Advanced Computer Algebra for Evaluating Determinants

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Journées Nationales de Calcul Formel

Introduction

In 2007, D. Zeilberger published the paper

The HOLONOMIC ANSATZ II.
Automatic DISCOVERY(!) and PROOF(!!)
of Holonomic Determinant Evaluations

in *Annals of Combinatorics* **11**:241–247.

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In this paper, he proposed an algorithmic method to produce proofs of determinant evaluations of the form

$$\det A_n = \det_{1 \le i,j \le n} a_{i,j} = b_n \qquad (n \ge 1)$$

where

- $a_{i,j}$ is a bivariate holonomic sequence, not depending on n,
- $b_n \neq 0$ for all $n \geq 1$.

Some Examples

$$\det_{1\le i,j\le n} \frac{1}{i+j-1} = \frac{1}{(2n-1)!} \prod_{k=1}^{n-1} \frac{(k!)^2}{(k+1)_{n-1}}$$

$$\det_{0\le i,j\le n-1} \binom{2i+2a}{j+b} = 2^{n(n-1)/2} \prod_{k=0}^{n-1} \frac{(2k+2a)!k!}{(k+b)!(2k+2a-b)!}$$

$$\det_{0\le i,j\le n-1} \sum_k \binom{i}{k}\binom{j}{k} 2^k = 2^{n(n-1)/2}$$

A Prominent Previous Application

C. Koutschan, M. Kauers, D. Zeilberger (2011):

Proof of George Andrews's and David Robbins's
q-TSPP Conjecture

Proc. Natl. Acad. Sci. (PNAS) **108**(6):2196–2199.

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By evaluating the determinant

$$\det_{1\leq i,j\leq n}\left(q^{i+j-1}\begin{bmatrix}i+j-2\\ i-1\end{bmatrix}_q + q^{i+j}\begin{bmatrix}i+j-1\\ i\end{bmatrix}_q + (1+q^i)\delta_{i,j} - \delta_{i,j+1}\right)$$

$$= \prod_{1\leq i\leq j\leq k\leq n}\left(\frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}\right)^2$$

a long-standing combinatorial problem (first stated in 1983) was solved, namely the q-enumeration of totally symmetric plane partitions.

Zeilberger's Holonomic Approach

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Then the following linear system has a unique solution:

$$\begin{pmatrix} a_{1,1} & \cdots & a_{1,n-1} & a_{1,n} \\ \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} c_{n,1} \\ \vdots \\ c_{n,n-1} \\ c_{n,n} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

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Prove the identity

$$b_{n-1} \sum_{j=1}^{n} c_{n,j} a_{n,j} = b_n$$

which is the Laplace expansion of A_n w.r.t. the last row.

Explanation for $c_{n,j}$

Why is $c_{n,j} = (-1)^{n+j} M_{n,j}/M_{n,n}$ the solution of the system

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Let $A_n^{(i)}$ denote the matrix that is obtained from A_n by replacing the last row by the i-th row $(1 \leq i < n)$.

Now again Laplace expansion w.r.t. the last row:

$$\sum_{j=1}^{n} M_{n,n} c_{n,j} a_{i,j} = \det A_n^{(i)} = 0.$$

This is just the i-th row in the above system.

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(at least not for symbolic n)

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linear recurrences
polynomial coefficients
finitely many initial values

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Instead:

- hope that $c_{n,j}$ is holonomic
- work with an implicit (recursive) description
- $c_{n,j}$ can be computed for concrete integers n and j
- recurrences can be guessed
  (e.g. with M. Kauers's program `Guess.m`)

Zeilberger's Holonomic Approach (Overview)

1. Compute some data for $c_{n,j}$ (e.g. for $1 \leq n, j \leq 100$).
2. Guess linear recurrences for $c_{n,j}$ from the data.
3. Prove the following identities using an implementation* of holonomic closure properties:

$$c_{n,n} = 1 \qquad (n \geq 1), \tag{1}$$

$$\sum_{j=1}^{n} c_{n,j} a_{i,j} = 0 \qquad (1 \leq i < n), \tag{2}$$

$$\sum_{j=1}^{n} c_{n,j} a_{n,j} = \frac{b_n}{b_{n-1}} \qquad (n \geq 1). \tag{3}$$

* there are F. Chyzak's Maple package `Mgfun` and CK's Mathematica package `HolonomicFunctions.m`; here we will use the latter one.

Part I

Solving Krattenthaler's Conjectures
by
Variations of Zeilberger's Approach

(joint work with T. Thanatipanonda)

Introduction

- In 1999, C. Krattenthaler published the classic

  Advanced Determinant Calculus

  in *Séminaire Lotharingien de Combinatoire* **42**:1–67.

  For example, the q-TSPP conjecture was mentioned therein.

