^{22}\,qk^8\,qn^2 +$$
$$7\,q^{23}\,qk^8\,qn^2 + 2\,q^{24}\,qk^8\,qn^2 - q^{23}\,qk^9\,qn^2 - 4\,q^{24}\,qk^9\,qn^2 - 3\,q^{25}\,qk^9\,qn^2 + q^{26}\,qk^{10}\,qn^2 +$$
$$q^{27}\,qk^{10}\,qn^2 + q^3\,qn^3 + 2\,q^4\,qn^3 + 2\,q^5\,qn^3 + q^6\,qn^3 - 3\,q^5\,qk\,qn^3 - 5\,q^6\,qk\,qn^3 - 5\,q^7\,qk\,qn^3 -$$
$$2\,q^8\,qk\,qn^3 + 6\,q^7\,qk^2\,qn^3 + 12\,q^8\,qk^2\,qn^3 + 15\,q^9\,qk^2\,qn^3 + 10\,q^{10}\,qk^2\,qn^3 + 5\,q^{11}\,qk^2\,qn^3 +$$
$$q^{12}\,qk^2\,qn^3 - 8\,q^9\,qk^3\,qn^3 - 20\,q^{10}\,qk^3\,qn^3 - 27\,q^{11}\,qk^3\,qn^3 - 21\,q^{12}\,qk^3\,qn^3 - 10\,q^{13}\,qk^3\,qn^3 -$$
$$2\,q^{14}\,qk^3\,qn^3 + 8\,q^{11}\,qk^4\,qn^3 + 25\,q^{12}\,qk^4\,qn^3 + 37\,q^{13}\,qk^4\,qn^3 + 33\,q^{14}\,qk^4\,qn^3 + 17\,q^{15}\,qk^4\,qn^3 +$$
$$6\,q^{16}\,qk^4\,qn^3 + q^{17}\,qk^4\,qn^3 - 6\,q^{13}\,qk^5\,qn^3 - 23\,q^{14}\,qk^5\,qn^3 - 40\,q^{15}\,qk^5\,qn^3 - 38\,q^{16}\,qk^5\,qn^3 -$$
$$23\,q^{17}\,qk^5\,qn^3 - 8\,q^{18}\,qk^5\,qn^3 - q^{19}\,qk^5\,qn^3 + 3\,q^{15}\,qk^6\,qn^3 + 14\,q^{16}\,qk^6\,qn^3 + 32\,q^{17}\,qk^6\,qn^3 +$$
$$33\,q^{18}\,qk^6\,qn^3 + 23\,q^{19}\,qk^6\,qn^3 + 7\,q^{20}\,qk^6\,qn^3 + q^{21}\,qk^6\,qn^3 - q^{17}\,qk^7\,qn^3 - 6\,q^{18}\,qk^7\,qn^3 -$$
$$20\,q^{19}\,qk^7\,qn^3 - 25\,q^{20}\,qk^7\,qn^3 - 18\,q^{21}\,qk^7\,qn^3 - 6\,q^{22}\,qk^7\,qn^3 - q^{23}\,qk^7\,qn^3 + q^{20}\,qk^8\,qn^3 +$$
$$8\,q^{21}\,qk^8\,qn^3 + 14\,q^{22}\,qk^8\,qn^3 + 10\,q^{23}\,qk^8\,qn^3 + 4\,q^{24}\,qk^8\,qn^3 - 2\,q^{23}\,qk^9\,qn^3 - 6\,q^{24}\,qk^9\,qn^3 -$$
$$6\,q^{25}\,qk^9\,qn^3 - 2\,q^{26}\,qk^9\,qn^3 + q^{26}\,qk^{10}\,qn^3 + 3\,q^{27}\,qk^{10}\,qn^3 - q^{29}\,qk^{11}\,qn^3 - q^5\,qk\,qn^4 -$$
$$3\,q^6\,qk\,qn^4 - 4\,q^7\,qk\,qn^4 - 3\,q^8\,qk\,qn^4 - q^9\,qk\,qn^4 + 3\,q^7\,qk^2\,qn^4 + 8\,q^8\,qk^2\,qn^4 + 10\,q^9\,qk^2\,qn^4 +$$
$$7\,q^{10}\,qk^2\,qn^4 + 2\,q^{11}\,qk^2\,qn^4 - 5\,q^9\,qk^3\,qn^4 - 14\,q^{10}\,qk^3\,qn^4 - 19\,q^{11}\,qk^3\,qn^4 - 15\,q^{12}\,qk^3\,qn^4 -$$
$$7\,q^{13}\,qk^3\,qn^4 - 2\,q^{14}\,qk^3\,qn^4 + 5\,q^{11}\,qk^4\,qn^4 + 17\,q^{12}\,qk^4\,qn^4 + 26\,q^{13}\,qk^4\,qn^4 + 23\,q^{14}\,qk^4\,qn^4 +$$

$12 q^{15} qk^4 qn^4 + 3 q^{16} qk^4 qn^4 - 3 q^{13} qk^5 qn^4 - 15 q^{14} qk^5 qn^4 - 27 q^{15} qk^5 qn^4 - 27 q^{16} qk^5 qn^4 -$
$15 q^{17} qk^5 qn^4 - 4 q^{18} qk^5 qn^4 - q^{19} qk^5 qn^4 + q^{15} qk^6 qn^4 + 10 q^{16} qk^6 qn^4 + 23 q^{17} qk^6 qn^4 +$
$25 q^{18} qk^6 qn^4 + 15 q^{19} qk^6 qn^4 + 5 q^{20} qk^6 qn^4 + q^{21} qk^6 qn^4 - 4 q^{18} qk^7 qn^4 - 14 q^{19} qk^7 qn^4 -$
$17 q^{20} qk^7 qn^4 - 11 q^{21} qk^7 qn^4 - 4 q^{22} qk^7 qn^4 + q^{20} qk^8 qn^4 + 6 q^{21} qk^8 qn^4 + 10 q^{22} qk^8 qn^4 +$
$7 q^{23} qk^8 qn^4 + 2 q^{24} qk^8 qn^4 - q^{23} qk^9 qn^4 - 4 q^{24} qk^9 qn^4 - 3 q^{25} qk^9 qn^4 + q^{26} qk^{10} qn^4 +$
$q^{27} qk^{10} qn^4 + q^8 qk^2 qn^5 + 2 q^9 qk^2 qn^5 + 2 q^{10} qk^2 qn^5 + q^{11} qk^2 qn^5 - 3 q^{10} qk^3 qn^5 -$
$5 q^{11} qk^3 qn^5 - 5 q^{12} qk^3 qn^5 - 2 q^{13} qk^3 qn^5 + 5 q^{12} qk^4 qn^5 + 8 q^{13} qk^4 qn^5 + 8 q^{14} qk^4 qn^5 +$
$3 q^{15} qk^4 qn^5 + q^{16} qk^4 qn^5 - 5 q^{14} qk^5 qn^5 - 9 q^{15} qk^5 qn^5 - 9 q^{16} qk^5 qn^5 - 4 q^{17} qk^5 qn^5 -$
$q^{18} qk^5 qn^5 + 3 q^{16} qk^6 qn^5 + 7 q^{17} qk^6 qn^5 + 7 q^{18} qk^6 qn^5 + 4 q^{19} qk^6 qn^5 - q^{18} qk^7 qn^5 -$
$4 q^{19} qk^7 qn^5 - 5 q^{20} qk^7 qn^5 - 2 q^{21} qk^7 qn^5 + q^{21} qk^8 qn^5 + 3 q^{22} qk^8 qn^5 - q^{24} qk^9 qn^5 \big) S_{qk,q} +$
$\big(q^{12} qk^3 + q^{13} qk^3 - 2 q^{13} qk^4 - 3 q^{14} qk^4 - 2 q^{15} qk^4 + q^{14} qk^5 + 3 q^{15} qk^5 + 4 q^{16} qk^5 +$
$q^{17} qk^5 + q^{18} qk^5 - q^{16} qk^6 - 2 q^{17} qk^6 - 2 q^{18} qk^6 - 2 q^{19} qk^6 - q^{20} qk^6 + q^{19} qk^7 + q^{20} qk^7 +$
$2 q^{21} qk^7 - q^{22} qk^8 - q^9 qk^2 qn - 2 q^{10} qk^2 qn - 2 q^{11} qk^2 qn - q^{12} qk^2 qn + 2 q^{10} qk^3 qn +$
$5 q^{11} qk^3 qn + 7 q^{12} qk^3 qn + 5 q^{13} qk^3 qn + 2 q^{14} qk^3 qn - q^{11} qk^4 qn - 4 q^{12} qk^4 qn -$
$9 q^{13} qk^4 qn - 10 q^{14} qk^4 qn - 8 q^{15} qk^4 qn - 3 q^{16} qk^4 qn - q^{17} qk^4 qn + q^{13} qk^5 qn + 5 q^{14} qk^5 qn +$
$10 q^{15} qk^5 qn + 13 q^{16} qk^5 qn + 10 q^{17} qk^5 qn + 5 q^{18} qk^5 qn + q^{19} qk^5 qn - q^{15} qk^6 qn -$
$5 q^{16} qk^6 qn - 10 q^{17} qk^6 qn - 12 q^{18} qk^6 qn - 9 q^{19} qk^6 qn - 4 q^{20} qk^6 qn - q^{21} qk^6 qn +$
$q^{17} qk^7 qn + 3 q^{18} qk^7 qn + 6 q^{19} qk^7 qn + 7 q^{20} qk^7 qn + 6 q^{21} qk^7 qn + 3 q^{22} qk^7 qn + q^{23} qk^7 qn -$
$q^{20} qk^8 qn - 2 q^{21} qk^8 qn - 4 q^{22} qk^8 qn - 3 q^{23} qk^8 qn - 2 q^{24} qk^8 qn + q^{23} qk^9 qn + q^{24} qk^9 qn +$
$q^{25} qk^9 qn + q^7 qk qn^2 + 2 q^8 qk qn^2 + 2 q^9 qk qn^2 + q^{10} qk qn^2 - 2 q^8 qk^2 qn^2 - 5 q^9 qk^2 qn^2 -$
$7 q^{10} qk^2 qn^2 - 5 q^{11} qk^2 qn^2 - 2 q^{12} qk^2 qn^2 + q^9 qk^3 qn^2 + 5 q^{10} qk^3 qn^2 + 11 q^{11} qk^3 qn^2 +$
$13 q^{12} qk^3 qn^2 + 11 q^{13} qk^3 qn^2 + 5 q^{14} qk^3 qn^2 + 2 q^{15} qk^3 qn^2 - 3 q^{11} qk^4 qn^2 - 10 q^{12} qk^4 qn^2 -$
$19 q^{13} qk^4 qn^2 - 23 q^{14} qk^4 qn^2 - 19 q^{15} qk^4 qn^2 - 10 q^{16} qk^4 qn^2 - 3 q^{17} qk^4 qn^2 + q^{12} qk^5 qn^2 +$
$5 q^{13} qk^5 qn^2 + 14 q^{14} qk^5 qn^2 + 23 q^{15} qk^5 qn^2 + 28 q^{16} qk^5 qn^2 + 21 q^{17} qk^5 qn^2 + 12 q^{18} qk^5 qn^2 +$
$3 q^{19} qk^5 qn^2 + q^{20} qk^5 qn^2 - q^{14} qk^6 qn^2 - 4 q^{15} qk^6 qn^2 - 11 q^{16} qk^6 qn^2 - 20 q^{17} qk^6 qn^2 -$
$24 q^{18} qk^6 qn^2 - 20 q^{19} qk^6 qn^2 - 11 q^{20} qk^6 qn^2 - 4 q^{21} qk^6 qn^2 - q^{22} qk^6 qn^2 + 2 q^{17} qk^7 qn^2 +$
$7 q^{18} qk^7 qn^2 + 15 q^{19} qk^7 qn^2 + 17 q^{20} qk^7 qn^2 + 15 q^{21} qk^7 qn^2 + 7 q^{22} qk^7 qn^2 + 3 q^{23} qk^7 qn^2 -$
$q^{19} qk^8 qn^2 - 4 q^{20} qk^8 qn^2 - 7 q^{21} qk^8 qn^2 - 9 q^{22} qk^8 qn^2 - 7 q^{23} qk^8 qn^2 - 4 q^{24} qk^8 qn^2 -$
$q^{25} qk^8 qn^2 + q^{22} qk^9 qn^2 + 2 q^{23} qk^9 qn^2 + 4 q^{24} qk^9 qn^2 + 3 q^{25} qk^9 qn^2 + 2 q^{26} qk^9 qn^2 -$
$q^{25} qk^{10} qn^2 - q^{26} qk^{10} qn^2 - q^{27} qk^{10} qn^2 - q^6 qn^3 - q^7 qn^3 + 2 q^7 qk qn^3 + 3 q^8 qk qn^3 +$
$2 q^9 qk qn^3 - 2 q^8 qk^2 qn^3 - 6 q^9 qk^2 qn^3 - 9 q^{10} qk^2 qn^3 - 6 q^{11} qk^2 qn^3 - 4 q^{12} qk^2 qn^3 -$
$q^{13} qk^2 qn^3 + 2 q^9 qk^3 qn^3 + 8 q^{10} qk^3 qn^3 + 16 q^{11} qk^3 qn^3 + 19 q^{12} qk^3 qn^3 + 16 q^{13} qk^3 qn^3 +$
$8 q^{14} qk^3 qn^3 + 2 q^{15} qk^3 qn^3 - q^{10} qk^4 qn^3 - 5 q^{11} qk^4 qn^3 - 14 q^{12} qk^4 qn^3 - 24 q^{13} qk^4 qn^3 -$
$28 q^{14} qk^4 qn^3 - 23 q^{15} qk^4 qn^3 - 12 q^{16} qk^4 qn^3 - 4 q^{17} qk^4 qn^3 - q^{18} qk^4 qn^3 + q^{12} qk^5 qn^3 +$
$6 q^{13} qk^5 qn^3 + 16 q^{14} qk^5 qn^3 + 28 q^{15} qk^5 qn^3 + 34 q^{16} qk^5 qn^3 + 28 q^{17} qk^5 qn^3 + 16 q^{18} qk^5 qn^3 +$
$6 q^{19} qk^5 qn^3 + q^{20} qk^5 qn^3 - q^{14} qk^6 qn^3 - 6 q^{15} qk^6 qn^3 - 16 q^{16} qk^6 qn^3 - 27 q^{17} qk^6 qn^3 -$
$32 q^{18} qk^6 qn^3 - 26 q^{19} qk^6 qn^3 - 14 q^{20} qk^6 qn^3 - 5 q^{21} qk^6 qn^3 - q^{22} qk^6 qn^3 + q^{16} qk^7 qn^3 +$
$4 q^{17} qk^7 qn^3 + 10 q^{18} qk^7 qn^3 + 18 q^{19} qk^7 qn^3 + 22 q^{20} qk^7 qn^3 + 18 q^{21} qk^7 qn^3 + 10 q^{22} qk^7 qn^3 +$
$4 q^{23} qk^7 qn^3 + q^{24} qk^7 qn^3 - q^{19} qk^8 qn^3 - 4 q^{20} qk^8 qn^3 - 10 q^{21} qk^8 qn^3 - 13 q^{22} qk^8 qn^3 -$
$10 q^{23} qk^8 qn^3 - 6 q^{24} qk^8 qn^3 - 2 q^{25} qk^8 qn^3 + 2 q^{22} qk^9 qn^3 + 4 q^{23} qk^9 qn^3 + 5 q^{24} qk^9 qn^3 +$
$4 q^{25} qk^9 qn^3 + 2 q^{26} qk^9 qn^3 - q^{25} qk^{10} qn^3 - q^{26} qk^{10} qn^3 - 2 q^{27} qk^{10} qn^3 + q^{28} qk^{11} qn^3 +$
$q^7 qk qn^4 + 2 q^8 qk qn^4 + 2 q^9 qk qn^4 + q^{10} qk qn^4 - 2 q^8 qk^2 qn^4 - 5 q^9 qk^2 qn^4 - 7 q^{10} qk^2 qn^4 -$
$5 q^{11} qk^2 qn^4 - 2 q^{12} qk^2 qn^4 + q^9 qk^3 qn^4 + 5 q^{10} qk^3 qn^4 + 11 q^{11} qk^3 qn^4 + 13 q^{12} qk^3 qn^4 +$
$11 q^{13} qk^3 qn^4 + 5 q^{14} qk^3 qn^4 + 2 q^{15} qk^3 qn^4 - 3 q^{11} qk^4 qn^4 - 10 q^{12} qk^4 qn^4 - 19 q^{13} qk^4 qn^4 -$
$23 q^{14} qk^4 qn^4 - 19 q^{15} qk^4 qn^4 - 10 q^{16} qk^4 qn^4 - 3 q^{17} qk^4 qn^4 + q^{12} qk^5 qn^4 + 5 q^{13} qk^5 qn^4 +$
$14 q^{14} qk^5 qn^4 + 23 q^{15} qk^5 qn^4 + 28 q^{16} qk^5 qn^4 + 21 q^{17} qk^5 qn^4 + 12 q^{18} qk^5 qn^4 + 3 q^{19} qk^5 qn^4 +$
$q^{20} qk^5 qn^4 - q^{14} qk^6 qn^4 - 4 q^{15} qk^6 qn^4 - 11 q^{16} qk^6 qn^4 - 20 q^{17} qk^6 qn^4 - 24 q^{18} qk^6 qn^4 -$
$20 q^{19} qk^6 qn^4 - 11 q^{20} qk^6 qn^4 - 4 q^{21} qk^6 qn^4 - q^{22} qk^6 qn^4 + 2 q^{17} qk^7 qn^4 + 7 q^{18} qk^7 qn^4 +$

