Creative Telescoping for D-Finite Functions

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Motivating Examples

Evaluate binomial sums and prove combinatorial identities, such as:

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{k+n}{k}^2 = \sum_{k=0}^{n} \binom{n}{k} \binom{k+n}{k} \sum_{j=0}^{k} \binom{k}{j}^3$$

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Discover and certify evaluations of hypergeometric functions, e.g.,

$${}_2F_1\left(2t, 2t+\frac{1}{3}, t+\frac{5}{6}; -\frac{1}{8}\right) = \left(\frac{16}{27}\right)^t \frac{\Gamma\left(t+\frac{5}{6}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(t+\frac{2}{3}\right)\Gamma\left(\frac{5}{6}\right)}.$$

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Prove special function identities:

$$\int_{-1}^{1} \left(1-x^2\right)^{\nu-\frac{1}{2}} e^{iax} C_n^{(\nu)}(x)\,\mathrm{d}x = \frac{\pi i^n \Gamma(n+2\nu) J_{n+\nu}(a)}{2^{\nu-1} a^\nu n!\,\Gamma(\nu)}$$

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Prove evaluations of infinite families of determinants:

$$\det_{0\leqslant i,j<n}\left(2^i\binom{i+2j+1}{2j+1}-\binom{i-1}{2j+1}\right)=2\prod_{i=1}^{n}\frac{2^{i-1}\,(4i-2)!}{(n+2i-1)!}$$

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Compute Feynman integrals, such as

$$\int_0^1\int_0^1\frac{w^{-1-\varepsilon/2}(1-z)^{\varepsilon/2}z^{-\varepsilon/2}}{(z+w-wz)^{1-\varepsilon}}\left(1-w^{n+1}-(1-w)^{n+1}\right)\,\mathrm{d}w\,\mathrm{d}z$$

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Or relativistic Coulomb integrals, also arising in physics:

$$\int_0^\infty r^{p+2}\left(F(r)^2\pm G(r)^2\right)\mathrm{d}r,\quad\text{where}$$

$$\begin{pmatrix}F(r)\\G(r)\end{pmatrix}=\frac{a^2(2a\beta r)^{\nu-1}}{\mathrm{e}^{a\beta r}}\sqrt{\frac{\beta^3 n!}{\gamma\,\Gamma(n+2\nu)}}\begin{pmatrix}\alpha_1&\alpha_2\\\beta_1&\beta_2\end{pmatrix}\begin{pmatrix}L_{n-1}^{(2\nu)}(2a\beta r)\\L_n^{(2\nu)}(2a\beta r)\end{pmatrix}$$

Selected Applications of Creative Telescoping

- Hypergeometric expressions for generating functions of walks with small steps in the quarter plane (Alin Bostan, Frédéric Chyzak, Mark van Hoeij, Manuel Kauers, Lucien Pech)

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- Uniqueness of the solution to Canham's problem which predicts the shape of biomembranes: show that the reduced volume $\mathrm{Iso}(z)$ of any stereographic projection of the Clifford torus to $\mathbb{R}^3$ is bijective (Alin Bostan, Sergey Yurkevich)

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- Computing efficiently the n-dimensional volume of a compact semi-algebraic set, i.e., the solution set of multivariate polynomial inequalities, up to a prescribed precision 2^{-p} (Pierre Lairez, Marc Mezzarobba, Mohab Safey El Din)

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- Irrationality measures of mathematical constants such as elliptic L-values (Wadim Zudilin), in the spirit of Apéry's proof of the irrationality of $\zeta(3)$.

Hypergeometric Terms

Definition: A term $f(n)$ is called hypergeometric if

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- $3^{a \cdot n+1}$
- $2^{n(n+1)/2}$
- $\binom{2n}{n}\frac{(7n+3)!}{(n+17)!}$
- $(a\,n+1)!$
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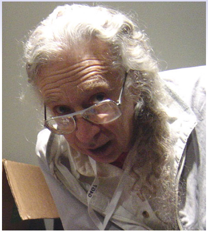

Proc. Natl. Acad. Sci. USA
Vol. 75, No. 1, pp. 40–42, January 1978
Mathematics

Decision procedure for indefinite hypergeometric summation

(algorithm/binomial coefficient identities/closed form/symbolic computation/linear recurrences)

R. WILLIAM GOSPER, JR.

Xerox Palo Alto Research Center, Palo Alto, California 94304

Communicated by Donald E. Knuth, September 26, 1977

ABSTRACT Given a summand a_n, we seek the "indefinite sum" $S(n)$ determined (within an additive constant) by

$$\sum_{n=1}^{m} a_n = S(m) - S(0) \qquad [0]$$

or, equivalently, by

$$a_n = S(n) - S(n-1). \qquad [1]$$

An algorithm is exhibited which, given a_n, finds those $S(n)$ with the property

$$\frac{S(n)}{S(n-1)} = \text{a rational function of } n. \qquad [2]$$

erate case where a_n is identically zero.) Express this ratio as

$$\frac{a_n}{a_{n-1}} = \frac{p_n}{p_{n-1}} \frac{q_n}{r_n}, \qquad [5]$$

where p_n, q_n, and r_n are polynomials in n subject to the following condition:

$$\gcd(q_n, r_{n+j}) = 1, \qquad [6]$$

for all non-negative integers j.