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- In 2005, C. Krattenthaler published

  Advanced Determinant Calculus: A Complement

  in *Linear Algebra and its Applications* **411**:68–166.

  We solved Conjectures 34, 35, and 36 from this paper which are related to combinatorial problems (rhombus tilings).

Conjecture 34 — Theorem 1

Let the determinant $D_1(n)$ be defined by

$$D_1(n) := \det_{1 \le i,j \le n} \left(\delta_{i,j} + \binom{\mu + i + j - 2}{j} \right)$$

where μ is an indeterminate and $\delta_{i,j}$ is the Kronecker delta. Then the following relation holds:

$$\frac{D_1(2n)}{D_1(2n-1)} = (-1)^{(n-1)(n-2)/2} 2^n \frac{\left(\frac{\mu}{2} + n\right)_{\lceil n/2 \rceil} \left(\frac{\mu}{2} + 2n + \frac{1}{2}\right)_{n-1}}{(n)_n \left(-\frac{\mu}{2} - 2n + \frac{3}{2}\right)_{\lceil (n-2)/2 \rceil}}.$$

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This conjecture was posed by G. Andrews in 1980; it appeared in the context of enumerating certain classes of plane partitions.

Result of Zeilberger's Approach

Some Data

$$
\begin{aligned}
D_1(1) &= \mu + 1 \\
D_1(2) &= (\mu+1)(\mu+2) \\
D_1(3) &= \tfrac{1}{12}(\mu+1)(\mu+2)(\mu+3)(\mu+14) \\
D_1(4) &= \tfrac{1}{72}(\mu+1)(\mu+2)(\mu+3)(\mu+4)(\mu+9)(\mu+14) \\
D_1(5) &= \tfrac{1}{8640}(\mu+1)(\mu+2)(\mu+3)(\mu+4)(\mu+5)(\mu+9) \\
&\quad \times(\mu^3+45\mu^2+722\mu+3432) \\
D_1(6) &= \tfrac{1}{518400}(\mu+1)(\mu+2)(\mu+3)(\mu+4)(\mu+5)(\mu+6) \\
&\quad \times(\mu+8)(\mu+13)(\mu+15)(\mu^3+45\mu^2+722\mu+3432) \\
D_1(7) &= \tfrac{1}{870912000}(\mu+1)\circ\circ\circ(\mu+34)(\mu^3+47\mu^2+954\mu+5928) \\
D_1(8) &= \tfrac{1}{731566080000}(\mu+1)\circ\circ\circ(\mu+34)(\mu^3+47\mu^2+954\mu+5928) \\
D_1(9) &= \tfrac{1}{221225582592000000}(\mu+1)(\mu+2)\circ\circ\circ(\mu+21)^2 \\
&\quad \times(\mu^6+142\mu^5+8505\mu^4+277100\mu^3+5253404\mu^2+52937808 \\
D_1(10) &= \tfrac{1}{334493080879104000000}(\mu+1)(\mu+2)\circ\circ\circ(\mu+25)(\mu+27) \\
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Modified Holonomic Approach

Idea: use the subsequence $\tilde{c}_{n,j} := c_{2n,j}$.

Modified Holonomic Approach

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Then we have to prove the following the identities:

$$\tilde{c}_{n,2n} = 1 \qquad (n \geq 1), \tag{1a}$$

$$\sum_{j=1}^{2n} \tilde{c}_{n,j} a_{i,j} = 0 \qquad (1 \leq i < 2n), \tag{2a}$$

$$\sum_{j=1}^{2n} \tilde{c}_{n,j} a_{2n,j} = \frac{b_{2n}}{b_{2n-1}} \qquad (n \geq 1). \tag{3a}$$

The Certificate

With the modified ansatz we obtain very small recurrences for $\tilde{c}_{n,j}$:

$$2n(j+1)(2n-1)(2j+\mu)(j-2n)(j-2n+1)$$
$$\times(\mu+4n-5)(\mu+4n-3)(j+\mu+2n-1)\tilde{c}_{n,j} =$$
$$j(j+\mu-1)(2j+\mu-1)(j-2n+3)(\mu+4n-3)$$
$$\times(j^4+2j^3\mu+\ldots\langle 24 \text{ terms}\rangle+12)\tilde{c}_{n-1,j+1} -$$
$$(j+1)(j+\mu+2n-3)(2j^6\mu+8j^6n+\ldots\langle 92 \text{ terms}\rangle-210\mu n)\tilde{c}_{n-1,j}$$

$$(j-1)(j+\mu-3)(j+\mu-2)(2j+\mu-4)(j-2n)$$
$$\times(j+\mu+2n-1)\tilde{c}_{n,j} =$$
$$j(j+\mu-3)(4j^4+8j^3\mu+\ldots\langle 26 \text{ terms}\rangle+16)\tilde{c}_{n,j-1} -$$
$$j(j-1)(j+\mu-2)(2j+\mu-2)(j-2n-2)(j+\mu+2n-3)\tilde{c}_{n,j-2}$$