$15 q^{19} qk^7 qn^4 + 17 q^{20} qk^7 qn^4 + 15 q^{21} qk^7 qn^4 + 7 q^{22} qk^7 qn^4 + 3 q^{23} qk^7 qn^4 - q^{19} qk^8 qn^4 -$
$4 q^{20} qk^8 qn^4 - 7 q^{21} qk^8 qn^4 - 9 q^{22} qk^8 qn^4 - 7 q^{23} qk^8 qn^4 - 4 q^{24} qk^8 qn^4 - q^{25} qk^8 qn^4 +$
$q^{22} qk^9 qn^4 + 2 q^{23} qk^9 qn^4 + 4 q^{24} qk^9 qn^4 + 3 q^{25} qk^9 qn^4 + 2 q^{26} qk^9 qn^4 - q^{25} qk^{10} qn^4 -$
$q^{26} qk^{10} qn^4 - q^{27} qk^{10} qn^4 - q^9 qk^2 qn^5 - 2 q^{10} qk^2 qn^5 - 2 q^{11} qk^2 qn^5 - q^{12} qk^2 qn^5 +$
$2 q^{10} qk^3 qn^5 + 5 q^{11} qk^3 qn^5 + 7 q^{12} qk^3 qn^5 + 5 q^{13} qk^3 qn^5 + 2 q^{14} qk^3 qn^5 - q^{11} qk^4 qn^5 -$
$4 q^{12} qk^4 qn^5 - 9 q^{13} qk^4 qn^5 - 10 q^{14} qk^4 qn^5 - 8 q^{15} qk^4 qn^5 - 3 q^{16} qk^4 qn^5 - q^{17} qk^4 qn^5 +$
$q^{13} qk^5 qn^5 + 5 q^{14} qk^5 qn^5 + 10 q^{15} qk^5 qn^5 + 13 q^{16} qk^5 qn^5 + 10 q^{17} qk^5 qn^5 + 5 q^{18} qk^5 qn^5 +$
$q^{19} qk^5 qn^5 - q^{15} qk^6 qn^5 - 5 q^{16} qk^6 qn^5 - 10 q^{17} qk^6 qn^5 - 12 q^{18} qk^6 qn^5 - 9 q^{19} qk^6 qn^5 -$
$4 q^{20} qk^6 qn^5 - q^{21} qk^6 qn^5 + q^{17} qk^7 qn^5 + 3 q^{18} qk^7 qn^5 + 6 q^{19} qk^7 qn^5 + 7 q^{20} qk^7 qn^5 +$
$6 q^{21} qk^7 qn^5 + 3 q^{22} qk^7 qn^5 + q^{23} qk^7 qn^5 - q^{20} qk^8 qn^5 - 2 q^{21} qk^8 qn^5 - 4 q^{22} qk^8 qn^5 -$
$3 q^{23} qk^8 qn^5 - 2 q^{24} qk^8 qn^5 + q^{23} qk^9 qn^5 + q^{24} qk^9 qn^5 + q^{25} qk^9 qn^5 + q^{12} qk^3 qn^6 +$
$q^{13} qk^3 qn^6 - 2 q^{13} qk^4 qn^6 - 3 q^{14} qk^4 qn^6 - 2 q^{15} qk^4 qn^6 + q^{14} qk^5 qn^6 + 3 q^{15} qk^5 qn^6 +$
$4 q^{16} qk^5 qn^6 + q^{17} qk^5 qn^6 + q^{18} qk^5 qn^6 - q^{16} qk^6 qn^6 - 2 q^{17} qk^6 qn^6 - 2 q^{18} qk^6 qn^6 -$
$2 q^{19} qk^6 qn^6 - q^{20} qk^6 qn^6 + q^{19} qk^7 qn^6 + q^{20} qk^7 qn^6 + 2 q^{21} qk^7 qn^6 - q^{22} qk^8 qn^6)\}$

In[21]:= **Support[annSmnd]**

Out[21]= $\{\{S_{qn,q}, 1\}, \{S^3_{qk,q}, S^2_{qk,q}, S_{qk,q}, 1\}\}$

In[22]:= **Timing[{{rec}, {cert}} = CreativeTelescoping[annSmnd, QK − 1, Support → Table[QN ^i, {i, 0, 5}]];]**

Out[22]= {30.0734, Null}

In[23]:= **rec**

Out[23]=

A very large output was generated. Here is a sample of it:

$(-1 - q + q\,qn + q^2 qn + 2 q^3 qn + q^4 qn + q^5 qn + q^6 qn + q\,qn^2 + \ll 133 \gg +$
$3 q^{17} qn^{11} + 3 q^{18} qn^{11} + q^{19} qn^{11} - q^{20} qn^{11} - 2 q^{22} qn^{11} - q^{23} qn^{11} - q^{18} qn^{12} -$
$q^{19} qn^{12} - q^{20} qn^{12} - q^{21} qn^{12} - 2 q^{22} qn^{12} - q^{23} qn^{12} + q^{23} qn^{13} + q^{24} qn^{13}) S^5_{qn,q} +$
$\ll 4 \gg + (-q^{18} qn^{11} - q^{19} qn^{11} + \ll 163 \gg + q^{60} qn^{24} + q^{61} qn^{24})$