Conjecture 35 — Theorem 2

Let μ be an indeterminate and n be a nonnegative integer.
If n is even, then the following determinant evaluation holds:

$$\det_{1\le i,j\le n}\left(-\delta_{i,j}+\binom{\mu+i+j-2}{j}\right)=$$

$$(-1)^{n/2}2^{n(n+2)/4}\frac{\left(\frac{\mu}{2}\right)_{n/2}}{\left(\frac{n}{2}\right)!}\left(\prod_{i=0}^{(n-2)/2}\frac{i!^2}{(2i)!^2}\right)\times$$

$$\left(\prod_{i=0}^{\lfloor (n-4)/4\rfloor}\left(\frac{\mu}{2}+3i+\frac{5}{2}\right)^2_{(n-4i-2)/2}\left(-\frac{\mu}{2}-\frac{3n}{2}+3i+3\right)^2_{(n-4i-4)/2}\right)$$

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$$(-1)^{(n-1)/2}2^{(n+3)(n+1)/4}\left(\frac{\mu-1}{2}\right)_{(n+1)/2}\left(\prod_{i=0}^{(n-1)/2}\frac{i!(i+1)!}{(2i)!(2i+2)!}\right)\times$$

$$\left(\prod_{i=0}^{\lfloor (n-3)/4\rfloor}\left(\frac{\mu}{2}+3i+\frac{5}{2}\right)^2_{(n-4i-3)/2}\left(-\frac{\mu}{2}-\frac{3n}{2}+3i+\frac{3}{2}\right)^2_{(n-4i-1)/2}\right)$$

Result of Zeilberger's Approach

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Instead of b_n/b_{n-1} we would like to study the quotients

$$\frac{b_{2n}}{b_{2n-2}} \quad \text{and} \quad \frac{b_{2n+1}}{b_{2n-1}}.$$

The Double Step Method

Based on the formula for the determinant of a block matrix

$$\det(M) = \det \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} = \det(M_1)\det(M_4 - M_3 M_1^{-1} M_2)$$

we obtain the following proof scheme:

$$c'_{n,n-1} = c''_{n,n} = 1, \qquad c'_{n,n} = c''_{n,n-1} = 0, \tag{1b}$$

$$\sum_{j=1}^{n} a_{i,j} c'_{n,j} = \sum_{j=1}^{n} a_{i,j} c''_{n,j} = 0, \qquad (1 \le i \le n-2) \tag{2b}$$

$$\begin{aligned} \frac{b_n}{b_{n-2}} = &\left(\sum_{j=1}^{n} a_{n-1,j}\, c'_{n,j}\right)\left(\sum_{j=1}^{n} a_{n,j}\, c''_{n,j}\right) \\ &- \left(\sum_{j=1}^{n} a_{n-1,j}\, c''_{n,j}\right)\left(\sum_{j=1}^{n} a_{n,j}\, c'_{n,j}\right). \end{aligned} \tag{3b}$$

The Desnanot-Jacobi Approach

Denote $b_n(I,J) := \det\limits_{\substack{I \le i \le n-1+I \\ J \le j \le n-1+J}} a_{i,j}$ (our determinant is $b_n(1,1)$).

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With this notation, the Desnanot-Jacobi identity reads

$$b_n(0,0)b_{n-2}(1,1) = b_{n-1}(0,0)b_{n-1}(1,1) - b_{n-1}(0,1)b_{n-1}(1,0)$$

The Desnanot-Jacobi Approach

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With this notation, the Desnanot-Jacobi identity reads

$$b_n(0,0)b_{n-2}(1,1) = b_{n-1}(0,0)b_{n-1}(1,1) - b_{n-1}(0,1)b_{n-1}(1,0)$$

Substitute $n \to 2n+1$ and $n \to 2n$, and use $b_{2n-1}(0,0) = 0$:

$$b_{2n}(0,0)b_{2n}(1,1) = b_{2n}(0,1)b_{2n}(1,0)$$
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From these two equations we obtain the desired quotient:

$$\frac{b_{2n}(1,1)}{b_{2n-2}(1,1)} = -\frac{b_{2n}(1,0)}{b_{2n-1}(1,0)}\frac{b_{2n}(0,1)}{b_{2n-1}(0,1)}$$

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Remarks:

- $b_{2n-1}(0,0) = 0$: show that the rows of this matrix are dependent, by guessing the coefficients of the corresponding linear combination.
- $b_{2n}(0,0) = -b_{2n-1}(1,1)$ can be shown by a variation of Zeilberger's approach.
- $b_{2n}(1,0)/b_{2n-1}(1,0)$ etc. can be treated (relatively easily) with the original approach.

Conjecture 36 — Theorem 3

Let μ be an indeterminate. For any odd nonnegative integer n there holds

$$\det_{1\le i,j\le n}\left(-\delta_{i,j}+\binom{\mu+i+j-2}{j+1}\right)=$$

$$(-1)^{(n-1)/2}2^{(n-1)(n+5)/4}(\mu+1)\frac{\left(\frac{\mu}{2}-1\right)_{(n+1)/2}}{\left(\frac{n+1}{2}\right)!}\left(\prod_{i=0}^{(n-1)/2}\frac{i!^2}{(2i)!^2}\right)$$

$$\times\left(\prod_{i=0}^{\lfloor (n-1)/4\rfloor}\left(\frac{\mu}{2}+3i+\frac{3}{2}\right)^2_{(n-4i-1)/2}\right)$$

$$\times\left(\prod_{i=0}^{\lfloor (n-3)/4\rfloor}\left(-\frac{\mu}{2}-\frac{3n}{2}+3i+\frac{5}{2}\right)^2_{(n-4i-3)/2}\right)$$

Proof

Relate this determinant to the previous problem:

$$\det_{1\le i,j\le 2n-1}\left(-\delta_{i,j}+\binom{\mu+i+j-2}{j+1}\right)=\det_{2\le i,j\le 2n}\left(-\delta_{i,j}+\binom{(\mu-2)+i+j-2}{j}\right)$$
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A variation of Zeilberger's approach yields a recurrence for

$$\frac{b_{2n}(1,1,\mu-2)}{b_{2n-1}(2,2,\mu-2)}.$$

But $b_{2n}(1,1,\mu-2)$ is already known (previous result).

Part II

Zeilberger's Holonomic Approach
adapted to the
Evaluation of Pfaffians

(joint work with M. Ishikawa)

Pfaffians

Consider a skew-symmetric matrix A, i.e., $A = -A^T$:

$$A = \begin{pmatrix} 0 & a_{1,2} & a_{1,3} & \cdots \\ -a_{1,2} & 0 & a_{2,3} & \cdots \\ -a_{1,3} & -a_{2,3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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It is easy to see that $\det A = 0$ if A has odd dimensions.

Now let A be a $2n \times 2n$ skew-symmetric matrix:

$$\mathrm{Pf}\, A := \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(2i-1),\sigma(2i)}$$

It follows that $(\mathrm{Pf}\, A)^2 = \det A$.

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- The double step method works in theory, but is complicated in practice.
- Come up with a new idea!

Laplace Expansion for Pfaffians

Let $A = (a_{i,j})_{1 \leq i,j \leq 2n}$ be a skew-symmetric matrix and denote by $A(i,j)$ the $(2n-2) \times (2n-2)$ matrix, which is obtained by removing the i-th and j-th rows and columns from A.

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Furthermore let

$$\Gamma_{i,j} := \begin{cases} (-1)^{j-i-1} \operatorname{Pf} A(i,j) & \text{if } i < j, \\ (-1)^{i-j} \operatorname{Pf} A(j,i) & \text{if } j < i, \\ 0 & \text{if } i = j. \end{cases}$$

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Then we have

$$\sum_{k=1}^{2n} a_{i,k} \Gamma_{j,k} = \sum_{k=1}^{2n} a_{k,i} \Gamma_{k,j} = \delta_{i,j} \operatorname{Pf} A.$$

The Holonomic Approach for Pfaffians

Using the Laplace expansion for Pfaffians, the holonomic approach can be formulated for Pfaffians:

$$c_{2n,2n-1} = 1 \qquad (n \geq 1), \tag{1c}$$

$$\sum_{i=1}^{2n-1} c_{2n,i} a_{i,j} = 0 \qquad (1 \leq j < 2n), \tag{2c}$$

$$\sum_{i=1}^{2n-1} c_{2n,i} a_{i,2n} = \frac{b_{2n}}{b_{2n-2}} \qquad (n \geq 1). \tag{3c}$$

Some Solved Problems

Let $M_n = \sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{2k}\binom{2k}{k}$ denote the n-th Motzkin number. Then

$$\mathrm{Pf}\left((j-i)M_{i+j-3}\right)_{1\le i,j\le 2n} = \prod_{k=0}^{n-1}(4k+1).$$

Let $D_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}$ denote the n-th central Delannoy number. Then

$$\mathrm{Pf}\left((j-i)D_{i+j-3}\right)_{1\le i,j\le 2n} = 2^{(n+1)(n-1)}(2n-1)\prod_{k=1}^{n-1}(4k-1).$$