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In[24]:= **Factor[%]**

Out[24]=
$(-1 + q\,qn)(1 + q\,qn)(-1 + q^2 qn)(1 + q^2 qn)(-1 + q^5 qn)(-1 + q\,qn^2)$
$(-1 + q^3 qn^2)(1 + q - q\,qn - q^2 qn - 2 q^3 qn - q^4 qn + q^2 qn^2 + 2 q^3 qn^2 + 2 q^4 qn^2 +$
$q^5 qn^2 + 2 q^6 qn^2 - q^5 qn^3 - q^6 qn^3 - 2 q^7 qn^3 - q^8 qn^3 + q^8 qn^4 + q^9 qn^4) S^5_{qn,q} +$
$q(-1 + q\,qn)(1 + q\,qn)(-1 + q^4 qn)^2 (1 + q^4 qn)(-1 + q\,qn^2)(-1 + q^3 qn^2)$
$(-2 - 3 q - q^2 + 2 q\,qn + 3 q^2 qn + 8 q^3 qn + 9 q^4 qn + 6 q^5 qn + q^6 qn - 2 q^2 qn^2 - 5 q^3 qn^2 -$
$7 q^4 qn^2 - 8 q^5 qn^2 - 10 q^6 qn^2 - 13 q^7 qn^2 - 8 q^8 qn^2 - 2 q^9 qn^2 + 3 q^{10} qn^2 + q^{11} qn^2 + 3 q^5 qn^3 +$
$9 q^6 qn^3 + 9 q^7 qn^3 + 9 q^8 qn^3 + 7 q^9 qn^3 + 10 q^{10} qn^3 + 7 q^{11} qn^3 - 3 q^{12} qn^3 - 6 q^{13} qn^3 -$
$5 q^{14} qn^3 - q^{15} qn^3 + 2 q^6 qn^4 + 4 q^7 qn^4 + 5 q^8 qn^4 - q^9 qn^4 - 3 q^{10} qn^4 - 3 q^{11} qn^4 - 4 q^{12} qn^4 +$
$q^{13} qn^4 - 2 q^{14} qn^4 + 4 q^{15} qn^4 + 9 q^{16} qn^4 + 7 q^{17} qn^4 + 4 q^{18} qn^4 - 3 q^9 qn^5 - 7 q^{10} qn^5 -$
$10 q^{11} qn^5 - 7 q^{12} qn^5 + 4 q^{13} qn^5 - q^{14} qn^5 - 3 q^{15} qn^5 - q^{16} qn^5 - 4 q^{17} qn^5 - q^{18} qn^5 -$
$8 q^{19} qn^5 - 8 q^{20} qn^5 - 3 q^{21} qn^5 - q^{22} qn^5 + 2 q^{12} qn^6 + 7 q^{13} qn^6 + 7 q^{14} qn^6 + 7 q^{15} qn^6 +$
$q^{16} qn^6 - 8 q^{17} qn^6 - 2 q^{19} qn^6 + 4 q^{21} qn^6 + q^{22} qn^6 + 5 q^{23} qn^6 + 2 q^{24} qn^6 + q^{25} qn^6 -$
$3 q^{16} qn^7 - 6 q^{17} qn^7 + 3 q^{19} qn^7 + 10 q^{20} qn^7 + 12 q^{21} qn^7 + 4 q^{23} qn^7 + 2 q^{24} qn^7 - q^{25} qn^7 -$
$2 q^{27} qn^7 + 2 q^{20} qn^8 + q^{21} qn^8 - 6 q^{22} qn^8 - 6 q^{23} qn^8 - 10 q^{24} qn^8 - 9 q^{25} qn^8 - 2 q^{26} qn^8 -$
$4 q^{27} qn^8 - q^{24} qn^9 + 2 q^{25} qn^9 + 7 q^{26} qn^9 + 5 q^{27} qn^9 + 6 q^{28} qn^9 + 3 q^{29} qn^9 + 2 q^{31} qn^9 -$

$2 q^{29} qn^{10} - 3 q^{30} qn^{10} - q^{31} qn^{10} - 2 q^{32} qn^{10} - q^{33} qn^{10} + q^{33} qn^{11} + q^{34} qn^{11}\Big) S^{4}_{qn,q} -$

$q^{2} (-1 + q\, qn) (1 + q\, qn) \left(-1 + q^{3} qn\right)^{2} \left(1 + q^{3} qn\right) \left(-1 + q\, qn^{2}\right) \left(-1 + q^{9} qn^{2}\right)$

$\Big(-1 - 3 q - 2 q^{2} + q\, qn + 3 q^{2} qn + 10 q^{3} qn + 11 q^{4} qn + 8 q^{5} qn + 2 q^{6} qn - q^{2} qn^{2} - 3 q^{3} qn^{2} -$

$8 q^{4} qn^{2} - 12 q^{5} qn^{2} - 23 q^{6} qn^{2} - 23 q^{7} qn^{2} - 10 q^{8} qn^{2} - 2 q^{9} qn^{2} + 2 q^{10} qn^{2} - q^{4} qn^{3} + 3 q^{5} qn^{3} +$

$6 q^{6} qn^{3} + 13 q^{7} qn^{3} + 20 q^{8} qn^{3} + 23 q^{9} qn^{3} + 22 q^{10} qn^{3} + 3 q^{11} qn^{3} - 9 q^{12} qn^{3} - 10 q^{13} qn^{3} -$

$2 q^{14} qn^{3} + q^{5} qn^{4} + 5 q^{6} qn^{4} + 11 q^{7} qn^{4} + 6 q^{8} qn^{4} - 2 q^{9} qn^{4} - 4 q^{10} qn^{4} - 16 q^{11} qn^{4} - 19 q^{12} qn^{4} -$

$8 q^{13} qn^{4} + 11 q^{14} qn^{4} + 25 q^{15} qn^{4} + 16 q^{16} qn^{4} + 5 q^{17} qn^{4} - 6 q^{8} qn^{5} - 18 q^{9} qn^{5} - 25 q^{10} qn^{5} -$

$22 q^{11} qn^{5} - 4 q^{12} qn^{5} + 3 q^{13} qn^{5} + 9 q^{14} qn^{5} + 13 q^{15} qn^{5} + q^{16} qn^{5} - 18 q^{17} qn^{5} - 30 q^{18} qn^{5} -$

$17 q^{19} qn^{5} - 2 q^{20} qn^{5} + 2 q^{21} qn^{5} - q^{9} qn^{6} - q^{10} qn^{6} + 5 q^{11} qn^{6} + 22 q^{12} qn^{6} + 20 q^{13} qn^{6} +$

$7 q^{14} qn^{6} - 8 q^{15} qn^{6} - 21 q^{16} qn^{6} - 20 q^{17} qn^{6} - 19 q^{18} qn^{6} - 2 q^{19} qn^{6} + 18 q^{20} qn^{6} + 23 q^{21} qn^{6} +$

$11 q^{22} qn^{6} - 3 q^{23} qn^{6} - 5 q^{24} qn^{6} + 3 q^{12} qn^{7} + 3 q^{13} qn^{7} - q^{14} qn^{7} - 14 q^{15} qn^{7} - 14 q^{16} qn^{7} +$

$12 q^{17} qn^{7} + 31 q^{18} qn^{7} + 39 q^{19} qn^{7} + 30 q^{20} qn^{7} + 23 q^{21} qn^{7} + 4 q^{22} qn^{7} - 16 q^{23} qn^{7} - 18 q^{24} qn^{7} -$

$3 q^{25} qn^{7} + 6 q^{26} qn^{7} + 4 q^{27} qn^{7} - 3 q^{15} qn^{8} - 3 q^{16} qn^{8} + 6 q^{17} qn^{8} + 13 q^{18} qn^{8} + 9 q^{19} qn^{8} -$

$16 q^{20} qn^{8} - 35 q^{21} qn^{8} - 36 q^{22} qn^{8} - 25 q^{23} qn^{8} - 11 q^{24} qn^{8} + 4 q^{25} qn^{8} + 14 q^{26} qn^{8} + 14 q^{27} qn^{8} +$

$3 q^{28} qn^{8} - 6 q^{29} qn^{8} - q^{30} qn^{8} + q^{18} qn^{9} - 10 q^{20} qn^{9} - 20 q^{21} qn^{9} - 15 q^{22} qn^{9} + 7 q^{23} qn^{9} +$

$24 q^{24} qn^{9} + 20 q^{25} qn^{9} + 10 q^{26} qn^{9} - 11 q^{28} qn^{9} - 17 q^{29} qn^{9} - 9 q^{30} qn^{9} - 2 q^{31} qn^{9} + 2 q^{32} qn^{9} +$

$2 q^{22} qn^{10} + 9 q^{23} qn^{10} + 16 q^{24} qn^{10} + 11 q^{25} qn^{10} - 6 q^{26} qn^{10} - 19 q^{27} qn^{10} - 14 q^{28} qn^{10} +$

$8 q^{30} qn^{10} + 8 q^{31} qn^{10} + 12 q^{32} qn^{10} + 6 q^{33} qn^{10} - q^{25} qn^{11} - 5 q^{26} qn^{11} - 8 q^{27} qn^{11} - 2 q^{28} qn^{11} +$

$11 q^{29} qn^{11} + 17 q^{30} qn^{11} + 11 q^{31} qn^{11} + 3 q^{32} qn^{11} - 4 q^{33} qn^{11} - 5 q^{34} qn^{11} - 4 q^{35} qn^{11} -$

$2 q^{36} qn^{11} + q^{29} qn^{12} + 2 q^{30} qn^{12} - 3 q^{31} qn^{12} - 12 q^{32} qn^{12} - 17 q^{33} qn^{12} - 10 q^{34} qn^{12} - 2 q^{35} qn^{12} -$

$2 q^{36} qn^{12} + 2 q^{37} qn^{12} + q^{38} qn^{12} + 3 q^{34} qn^{13} + 8 q^{35} qn^{13} + 9 q^{36} qn^{13} + 7 q^{37} qn^{13} + q^{38} qn^{13} +$

$q^{39} qn^{13} - q^{37} qn^{14} - 4 q^{38} qn^{14} - 3 q^{39} qn^{14} - 2 q^{40} qn^{14} - q^{41} qn^{14} + q^{41} qn^{15} + q^{42} qn^{15}\Big) S^{3}_{qn,q} +$

$q^{4} \left(-1 + q^{2} qn\right)^{2} \left(1 + q^{2} qn\right) \left(-1 + q^{4} qn\right) \left(1 + q^{4} qn\right) \left(-1 + q\, qn^{2}\right) \left(-1 + q^{9} qn^{2}\right)$

$\Big(-1 - q + q\, qn + 4 q^{2} qn + 3 q^{3} qn + 2 q^{4} qn + q^{5} qn - 3 q^{3} qn^{2} - 8 q^{4} qn^{2} - 9 q^{5} qn^{2} - 7 q^{6} qn^{2} -$

$q^{7} qn^{2} - q^{8} qn^{2} - q^{3} qn^{3} - 2 q^{4} qn^{3} + 3 q^{5} qn^{3} + 12 q^{6} qn^{3} + 17 q^{7} qn^{3} + 10 q^{8} qn^{3} + 2 q^{9} qn^{3} +$

$2 q^{10} qn^{3} - 2 q^{11} qn^{3} - q^{12} qn^{3} + q^{4} qn^{4} + 5 q^{5} qn^{4} + 8 q^{6} qn^{4} + 2 q^{7} qn^{4} - 11 q^{8} qn^{4} -$

$17 q^{9} qn^{4} - 11 q^{10} qn^{4} - 3 q^{11} qn^{4} + 4 q^{12} qn^{4} + 5 q^{13} qn^{4} + 4 q^{14} qn^{4} + 2 q^{15} qn^{4} - 2 q^{6} qn^{5} -$

$9 q^{7} qn^{5} - 16 q^{8} qn^{5} - 11 q^{9} qn^{5} + 6 q^{10} qn^{5} + 19 q^{11} qn^{5} + 14 q^{12} qn^{5} - 8 q^{14} qn^{5} - 8 q^{15} qn^{5} -$

$12 q^{16} qn^{5} - 6 q^{17} qn^{5} - q^{7} qn^{6} + 10 q^{9} qn^{6} + 20 q^{10} qn^{6} + 15 q^{11} qn^{6} - 7 q^{12} qn^{6} - 24 q^{13} qn^{6} -$

$20 q^{14} qn^{6} - 10 q^{15} qn^{6} + 11 q^{17} qn^{6} + 17 q^{18} qn^{6} + 9 q^{19} qn^{6} + 2 q^{20} qn^{6} - 2 q^{21} qn^{6} + 3 q^{9} qn^{7} +$

$3 q^{10} qn^{7} - 6 q^{11} qn^{7} - 13 q^{12} qn^{7} - 9 q^{13} qn^{7} + 16 q^{14} qn^{7} + 35 q^{15} qn^{7} + 36 q^{16} qn^{7} + 25 q^{17} qn^{7} +$

$11 q^{18} qn^{7} - 4 q^{19} qn^{7} - 14 q^{20} qn^{7} - 14 q^{21} qn^{7} - 3 q^{22} qn^{7} + 6 q^{23} qn^{7} + q^{24} qn^{7} - 3 q^{11} qn^{8} -$

$3 q^{12} qn^{8} + q^{13} qn^{8} + 14 q^{14} qn^{8} + 14 q^{15} qn^{8} - 12 q^{16} qn^{8} - 31 q^{17} qn^{8} - 39 q^{18} qn^{8} - 30 q^{19} qn^{8} -$

$23 q^{20} qn^{8} - 4 q^{21} qn^{8} + 16 q^{22} qn^{8} + 18 q^{23} qn^{8} + 3 q^{24} qn^{8} - 6 q^{25} qn^{8} - 4 q^{26} qn^{8} + q^{13} qn^{9} +$

$q^{14} qn^{9} - 5 q^{15} qn^{9} - 22 q^{16} qn^{9} - 20 q^{17} qn^{9} - 7 q^{18} qn^{9} + 8 q^{19} qn^{9} + 21 q^{20} qn^{9} + 20 q^{21} qn^{9} +$

$19 q^{22} qn^{9} + 2 q^{23} qn^{9} - 18 q^{24} qn^{9} - 23 q^{25} qn^{9} - 11 q^{26} qn^{9} + 3 q^{27} qn^{9} + 5 q^{28} qn^{9} +$

$6 q^{17} qn^{10} + 18 q^{18} qn^{10} + 25 q^{19} qn^{10} + 22 q^{20} qn^{10} + 4 q^{21} qn^{10} - 3 q^{22} qn^{10} - 9 q^{23} qn^{10} -$

$13 q^{24} qn^{10} - q^{25} qn^{10} + 18 q^{26} qn^{10} + 30 q^{27} qn^{10} + 17 q^{28} qn^{10} + 2 q^{29} qn^{10} - 2 q^{30} qn^{10} -$

$q^{19} qn^{11} - 5 q^{20} qn^{11} - 11 q^{21} qn^{11} - 6 q^{22} qn^{11} + 2 q^{23} qn^{11} + 4 q^{24} qn^{11} + 16 q^{25} qn^{11} +$

$19 q^{26} qn^{11} + 8 q^{27} qn^{11} - 11 q^{28} qn^{11} - 25 q^{29} qn^{11} - 16 q^{30} qn^{11} - 5 q^{31} qn^{11} + q^{23} qn^{12} -$

$3 q^{24} qn^{12} - 6 q^{25} qn^{12} - 13 q^{26} qn^{12} - 20 q^{27} qn^{12} - 23 q^{28} qn^{12} - 22 q^{29} qn^{12} - 3 q^{30} qn^{12} +$

$9 q^{31} qn^{12} + 10 q^{32} qn^{12} + 2 q^{33} qn^{12} + q^{26} qn^{13} + 3 q^{27} qn^{13} + 8 q^{28} qn^{13} + 12 q^{29} qn^{13} +$

$23 q^{30} qn^{13} + 23 q^{31} qn^{13} + 10 q^{32} qn^{13} + 2 q^{33} qn^{13} - 2 q^{34} qn^{13} - q^{30} qn^{14} - 3 q^{31} qn^{14} -$

$10 q^{32} qn^{14} - 11 q^{33} qn^{14} - 8 q^{34} qn^{14} - 2 q^{35} qn^{14} + q^{34} qn^{15} + 3 q^{35} qn^{15} + 2 q^{36} qn^{15}\Big) S^{2}_{qn,q} -$

$q^{12} qn^{4} (-1 + q\, qn)^{2} (1 + q\, qn) \left(-1 + q^{4} qn\right) \left(1 + q^{4} qn\right) \left(-1 + q^{7} qn^{2}\right) \left(-1 + q^{9} qn^{2}\right)$

$\Big(-1 - q + 2 q\, qn + 3 q^{2} qn + q^{3} qn + 2 q^{4} qn + q^{5} qn + q\, qn^{2} - 2 q^{2} qn^{2} - 7 q^{3} qn^{2} - 5 q^{4} qn^{2} -$

$6 q^{5} qn^{2} - 3 q^{6} qn^{2} - 2 q^{8} qn^{2} - 2 q^{2} qn^{3} - q^{3} qn^{3} + 6 q^{4} qn^{3} + 6 q^{5} qn^{3} + 10 q^{6} qn^{3} + 9 q^{7} qn^{3} +$

$2\,q^{8}\,qn^{3}+4\,q^{9}\,qn^{3}+3\,q^{3}\,qn^{4}+6\,q^{4}\,qn^{4}-3\,q^{6}\,qn^{4}-10\,q^{7}\,qn^{4}-12\,q^{8}\,qn^{4}-4\,q^{10}\,qn^{4}-$
$2\,q^{11}\,qn^{4}+q^{12}\,qn^{4}+2\,q^{14}\,qn^{4}-2\,q^{4}\,qn^{5}-7\,q^{5}\,qn^{5}-7\,q^{6}\,qn^{5}-7\,q^{7}\,qn^{5}-q^{8}\,qn^{5}+8\,q^{9}\,qn^{5}+$
$2\,q^{11}\,qn^{5}-4\,q^{13}\,qn^{5}-q^{14}\,qn^{5}-5\,q^{15}\,qn^{5}-2\,q^{16}\,qn^{5}-q^{17}\,qn^{5}+3\,q^{6}\,qn^{6}+7\,q^{7}\,qn^{6}+$
$10\,q^{8}\,qn^{6}+7\,q^{9}\,qn^{6}-4\,q^{10}\,qn^{6}+q^{11}\,qn^{6}+3\,q^{12}\,qn^{6}+q^{13}\,qn^{6}+4\,q^{14}\,qn^{6}+q^{15}\,qn^{6}+$
$8\,q^{16}\,qn^{6}+8\,q^{17}\,qn^{6}+3\,q^{18}\,qn^{6}+q^{19}\,qn^{6}-2\,q^{8}\,qn^{7}-4\,q^{9}\,qn^{7}-5\,q^{10}\,qn^{7}+q^{11}\,qn^{7}+$
$3\,q^{12}\,qn^{7}+3\,q^{13}\,qn^{7}+4\,q^{14}\,qn^{7}-q^{15}\,qn^{7}+2\,q^{16}\,qn^{7}-4\,q^{17}\,qn^{7}-9\,q^{18}\,qn^{7}-7\,q^{19}\,qn^{7}-$
$4\,q^{20}\,qn^{7}-3\,q^{12}\,qn^{8}-9\,q^{13}\,qn^{8}-9\,q^{14}\,qn^{8}-9\,q^{15}\,qn^{8}-7\,q^{16}\,qn^{8}-10\,q^{17}\,qn^{8}-7\,q^{18}\,qn^{8}+$
$3\,q^{19}\,qn^{8}+6\,q^{20}\,qn^{8}+5\,q^{21}\,qn^{8}+q^{22}\,qn^{8}+2\,q^{14}\,qn^{9}+5\,q^{15}\,qn^{9}+7\,q^{16}\,qn^{9}+8\,q^{17}\,qn^{9}+$
$10\,q^{18}\,qn^{9}+13\,q^{19}\,qn^{9}+8\,q^{20}\,qn^{9}+2\,q^{21}\,qn^{9}-3\,q^{22}\,qn^{9}-q^{23}\,qn^{9}-2\,q^{18}\,qn^{10}-3\,q^{19}\,qn^{10}-$
$8\,q^{20}\,qn^{10}-9\,q^{21}\,qn^{10}-6\,q^{22}\,qn^{10}-q^{23}\,qn^{10}+2\,q^{22}\,qn^{11}+3\,q^{23}\,qn^{11}+q^{24}\,qn^{11}\big)\,S_{qn\,,\,q}+$
$q^{18}\,(-1+qn)\,qn^{11}\,(-1+q^{3}\,qn)\,(1+q^{3}\,qn)\,(-1+q^{4}\,qn)\,(1+q^{4}\,qn)$
$(-1+q^{7}\,qn^{2})$
$(-1+q^{9}\,qn^{2})$
$(1+q-q^{2}\,qn-q^{3}\,qn-2\,q^{4}\,qn-q^{5}\,qn+q^{4}\,qn^{2}+2\,q^{5}\,qn^{2}+2\,q^{6}\,qn^{2}+$
$q^{7}\,qn^{2}+2\,q^{8}\,qn^{2}-q^{8}\,qn^{3}-q^{9}\,qn^{3}-2\,q^{10}\,qn^{3}-q^{11}\,qn^{3}+q^{12}\,qn^{4}+q^{13}\,qn^{4})$

3D Integral (FCC Lattice)

In[25]:=
```
TraditionalForm[
 HoldForm[Integrate[1 / (1 - z / 3 * (Cos[k1] * Cos[k2] + Cos[k1] * Cos[k3] + Cos[k2] * Cos[k3])),
   {k1, 0, Pi}, {k2, 0, Pi}, {k3, 0, Pi}]]]
```

Out[25]//TraditionalForm=

$$\int_0^{\pi}\int_0^{\pi}\int_0^{\pi} 1\Big/\left(1-\frac{1}{3}\,z\,(\cos(\mathrm{k1})\,\cos(\mathrm{k2})+\cos(\mathrm{k1})\,\cos(\mathrm{k3})+\cos(\mathrm{k2})\,\cos(\mathrm{k3}))\right)d\,\mathrm{k3}\;d\,\mathrm{k2}\;d\,\mathrm{k1}$$

After the substitutions xi=cos(ki) the integrand transforms to:

In[26]:=
```
integrand = 1 / (1 - z / 3 * (x1 * x2 + x1 * x3 + x2 * x3)) / (Sqrt[1 - x1^2] Sqrt[1 - x2^2] Sqrt[1 - x3^2])
```

Out[26]=

$$1\Big/\left(\sqrt{1-x1^{2}}\,\sqrt{1-x2^{2}}\,\sqrt{1-x3^{2}}\,\left(1-\frac{1}{3}\,(x1\,x2+x1\,x3+x2\,x3)\,z\right)\right)$$

In[27]:=
```
{ann1, delta1} = CreativeTelescoping[integrand, Der[x1], {Der[x2], Der[x3], Der[z]}];
ann1
```

Out[28]=

$\{(9-6\,x2\,x3\,z-x2^{2}\,z^{2}-2\,x2\,x3\,z^{2}-x3^{2}\,z^{2}+x2^{2}\,x3^{2}\,z^{2})\,D_{z}+$
$(-3\,x2\,x3-x2^{2}\,z-2\,x2\,x3\,z-x3^{2}\,z+x2^{2}\,x3^{2}\,z),$
$(-9+9\,x3^{2}+6\,x2\,x3\,z-6\,x2\,x3^{3}\,z+x2^{2}\,z^{2}+2\,x2\,x3\,z^{2}+x3^{2}\,z^{2}-$
$2\,x2^{2}\,x3^{2}\,z^{2}-2\,x2\,x3^{3}\,z^{2}-x3^{4}\,z^{2}+x2^{2}\,x3^{4}\,z^{2})\,D_{x3}+$
$(9\,x3+3\,x2\,z-9\,x2\,x3^{2}\,z+x2\,z^{2}+x3\,z^{2}-2\,x2^{2}\,x3\,z^{2}-3\,x2\,x3^{2}\,z^{2}-2\,x3^{3}\,z^{2}+2\,x2^{2}\,x3^{3}\,z^{2}),$
$(-9+9\,x2^{2}+6\,x2\,x3\,z-6\,x2^{3}\,x3\,z+x2^{2}\,z^{2}-x2^{4}\,z^{2}+2\,x2\,x3\,z^{2}-$
$2\,x2^{3}\,x3\,z^{2}+x3^{2}\,z^{2}-2\,x2^{2}\,x3^{2}\,z^{2}+x2^{4}\,x3^{2}\,z^{2})\,D_{x2}+$
$(9\,x2+3\,x3\,z-9\,x2^{2}\,x3\,z+x2\,z^{2}-2\,x2^{3}\,z^{2}+x3\,z^{2}-3\,x2^{2}\,x3\,z^{2}-2\,x2\,x3^{2}\,z^{2}+2\,x2^{3}\,x3^{2}\,z^{2})\}$

In[29]:=
```
{ann2, delta2} = CreativeTelescoping[ann1, Der[x2]];
ann2
```

Out[30]= $\{(9-9\,x3^2-3\,z+9\,x3^2\,z-6\,x3^4\,z)\,D_{x3}+(-6\,x3\,z^2+6\,x3^3\,z^2-2\,x3\,z^3+2\,x3^3\,z^3)\,D_z+(-9\,x3+3\,x3\,z-6\,x3^3\,z-2\,x3\,z^2+2\,x3^3\,z^2),$
$(243\,z-162\,z^2+243\,x3^2\,z^2-216\,x3^2\,z^3+54\,x3^4\,z^3+18\,z^4-18\,x3^2\,z^4-126\,x3^4\,z^4-3\,z^5+24\,x3^2\,z^5-30\,x3^4\,z^5-24\,x3^6\,z^5-x3^2\,z^6+6\,x3^4\,z^6-8\,x3^6\,z^6)\,D_z^2+$
$(243-162\,z+162\,x3^2\,z-54\,z^2-486\,x3^2\,z^2+54\,x3^4\,z^2+54\,z^3-126\,x3^2\,z^3-324\,x3^4\,z^3-9\,z^4+78\,x3^2\,z^4-114\,x3^4\,z^4-72\,x3^6\,z^4-4\,x3^2\,z^5+24\,x3^4\,z^5-32\,x3^6\,z^5)\,D_z+$
$(-27\,z-108\,x3^2\,z+18\,z^2-72\,x3^2\,z^2-90\,x3^4\,z^2-3\,z^3+30\,x3^2\,z^3-54\,x3^4\,z^3-24\,x3^6\,z^3-2\,x3^2\,z^4+12\,x3^4\,z^4-16\,x3^6\,z^4)\}$

In[31]:=
```
{ann3, delta3} = CreativeTelescoping[ann2, Der[x3]];
ann3
```

Out[32]= $\{(-18\,z^2+6\,z^3+10\,z^4+2\,z^5)\,D_z^3+(-54\,z+27\,z^2+60\,z^3+15\,z^4)\,D_z^2+(-18+18\,z+72\,z^2+24\,z^3)\,D_z+(12\,z+6\,z^2)\}$

The same differential equation, using the notation $\theta_x = x\,D_x$:

In[33]:=
```
ChangeOreAlgebra[First[ann3], OreAlgebra[Euler[z]]] // Factor
```

Out[33]= $\frac{2\,(-1+z)\,(3+z)^2}{z}\,\theta_z^3+3\,(3+z)\,(1+3\,z)\,\theta_z^2+(3+32\,z+13\,z^2)\,\theta_z+6\,z\,(2+z)$

In[34]:=
```
z ** %
```

Out[34]= $2\,(-1+z)\,(3+z)^2\,\theta_z^3+3\,z\,(3+z)\,(1+3\,z)\,\theta_z^2+z\,(3+32\,z+13\,z^2)\,\theta_z+6\,z^2\,(2+z)$

Find Relations

In[35]:=
```
TraditionalForm[
 HoldForm[F[m, n] := HypergeometricPFQ[{m + n + 1, m + n + 2}, {m + 1, n + 1, m + n + 1}, -z^2]]]
```

Out[35]//TraditionalForm=

$F(m,n) := {}_2F_3(m+n+1, m+n+2; m+1, n+1, m+n+1; -z^2)$

Task: Find a contiguous relation between $F(m, n)$, $F(m+2, n)$, $F(m, n+2)$, and $F(m+2, n+2)$.

In[36]:=
```
ann =
 Annihilator[HypergeometricPFQ[{m + n + 1, m + n + 2}, {m + 1, n + 1, m + n + 1}, -z^2], {S[m], S[n]}]
```

Out[36]= $\{(3\,m\,z^2+m^2\,z^2-3\,n\,z^2-n^2\,z^2)\,S_n^2+(2\,z^2+3\,n\,z^2+n^2\,z^2)\,S_m+(2\,m-2\,m^2-2\,n+7\,m\,n-3\,m^2\,n-5\,n^2+7\,m\,n^2-m^2\,n^2-4\,n^3+2\,m\,n^3-n^4-2\,z^2-4\,m\,z^2+n\,z^2-2\,m\,n\,z^2+n^2\,z^2)\,S_n+(-2\,m+2\,m^2+2\,n-7\,m\,n+3\,m^2\,n+5\,n^2-7\,m\,n^2+m^2\,n^2+4\,n^3-2\,m\,n^3+n^4),$
$(3\,m+m^2-3\,n-n^2)\,S_m\,S_n+(2+3\,n+n^2)\,S_m+(-2-3\,m-m^2)\,S_n,$
$(-3\,m\,z^2-m^2\,z^2+3\,n\,z^2+n^2\,z^2)\,S_m^2+(-2\,m-5\,m^2-4\,m^3-m^4+2\,n+7\,m\,n+7\,m^2\,n+2\,m^3\,n-2\,n^2-3\,m\,n^2-m^2\,n^2-2\,z^2+m\,z^2+m^2\,z^2-4\,n\,z^2-2\,m\,n\,z^2)\,S_m+(2\,z^2+3\,m\,z^2+m^2\,z^2)\,S_n+(2\,m+5\,m^2+4\,m^3+m^4-2\,n-7\,m\,n-7\,m^2\,n-2\,m^3\,n+2\,n^2+3\,m\,n^2+m^2\,n^2)\}$

In[37]:=
```
UnderTheStaircase[ann]
```

Out[37]= $\{1, S_n, S_m\}$

In[38]:=
```
FindRelation[ann, Support → {1, S[m]^2, S[n]^2, S[m]^2*S[n]^2}]
```

Out[38]=

$$\{(20 m + 29 m^2 - 10 m^3 - 28 m^4 - 10 m^5 - m^6 - 20 n + 90 m^2 n + 69 m^3 n - 9 m^4 n - 9 m^5 n - m^6 n - 29 n^2 - 90 m n^2 + 81 m^3 n^2 + 21 m^4 n^2 + m^5 n^2 + 10 n^3 - 69 m n^3 - 81 m^2 n^3 + 2 m^4 n^3 + 28 n^4 + 9 m n^4 - 21 m^2 n^4 - 2 m^3 n^4 + 10 n^5 + 9 m n^5 - m^2 n^5 + n^6 + m n^6 - 120 m z^2 - 134 m^2 z^2 - 82 m^3 z^2 - 22 m^4 z^2 - 2 m^5 z^2 + 120 n z^2 + 38 m^2 n z^2 + 10 m^3 n z^2 + 134 n^2 z^2 - 38 m n^2 z^2 + 2 m^3 n^2 z^2 + 82 n^3 z^2 - 10 m n^3 z^2 - 2 m^2 n^3 z^2 + 22 n^4 z^2 + 2 n^5 z^2 + 40 m z^4 + 18 m^2 z^4 + 2 m^3 z^4 - 40 n z^4 + 2 m^2 n z^4 - 18 n^2 z^4 - 2 m n^2 z^4 - 2 n^3 z^4) S_m^2 S_n^2 +$$

$$(-12 + 12 m^2 - 52 n - 12 m n + 40 m^2 n - 91 n^2 - 40 m n^2 + 51 m^2 n^2 - 82 n^3 - 51 m n^3 + 31 m^2 n^3 - 40 n^4 - 31 m n^4 + 9 m^2 n^4 - 10 n^5 - 9 m n^5 + m^2 n^5 - n^6 - m n^6 + 24 m z^2 - 24 n z^2 + 56 m n z^2 - 56 n^2 z^2 + 46 m n^2 z^2 - 46 n^3 z^2 + 16 m n^3 z^2 - 16 n^4 z^2 + 2 m n^4 z^2 - 2 n^5 z^2 + 12 z^4 + 22 n z^4 + 12 n^2 z^4 + 2 n^3 z^4) S_m^2 +$$

$$(12 + 52 m + 91 m^2 + 82 m^3 + 40 m^4 + 10 m^5 + m^6 + 12 m n + 40 m^2 n + 51 m^3 n + 31 m^4 n + 9 m^5 n + m^6 n - 12 n^2 - 40 m n^2 - 51 m^2 n^2 - 31 m^3 n^2 - 9 m^4 n^2 - m^5 n^2 + 24 m z^2 + 56 m^2 z^2 + 46 m^3 z^2 + 16 m^4 z^2 + 2 m^5 z^2 - 24 n z^2 - 56 m n z^2 - 46 m^2 n z^2 - 16 m^3 n z^2 - 2 m^4 n z^2 - 12 z^4 - 22 m z^4 - 12 m^2 z^4 - 2 m^3 z^4) S_n^2 +$$

$$(-72 m - 132 m^2 - 72 m^3 - 12 m^4 + 72 n - 170 m^2 n - 120 m^3 n - 22 m^4 n + 132 n^2 + 170 m n^2 - 50 m^3 n^2 - 12 m^4 n^2 + 72 n^3 + 120 m n^3 + 50 m^2 n^3 - 2 m^4 n^3 + 12 n^4 + 22 m n^4 + 12 m^2 n^4 + 2 m^3 n^4)\}$$

In[39]:=
```
TraditionalForm[
 HoldForm[φ[i, j, x, y] := LegendreP[i, 2 y / (1 - x) - 1] (1 - x)^i JacobiP[j, 2 i + 1, 0, 2 x - 1]]]
```

Out[39]//TraditionalForm=

$$\varphi(i, j, x, y) := P_i\left(\frac{2 y}{1 - x} - 1\right) (1 - x)^i \, P_j^{(2 i + 1, 0)}(2 x - 1)$$

In[40]:=
```
ann = Annihilator[
  LegendreP[i, 2 y / (1 - x) - 1] (1 - x)^i JacobiP[j, 2 i + 1, 0, 2 x - 1], {S[i], S[j], Der[x]}]
```

Out[40]=

A very large output was generated. Here is a sample of it:

$$\{(-15 x^2 - 22 i x^2 - 8 i^2 x^2 - 16 j x^2 - 12 i j x^2 - 4 j^2 x^2 + 39 x^3 + 56 i x^3 + 20 i^2 x^3 + 44 j x^3 + 32 i j x^3 + \ll 37 \gg + 9 x^4 y + 12 i x^4 y + 4 i^2 x^4 y + 12 j x^4 y + 8 i j x^4 y + 4 j^2 x^4 y) D_x^2 + (\ll 1 \gg) S_i + (\ll 1 \gg) S_j + (\ll 1 \gg) D_x + (\ll 139 \gg + 20 i^2 j^2 x^2 y + 24 j^3 x^2 y + 16 i j^3 x^2 y + 4 j^4 x^2 y), \ll 5 \gg\}$$

Show Less | Show More | Show Full Output | Set Size Limit...

In[41]:=
```
FindRelation[ann, Eliminate → {x, y}, Pattern → {_, _, 0 | 1}] // Factor
```

Out[41]=

$$\{-(5 + 2 i + j) (5 + 2 i + 2 j) S_i S_j^2 D_x - (3 + j) (5 + 2 i + 2 j) S_j^3 D_x - 2 (3 + 2 i) (3 + i + j) S_i S_j D_x + 2 (1 + 2 i) (3 + i + j) S_j^2 D_x + 2 (3 + i + j) (5 + 2 i + 2 j) (7 + 2 i + 2 j) S_i S_j + (1 + j) (7 + 2 i + 2 j) S_i D_x + 2 (3 + i + j) (5 + 2 i + 2 j) (7 + 2 i + 2 j) S_j^2 + (3 + 2 i + j) (7 + 2 i + 2 j) S_j D_x\}$$

In[42]:=
$$\text{expr} = \frac{1}{\pi^{1/4}\sqrt{2^n\, n!}} \text{Exp}\left[\left(i\left(-t\,\delta^2 + 2\,x\,(x\,\alpha+\delta)\right) - (x\,\beta+\varepsilon)^2 - 2\,t\,\varepsilon\,(-\beta\,\delta+\alpha\,\varepsilon)\right) / \left(2\left(1+2\,t\,\alpha+i\,t\,\beta^2\right)\right) + i\,(1+2\,n)\left(\gamma - \frac{1}{2}\text{ArcTan}\left[\frac{t\,\beta^2}{1+2\,t\,\alpha}\right]\right)\right] \sqrt{\frac{\beta}{\sqrt{(1+2\,t\,\alpha)^2+t^2\,\beta^4}}}\ \text{HermiteH}\left[n, \frac{x\,\beta - t\,\beta\,\delta + \varepsilon + 2\,t\,\alpha\,\varepsilon}{\sqrt{(1+2\,t\,\alpha)^2+t^2\,\beta^4}}\right];$$

In[43]:= **ann = Annihilator[expr, {Der[t], Der[x]}]**

Out[43]=
$$\left\{\left(2+8\,t\,\alpha+8\,t^2\,\alpha^2+2\,t^2\,\beta^4\right) D_t + \left(4\,x\,\alpha+8\,t\,x\,\alpha^2+2\,t\,x\,\beta^4+2\,\delta+4\,t\,\alpha\,\delta+2\,t\,\beta^3\,\varepsilon\right) D_x + \left(2\,\alpha+4\,t\,\alpha^2-4\,i\,x^2\,\alpha^2+i\,\beta^2+2\,i\,n\,\beta^2+t\,\beta^4-i\,x^2\,\beta^4-4\,i\,x\,\alpha\,\delta-i\,\delta^2-2\,i\,x\,\beta^3\,\varepsilon-i\,\beta^2\,\varepsilon^2\right),\right.$$
$$\left.\left(1+4\,t\,\alpha+4\,t^2\,\alpha^2+t^2\,\beta^4\right) D_x^2 + \left(-4\,i\,x\,\alpha-8\,i\,t\,x\,\alpha^2-2\,i\,t\,x\,\beta^4-2\,i\,\delta-4\,i\,t\,\alpha\,\delta-2\,i\,t\,\beta^3\,\varepsilon\right) D_x + \left(-2\,i\,\alpha-4\,i\,t\,\alpha^2-4\,x^2\,\alpha^2+\beta^2+2\,n\,\beta^2-i\,t\,\beta^4-x^2\,\beta^4-4\,x\,\alpha\,\delta-\delta^2-2\,x\,\beta^3\,\varepsilon-\beta^2\,\varepsilon^2\right)\right\}$$

In[44]:= **FindRelation[ann, Eliminate → t]**

Out[44]= $\left\{D_x^2 + 2\,i\,D_t\right\}$

Non-Holonomic Example

Chyzak+Kauers+Salvy: A non-holonomic systems approach to special function identities (ISSAC 2009)

In[45]:=
```
TraditionalForm[HoldForm[
  Integrate[x^(k-1) Zeta[n, a+b x], {x, 0, Infinity}] == b^(-k) Beta[k, n-k] Zeta[n-k, a]]]
```

Out[45]//TraditionalForm=
$$\int_0^\infty x^{k-1}\,\zeta(n, a+b\,x)\,dx = b^{-k}\,\mathrm{B}(k, n-k)\,\zeta(n-k, a)$$

In[46]:=
```
CreativeTelescoping[x^(k-1) Zeta[n, a+b x], Der[x], {S[a], Der[a], Der[b], S[k], S[n]}]
```

Annihilator::nondf : The expression Zeta[n, a + b∗x] is not recognized to be ∂−finite. The result might not generate a zero−dimensional ideal.

Out[46]=
$$\{\{-b\,D_b - k,\ D_a + n\,S_n,\ -b\,n\,S_k\,S_n + k,\ a\,n\,S_a\,S_n + (k-n)\,S_a - a\,n\,S_n + (-k+n),\ (b+b\,k-b\,n)\,S_a\,S_k + a\,k\,S_a + (-b-b\,k+b\,n)\,S_k - a\,k\},$$
$$\{x, 0, -x, -x\,S_a + x, -x\,(a+b\,x)\,S_a + \left(a\,x+b\,x^2\right)\}\}$$

In[47]:=
```
Annihilator[b^(-k) Beta[k, n-k] Zeta[n-k, a], {S[a], Der[a], Der[b], S[k], S[n]}]
```

Annihilator::nondf : The expression Zeta[−k + n, a] is not recognized to be ∂−finite. The result might not generate a zero−dimensional ideal.

Out[47]=
$$\{b\,D_b + k,\ D_a + n\,S_n,\ b\,n\,S_k\,S_n - k,\ a\,n\,S_a\,S_n + (k-n)\,S_a - a\,n\,S_n + (-k+n),\ (b+b\,k-b\,n)\,S_a\,S_k + a\,k\,S_a + (-b-b\,k+b\,n)\,S_k - a\,k\}$$

In[48]:= **GBEqual[First[%%], %]**

Out[48]= True

Different Algorithms for Creative Telescoping

In[49]:=
```
expr = (x^2 + y^2) / y * Exp[(x^2 + x y + y^2) / (x - y)]
```

Out[49]= $$\frac{e^{\frac{x^2+x\,y+y^2}{x-y}}\left(x^2+y^2\right)}{y}$$

Chyzak: An extension of Zeilberger's fast algorithm to general holonomic functions (2000)

In[50]:=
```
First[CreativeTelescoping[expr, Der[y], Der[x], Method → "Chyzak"]]
```

Out[50]= $$\left\{\left(12 x^2-44 x^3+25 x^4+52 x^5\right) D_x^3+\left(-24 x+192 x^2-320 x^3-135 x^4+260 x^5\right) D_x^2+\left(24-216 x+332 x^2+654 x^3-1161 x^4-468 x^5\right) D_x+\left(48-360 x-180 x^2+1590 x^3+855 x^4+156 x^5\right)\right\}$$

Bostan+Chen+Chyzak+Li+Xin: Hermite reduction and creative telescoping for hyperexponential functions (ISSAC 2013)

In[51]:=
```
First[HermiteTelescoping[expr, Der[y], Der[x]]]
```

Out[51]= $$\left\{\left(12 x^2-44 x^3+25 x^4+52 x^5\right) D_x^3+\left(-24 x+192 x^2-320 x^3-135 x^4+260 x^5\right) D_x^2+\left(24-216 x+332 x^2+654 x^3-1161 x^4-468 x^5\right) D_x+\left(48-360 x-180 x^2+1590 x^3+855 x^4+156 x^5\right)\right\}$$

CK: A fast approach to creative telescoping (2010)

In[52]:=
```
First[FindCreativeTelescoping[expr, Der[y], Der[x]]]
```

Out[52]= $$\left\{\left(12 x^2-44 x^3+25 x^4+52 x^5\right) D_x^3+\left(-24 x+192 x^2-320 x^3-135 x^4+260 x^5\right) D_x^2+\left(24-216 x+332 x^2+654 x^3-1161 x^4-468 x^5\right) D_x+\left(48-360 x-180 x^2+1590 x^3+855 x^4+156 x^5\right)\right\}$$

Takayama: An algorithm of constructing the integral of a module - an infinite dimensional analog of Gröbner basis

In[53]:=
```
Takayama[Annihilator[expr, {Der[y], Der[x]}], {y}]
```

Out[53]= $$\left\{\left(3312 x^4+1128 x^5+1258 x^6+520 x^7\right) D_x^5+\left(-6624 x^3+16488 x^4+1736 x^5+4948 x^6+3120 x^7\right) D_x^4+\left(19872 x^2-8640 x^3-6228 x^4-14576 x^5-16363 x^6-1820 x^7\right) D_x^3+\left(-39744 x+30528 x^2-80712 x^3+3144 x^4+25736 x^5-7263 x^6-1820 x^7\right) D_x^2+\left(39744-70272 x+78120 x^2-38016 x^3-69896 x^4+13026 x^5+1233 x^6-780 x^7\right) D_x+\left(79488+5184 x+142176 x^2+126144 x^3+14640 x^4+12270 x^5+5787 x^6+780 x^7\right)\right\}$$

In[54]:=
```
FindRelation[Annihilator[expr, {Der[y], Der[x]}], Eliminate → y]
```

Out[54]= $$\left\{\left(-12032 x^3-427648 x^4+294144 x^5+99264 x^6+503616 x^7+2830224 x^8+976920 x^9-292584 x^{10}-999000 x^{11}-390960 x^{12}-95229 x^{13}-25110 x^{14}\right) D_y^4 D_x+\left(-114688 x^3-1059328 x^4+2181888 x^5-2211456 x^6+4780224 x^7+4160928 x^8-1312224 x^9-4314864 x^{10}-2210976 x^{11}-162756 x^{12}+92772 x^{13}+18630 x^{14}\right) D_y^3 D_x^2+\left(-247296 x^3-252672 x^4+1628928 x^5-3220224 x^6+7971264 x^7+1724544 x^8-5156208 x^9-6928704 x^{10}-2388528 x^{11}+499284 x^{12}+369522 x^{13}+75330 x^{14}\right) D_y^2 D_x^3+\left(-198656 x^3+962048 x^4-2111232 x^5+590976 x^6+3616320 x^7+2287200 x^8-2468064 x^9-2083152 x^{10}-2140128 x^{11}-77004 x^{12}+79812 x^{13}-5670 x^{14}\right) D_y D_x^4+\left(-54016 x^3+583040 x^4-1852416 x^5+1500480 x^6-78336 x^7+1893360 x^8+399000 x^9+823272 x^{10}-963576 x^{11}-348084 x^{12}-101709 x^{13}-37260 x^{14}\right) D_x^5+\left(24064 x^2+867328 x^3-160640 x^4-492672 x^5-1106496 x^6-6164064 x^7-4784064 x^8-391752 x^9+2290584 x^{10}+1780920 x^{11}+581418 x^{12}+145449 x^{13}+25110 x^{14}\right) D_y^4+$$

$$\left(53248\,x^2+2698240\,x^3+2850304\,x^4-6914688\,x^5+923328\,x^6-28194048\,x^7-27320352\,x^8+\right.$$
$$\left.619968\,x^9+16541544\,x^{10}+11919312\,x^{11}+2468772\,x^{12}+304452\,x^{13}+83430\,x^{14}\right)D_y^3\,D_x+$$
$$\left(98304\,x^2+4297728\,x^3-10589184\,x^4-1135872\,x^5+9442944\,x^6-36469152\,x^7+31260096\,x^8+\right.$$
$$\left.47046528\,x^9+35325288\,x^{10}-1517508\,x^{11}-7898796\,x^{12}-3064716\,x^{13}-663390\,x^{14}\right)D_y^2\,D_x^2+$$
$$\left(59392\,x^2+1945600\,x^3-15257600\,x^4+35447424\,x^5-33627456\,x^6+21593664\,x^7-11839200\,x^8-\right.$$
$$\left.12605760\,x^9-27859560\,x^{10}+4323240\,x^{11}+5524524\,x^{12}+2497932\,x^{13}+739530\,x^{14}\right)D_y\,D_x^3+$$
$$\left(-9728\,x^2-226304\,x^3+2655616\,x^4-7661184\,x^5+7076160\,x^6-4139712\,x^7+9041664\,x^8-\right.$$
$$\left.2936760\,x^9+2190720\,x^{10}-5016060\,x^{11}-987822\,x^{12}-40365\,x^{13}-113400\,x^{14}\right)D_x^4+$$
$$\left(122880\,x-2872320\,x^2-8410624\,x^3+7733760\,x^4+2171520\,x^5+30233472\,x^6+\right.$$
$$64035264\,x^7+35798112\,x^8-10709424\,x^9-26895888\,x^{10}-$$
$$\left.14180400\,x^{11}-3323268\,x^{12}-638604\,x^{13}-102060\,x^{14}\right)D_y^3+$$
$$\left(-3492864\,x^2+10688256\,x^3+4991616\,x^4+9968256\,x^5-12510144\,x^6-29668320\,x^7-91896624\,x^8-\right.$$
$$\left.79391520\,x^9-33915960\,x^{10}+12286188\,x^{11}+16398450\,x^{12}+5818311\,x^{13}+979290\,x^{14}\right)D_y^2\,D_x+$$
$$\left(-196608\,x-193536\,x^2-1391616\,x^3+32235264\,x^4-56668032\,x^5+20196288\,x^6-\right.$$
$$66149568\,x^7+14796288\,x^8+87828768\,x^9+96334344\,x^{10}+$$
$$\left.9238104\,x^{11}-15951492\,x^{12}-8207244\,x^{13}-1620810\,x^{14}\right)D_y\,D_x^2+$$
$$\left(-116736\,x^2+4014848\,x^3-22288512\,x^4+34805376\,x^5-2422272\,x^6-2086368\,x^7-1960272\,x^8-\right.$$
$$\left.16686576\,x^9-24438096\,x^{10}+2541348\,x^{11}+4759074\,x^{12}+2797821\,x^{13}+724140\,x^{14}\right)D_x^3+$$
$$\left(-196608-519168\,x-10994688\,x^2+11649024\,x^3-18880896\,x^4+21219264\,x^5+\right.$$
$$90805632\,x^6+43416000\,x^7+31367808\,x^8+22970376\,x^9+14150088\,x^{10}-$$
$$\left.8665272\,x^{11}-9719028\,x^{12}-2880117\,x^{13}-391230\,x^{14}\right)D_y^2+$$
$$\left(196608+310272\,x-3101184\,x^2+7123968\,x^3-63292416\,x^4+115255296\,x^5-\right.$$
$$8165184\,x^6+70553664\,x^7-53233920\,x^8-104229936\,x^9-94386960\,x^{10}-$$
$$\left.9047808\,x^{11}+14685300\,x^{12}+5947020\,x^{13}+886950\,x^{14}\right)D_y\,D_x+$$
$$\left(506880\,x^2-10388736\,x^3+53439360\,x^4-96353280\,x^5+39116544\,x^6-36720576\,x^7+18697680\,x^8+\right.$$
$$\left.44586864\,x^9+51973200\,x^{10}+4412340\,x^{11}-8073594\,x^{12}-4000347\,x^{13}-646380\,x^{14}\right)D_x^2+$$
$$\left(-196608-764928\,x+2760192\,x^2+12409344\,x^3+43382016\,x^4-5282496\,x^5-\right.$$
$$23970240\,x^6-73064160\,x^7-86468256\,x^8-56993760\,x^9-$$
$$\left.7274016\,x^{10}+7881624\,x^{11}+4258818\,x^{12}+865080\,x^{13}+72900\,x^{14}\right),$$
$$\left(24064\,x^3+913408\,x^4+1216896\,x^5+2027520\,x^6+3004896\,x^7+1189248\,x^8-269760\,x^9-\right.$$
$$\left.949008\,x^{10}-258336\,x^{11}-202284\,x^{12}-71577\,x^{13}-10530\,x^{14}\right)D_y^4\,D_x+$$
$$\left(229376\,x^3+2593792\,x^4+720384\,x^5+5488896\,x^6+2572800\,x^7-1841088\,x^8-\right.$$
$$\left.4695648\,x^9-2189712\,x^{10}-326160\,x^{11}+2268\,x^{12}+2376\,x^{13}-810\,x^{14}\right)D_y^3\,D_x^2+$$
$$\left(494592\,x^3+1459200\,x^4-458496\,x^5+5978880\,x^6-1055808\,x^7-5199552\,x^8-\right.$$
$$\left.7118496\,x^9-1568592\,x^{10}+184464\,x^{11}+254988\,x^{12}+68526\,x^{13}+7290\,x^{14}\right)D_y^2\,D_x^3+$$
$$\left(397312\,x^3-1209344\,x^4+1789440\,x^5+3601152\,x^6+2189568\,x^7-118848\,x^8-\right.$$
$$\left.1229088\,x^9-364080\,x^{10}+314064\,x^{11}-305964\,x^{12}-156384\,x^{13}-25110\,x^{14}\right)D_y\,D_x^4+$$
$$\left(108032\,x^3-988160\,x^4+1751424\,x^5+1083648\,x^6+2813280\,x^7+2050368\,x^8+\right.$$
$$\left.1463520\,x^9-36192\,x^{10}+61776\,x^{11}-356400\,x^{12}-150957\,x^{13}-22680\,x^{14}\right)D_x^5+$$
$$\left(-48128\,x^2-1850880\,x^3-3347200\,x^4-5271936\,x^5-8037312\,x^6-5383392\,x^7-649728\,x^8+\right.$$
$$\left.2167776\,x^9+1465680\,x^{10}+662904\,x^{11}+345438\,x^{12}+92637\,x^{13}+10530\,x^{14}\right)D_y^4+$$
$$\left(-106496\,x^2-5662720\,x^3-16981504\,x^4-18701568\,x^5-37226496\,x^6-25744128\,x^7+349920\,x^8+\right.$$
$$\left.16409568\,x^9+9894072\,x^{10}+2443392\,x^{11}+1399140\,x^{12}+460296\,x^{13}+63990\,x^{14}\right)D_y^3\,D_x+$$

$$\left(-196608\,x^2-9086976\,x^3+3511296\,x^4+11384064\,x^5-1400832\,x^6+51189120\,x^7+53841600\,x^8+36253008\,x^9-4152600\,x^{10}-4716036\,x^{11}-3956688\,x^{12}-1248048\,x^{13}-167670\,x^{14}\right)D_y^2\,D_x^2+$$

$$\left(-118784\,x^2-4171776\,x^3+22865408\,x^4-25103616\,x^5+7444992\,x^6-33192960\,x^7-35947296\,x^8-41469696\,x^9-7526616\,x^{10}-405432\,x^{11}+5905116\,x^{12}+2326536\,x^{13}+331290\,x^{14}\right)D_y\,D_x^3+$$

$$\left(19456\,x^2+513536\,x^3-4380928\,x^4+6263424\,x^5-215616\,x^6+6319200\,x^7+721152\,x^8+717744\,x^9-3267672\,x^{10}+1446228\,x^{11}-670734\,x^{12}-474525\,x^{13}-89100\,x^{14}\right)D_x^4+$$

$$\left(-245760\,x+5326848\,x^2+27580416\,x^3+37931520\,x^4+65332992\,x^5+76616832\,x^6+39227136\,x^7-8165952\,x^8-22090896\,x^9-11291040\,x^{10}-4929336\,x^{11}-2329884\,x^{12}-587412\,x^{13}-63180\,x^{14}\right)D_y^3+$$

$$\left(7084032\,x^2-7633920\,x^3-27234048\,x^4-70868736\,x^5-97771968\,x^6-131911488\,x^7-89546976\,x^8-28346184\,x^9+18542880\,x^{10}+15401448\,x^{11}+7457832\,x^{12}+1795203\,x^{13}+191970\,x^{14}\right)D_y^2\,D_x+$$

$$\left(393216\,x+1173504\,x^2+5296128\,x^3-55117824\,x^4+2135808\,x^5-24439680\,x^6+87229440\,x^7+148177152\,x^8+133362432\,x^9+32934888\,x^{10}-10287432\,x^{11}-14700852\,x^{12}-4425192\,x^{13}-512730\,x^{14}\right)D_y\,D_x^2+$$

$$\left(135168\,x^2-8205312\,x^3+30802176\,x^4-946176\,x^5+7883328\,x^6-17431104\,x^7-32838624\,x^8-34702920\,x^9-16949448\,x^{10}-1368792\,x^{11}+6525036\,x^{12}+2479653\,x^{13}+325620\,x^{14}\right)D_x^3+$$

$$\left(393216+1824768\,x+25982976\,x^2+27876864\,x^3+88144128\,x^4+129173760\,x^5+82109376\,x^6+46908000\,x^7+13012992\,x^8-3526848\,x^9-19483776\,x^{10}-9403776\,x^{11}-2473902\,x^{12}-372681\,x^{13}-31590\,x^{14}\right)D_y^2+$$

$$\left(-393216-1406976\,x+3351552\,x^2-8091648\,x^3+112200192\,x^4+5010048\,x^5+19435968\,x^6-126281664\,x^7-154588032\,x^8-124331472\,x^9-15380352\,x^{10}+14524272\,x^{11}+9221364\,x^{12}+2106000\,x^{13}+206550\,x^{14}\right)D_y\,D_x+$$

$$\left(196608\,x-52224\,x^2+17599488\,x^3-83257344\,x^4-10278144\,x^5-31105152\,x^6+48466080\,x^7+85929408\,x^8+101604312\,x^9+33824592\,x^{10}-7508376\,x^{11}-9987624\,x^{12}-2732211\,x^{13}-286740\,x^{14}\right)D_x^2+\left(-196608-764928\,x+2760192\,x^2+12409344\,x^3+43382016\,x^4-5282496\,x^5-23970240\,x^6-73064160\,x^7-86468256\,x^8-56993760\,x^9-7274016\,x^{10}+7881624\,x^{11}+4258818\,x^{12}+865080\,x^{13}+72900\,x^{14}\right)D_x,$$

$$\left(-10112\,x^3-80384\,x^4-110400\,x^5-582912\,x^6-239760\,x^7+189024\,x^8-1384824\,x^9-735600\,x^{10}-701028\,x^{11}+61128\,x^{12}+71469\,x^{13}+8910\,x^{14}\right)D_y^4\,D_x+$$

$$\left(-40960\,x^3-244736\,x^4-185856\,x^5-1520448\,x^6+2688384\,x^7-1206912\,x^8-2539824\,x^9-2084160\,x^{10}-577152\,x^{11}+164268\,x^{12}+94068\,x^{13}+13770\,x^{14}\right)D_y^3\,D_x^2+$$

$$\left(-49920\,x^3-231936\,x^4-243072\,x^5-686016\,x^6+5825376\,x^7-3749760\,x^8-2583360\,x^9-2949696\,x^{10}-361800\,x^{11}+321084\,x^{12}+132678\,x^{13}+21870\,x^{14}\right)D_y^2\,D_x^3+$$

$$\left(-17408\,x^3-51200\,x^4-370176\,x^5+857664\,x^6+2626560\,x^7-3122688\,x^8-3086544\,x^9-2589312\,x^{10}-1796256\,x^{11}+393876\,x^{12}+269028\,x^{13}+38070\,x^{14}\right)D_y\,D_x^4+$$

$$\left(1664\,x^3+16384\,x^4-202560\,x^5+606144\,x^6-270672\,x^7-768864\,x^8-1658184\,x^9-988176\,x^{10}-1310580\,x^{11}+175932\,x^{12}+158949\,x^{13}+21060\,x^{14}\right)D_x^5+$$

$$\left(20224\,x^2+170880\,x^3+301184\,x^4+1276224\,x^5+1062432\,x^6-138288\,x^7+2580624\,x^8+2856024\,x^9+2137656\,x^{10}+578772\,x^{11}-204066\,x^{12}-89289\,x^{13}-8910\,x^{14}\right)D_y^4+$$

$$\left(59392\,x^2+598016\,x^3+1450496\,x^4+3805248\,x^5+6610176\,x^6-5418624\,x^7+10231200\,x^8+17159664\,x^9+13094040\,x^{10}+5024052\,x^{11}-1310310\,x^{12}-744012\,x^{13}-91530\,x^{14}\right)D_y^3\,D_x+$$

$$\left(98304\,x^2+693504\,x^3-94464\,x^4+2865024\,x^5-10635840\,x^6-28356768\,x^7+21403152\,x^8-222480\,x^9+6989976\,x^{10}-7319376\,x^{11}-1072278\,x^{12}+197316\,x^{13}+7290\,x^{14}\right)D_y^2\,D_x^2+$$

$$\left(62464\,x^2+136704\,x^3-1490176\,x^4+3441984\,x^5-18350208\,x^6+21812928\,x^7+2666880\,x^8+\right.$$

$18463248 x^9 + 8721768 x^{10} + 15761628 x^{11} - 2000970 x^{12} - 1809972 x^{13} - 213030 x^{14}) D_y D_x^3 +$
$(3328 x^2 + 17792 x^3 - 6784 x^4 - 1068864 x^5 + 2367648 x^6 + 749328 x^7 + 975744 x^8 -$
$6284616 x^9 - 641616 x^{10} - 6511860 x^{11} + 305640 x^{12} + 689445 x^{13} + 105300 x^{14}) D_x^4 +$
$(-36864 x - 602112 x^2 - 2107392 x^3 - 5031936 x^4 - 16134720 x^5 - 2592192 x^6 - 7824864 x^7 -$
$29016000 x^8 - 31316976 x^9 - 19077912 x^{10} - 2671488 x^{11} + 2230254 x^{12} + 777924 x^{13} + 77760 x^{14})$
$D_y^3 + (-98304 x - 556032 x^2 + 1159296 x^3 + 1078656 x^4 + 20759040 x^5 + 30262752 x^6 + 25074000 x^7 -$
$4006224 x^8 + 5417856 x^9 - 2319588 x^{10} + 6493068 x^{11} + 1771146 x^{12} + 347409 x^{13} + 89910 x^{14}) D_y^2$
$D_x + (-98304 x - 342528 x^2 + 1404672 x^3 + 5654784 x^4 + 3569472 x^5 - 2067840 x^6 - 39043008 x^7 +$
$997344 x^8 - 783504 x^9 - 7419312 x^{10} - 14579460 x^{11} + 1517454 x^{12} + 1069524 x^{13} + 41310 x^{14}) D_y D_x^2 +$
$(39936 x^2 + 158592 x^3 - 1605504 x^4 + 6077568 x^5 - 15384288 x^6 - 8467248 x^7 + 12839760 x^8 +$
$5277312 x^9 + 19564308 x^{10} + 15221520 x^{11} - 2894130 x^{12} - 1809621 x^{13} - 189540 x^{14}) D_x^3 +$
$(-270336 x - 2754816 x^2 - 6462336 x^3 - 26884224 x^4 - 15485760 x^5 - 10921824 x^6 - 41699088 x^7 -$
$4374576 x^8 + 8574984 x^9 + 16959528 x^{10} - 1182276 x^{11} - 3445902 x^{12} - 1017603 x^{13} - 119070 x^{14})$
$D_y^2 + (98304 + 204288 x - 1648128 x^2 - 1682688 x^3 - 12024576 x^4 + 61634880 x^5 -$
$23178816 x^6 + 21263616 x^7 - 73453680 x^8 - 51464736 x^9 - 32958432 x^{10} +$
$4666896 x^{11} + 6053130 x^{12} + 1731780 x^{13} + 206550 x^{14}) D_y D_x +$
$(-139776 x^2 - 452736 x^3 + 5091840 x^4 - 15078528 x^5 + 41350176 x^6 - 20300112 x^7 + 13203072 x^8 -$
$12382128 x^9 - 15724620 x^{10} - 5700132 x^{11} + 2320326 x^{12} + 792747 x^{13} + 63180 x^{14}) D_x^2 +$
$(196608 + 764928 x - 2760192 x^2 - 12409344 x^3 - 43382016 x^4 + 5282496 x^5 +$
$23970240 x^6 + 73064160 x^7 + 86468256 x^8 + 56993760 x^9 +$
$7274016 x^{10} - 7881624 x^{11} - 4258818 x^{12} - 865080 x^{13} - 72900 x^{14}) D_y \}$

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In[55]:= OrePolynomialSubstitute[%, {Der[y] → 0}]
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Out[55]=

$$\begin{aligned}
\{ & (-54016\, x^{3} + 583040\, x^{4} - 1852416\, x^{5} + 1500480\, x^{6} - 78336\, x^{7} + 1893360\, x^{8} + \\
& \quad 399000\, x^{9} + 823272\, x^{10} - 963576\, x^{11} - 348084\, x^{12} - 101709\, x^{13} - 37260\, x^{14})\, D_x^{5} + \\
& (-9728\, x^{2} - 226304\, x^{3} + 2655616\, x^{4} - 7661184\, x^{5} + 7076160\, x^{6} - 4139712\, x^{7} + 9041664\, x^{8} - \\
& \quad 2936760\, x^{9} + 2190720\, x^{10} - 5016060\, x^{11} - 987822\, x^{12} - 40365\, x^{13} - 113400\, x^{14})\, D_x^{4} + \\
& (-116736\, x^{2} + 4014848\, x^{3} - 22288512\, x^{4} + 34805376\, x^{5} - 2422272\, x^{6} - 2086368\, x^{7} - 1960272\, x^{8} - \\
& \quad 16686576\, x^{9} - 24438096\, x^{10} + 2541348\, x^{11} + 4759074\, x^{12} + 2797821\, x^{13} + 724140\, x^{14})\, D_x^{3} + \\
& (506880\, x^{2} - 10388736\, x^{3} + 53439360\, x^{4} - 96353280\, x^{5} + 39116544\, x^{6} - 36720576\, x^{7} + 18697680\, x^{8} + \\
& \quad 44586864\, x^{9} + 51973200\, x^{10} + 4412340\, x^{11} - 8073594\, x^{12} - 4000347\, x^{13} - 646380\, x^{14})\, D_x^{2} + \\
& (-196608 - 764928\, x + 2760192\, x^{2} + 12409344\, x^{3} + 43382016\, x^{4} - 5282496\, x^{5} - \\
& \quad 23970240\, x^{6} - 73064160\, x^{7} - 86468256\, x^{8} - 56993760\, x^{9} - \\
& \quad 7274016\, x^{10} + 7881624\, x^{11} + 4258818\, x^{12} + 865080\, x^{13} + 72900\, x^{14}), \\
& (108032\, x^{3} - 988160\, x^{4} + 1751424\, x^{5} + 1083648\, x^{6} + 2813280\, x^{7} + 2050368\, x^{8} + \\
& \quad 1463520\, x^{9} - 36192\, x^{10} + 61776\, x^{11} - 356400\, x^{12} - 150957\, x^{13} - 22680\, x^{14})\, D_x^{5} + \\
& (19456\, x^{2} + 513536\, x^{3} - 4380928\, x^{4} + 6263424\, x^{5} - 215616\, x^{6} + 6319200\, x^{7} + 721152\, x^{8} + \\
& \quad 717744\, x^{9} - 3267672\, x^{10} + 1446228\, x^{11} - 670734\, x^{12} - 474525\, x^{13} - 89100\, x^{14})\, D_x^{4} + \\
& (135168\, x^{2} - 8205312\, x^{3} + 30802176\, x^{4} - 946176\, x^{5} + 7883328\, x^{6} - 17431104\, x^{7} - 32838624\, x^{8} - \\
& \quad 34702920\, x^{9} - 16949448\, x^{10} - 1368792\, x^{11} + 6525036\, x^{12} + 2479653\, x^{13} + 325620\, x^{14})\, D_x^{3} + \\
& (196608\, x - 52224\, x^{2} + 17599488\, x^{3} - 83257344\, x^{4} - 10278144\, x^{5} - 31105152\, x^{6} + 48466080\, x^{7} + \\
& \quad 85929408\, x^{8} + 101604312\, x^{9} + 33824592\, x^{10} - 7508376\, x^{11} - 9987624\, x^{12} - 2732211\, x^{13} - 286740\, x^{14}) \\
& \quad D_x^{2} + (-196608 - 764928\, x + 2760192\, x^{2} + 12409344\, x^{3} + 43382016\, x^{4} - \\
& \quad 5282496\, x^{5} - 23970240\, x^{6} - 73064160\, x^{7} - 86468256\, x^{8} - 56993760\, x^{9} - \\
& \quad 7274016\, x^{10} + 7881624\, x^{11} + 4258818\, x^{12} + 865080\, x^{13} + 72900\, x^{14})\, D_x, \\
& (1664\, x^{3} + 16384\, x^{4} - 202560\, x^{5} + 606144\, x^{6} - 270672\, x^{7} - 768864\, x^{8} - 1658184\, x^{9} - \\
& \quad 988176\, x^{10} - 1310580\, x^{11} + 175932\, x^{12} + 158949\, x^{13} + 21060\, x^{14})\, D_x^{5} + \\
& (3328\, x^{2} + 17792\, x^{3} - 6784\, x^{4} - 1068864\, x^{5} + 2367648\, x^{6} + 749328\, x^{7} + 975744\, x^{8} - \\
& \quad 6284616\, x^{9} - 641616\, x^{10} - 6511860\, x^{11} + 305640\, x^{12} + 689445\, x^{13} + 105300\, x^{14})\, D_x^{4} + \\
& (39936\, x^{2} + 158592\, x^{3} - 1605504\, x^{4} + 6077568\, x^{5} - 15384288\, x^{6} - 8467248\, x^{7} + 12839760\, x^{8} + \\
& \quad 5277312\, x^{9} + 19564308\, x^{10} + 15221520\, x^{11} - 2894130\, x^{12} - 1809621\, x^{13} - 189540\, x^{14})\, D_x^{3} + \\
& (-139776\, x^{2} - 452736\, x^{3} + 5091840\, x^{4} - 15078528\, x^{5} + 41350176\, x^{6} - 20300112\, x^{7} + 13203072\, x^{8} - \\
& \quad 12382128\, x^{9} - 15724620\, x^{10} - 5700132\, x^{11} + 2320326\, x^{12} + 792747\, x^{13} + 63180\, x^{14})\, D_x^{2} \}
\end{aligned}$$

In[56]:= **OreGroebnerBasis[%]**

Out[56]=

$$\begin{aligned}
\{ & (12\, x^{2} - 44\, x^{3} + 25\, x^{4} + 52\, x^{5})\, D_x^{3} + (-24\, x + 192\, x^{2} - 320\, x^{3} - 135\, x^{4} + 260\, x^{5})\, D_x^{2} + \\
& (24 - 216\, x + 332\, x^{2} + 654\, x^{3} - 1161\, x^{4} - 468\, x^{5})\, D_x + (48 - 360\, x - 180\, x^{2} + 1590\, x^{3} + 855\, x^{4} + 156\, x^{5}) \}
\end{aligned}$$