It is always possible to put a rational function in this form, for if $\gcd(q_n, r_{n+j}) = g(n)$, then this common factor can be

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Motivation: such $g(n)$ yields a closed form for the indefinite sum

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From $f(n) = g(n+1) - g(n)$ it follows that if such $g(n)$ exists, then it must be a rational function multiple of $f(n)$:

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$$\frac{f(n)}{g(n)} = \underbrace{\frac{g(n+1)}{g(n)}}_{\in \mathbb{K}(n)} - 1 \qquad \Longrightarrow \qquad \boxed{g(n) = \underbrace{y(n)}_{\in \mathbb{K}(n)} \cdot f(n).}$$

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Key idea: write the rational function $r(n)$ in **Gosper form:**

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for polynomials $a, b, c \in \mathbb{K}[n]$

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The equation turns into:

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Recall: difference equation for the unknown rational function $y(n)$:

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This is called **Gosper's equation**.

The Miracle

Theorem (Gosper): if there exists $x(n) \in \mathbb{K}(n)$ that solves

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- Degree bounding, ansatz, solving a linear system.

Examples:

- $$\sum_{k=0}^{n}(4k+1)\frac{k!}{(2k+1)!} = 2 - \frac{n!}{(2n+1)!}$$

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$$\sum_{k=0}^{n} f(n,k) = \,? \qquad \text{Such as } \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Bivariate Hypergeometric Terms

Definition: A bivariate term $f(n,k)$ is called hypergeometric (w.r.t. n and k) if

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Conjecture of the form $\sum_k f(n,k) = h(n)$ with $\dfrac{h(n+1)}{h(n)} = \dfrac{p(n)}{q(n)}$.

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(Assume $f(n,k)$ is a hypergeometric term and has **finite support**, hence the sum can be taken for all $k \in \mathbb{Z}$.)

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- Check that $h(0) = S(0)$. Hence $S(n) = h(n)$ for all n.

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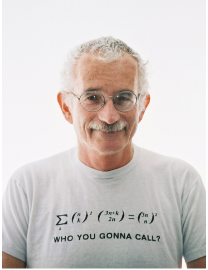
This yields $S(n+1) - 2S(n) = 0$ and the original identity follows.

Zeilberger's (Fast) Algorithm



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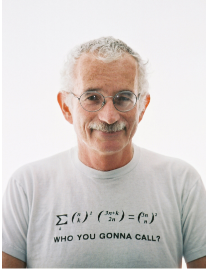

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Communication

A fast algorithm for proving terminating hypergeometric identities

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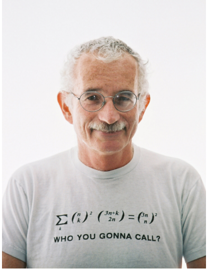

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J. Symbolic Computation (1991) **11**, 195–204

The Method of Creative Telescoping

DORON ZEILBERGER

Department of Mathematics and Computer Science, Temple University, Philadelphia, PA 19122, USA

In memory of John Riordan, master of ars combinatorica

(*Received* 1 *June* 1989)

An algorithm for definite hypergeometric summation is given. It is based, in a non-obvious way, on Gosper's algorithm for definite hypergeometric summation, and its theoretical justification relies on Bernstein's theory of holonomic systems.

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- to deal with parametrized definite sums and integrals

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Creative telescoping is a method

- to deal with parametrized definite sums and integrals
- that yields differential/recurrence equations for them
- that became popular in computer algebra in the past 30 years

Example:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \qquad \text{Bad: no parameter!}$$

$$\underbrace{\sum_{k=1}^{\infty} \frac{1}{k(k+n)}}_{=:\, F_n} \rightsquigarrow (n+2)^2 F_{n+2} = (n+1)(2n+3)F_{n+1} - n(n+1)F_n$$

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Method for doing sums and integrals
(aka Feynman's differentiating under the integral sign)

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Consider the following integration problem: $F(x) := \int_a^b f(x,y)\,\mathrm{d}y$

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Then $F(n) = \int_a^b \left(\frac{\mathrm{d}}{\mathrm{d}y} g(x,y)\right)\mathrm{d}y \qquad = g(x,b) - g(x,a).$

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A reasonable choice for where to search for $g(n,k)$ is:

hypergeometric terms,
i.e., rational function multiples of $f(n,k)$.

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- Write recurrence with undetermined coefficients $p_i \in \mathbb{K}(n)$:

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- Apply a parametrized version of Gosper's algorithm to

$$p_r(n)f(n+r,k) + \cdots + p_1(n)f(n+1,k) + p_0(n)f(n,k).$$

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The parametrized Gosper is applied to the **hypergeometric** term

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- The algorithm always finds the telescoper of minimal order.

Examples for Zeilberger's Algorithm

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

$$\sum_{k=-n}^{n} (-1)^k \binom{2n}{n+k}^2 = \frac{(2n)!}{(n!)^2}$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \quad \rightsquigarrow \quad \text{second-order recurrence}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{dk}{n} = (-d)^n$$

$${}_2F_1(a,b,c;z) := \sum_{k=0}^{\infty} \frac{(a)_k\,(b)_k}{(c)_k\,k!} z^k \quad \rightsquigarrow \quad \text{second-order recurrence}$$

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with $a_j, a'_j, b_j, b'_j, c_j, c'_j, d_j, d'_j \in \mathbb{N}$.

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Then there exist polynomials $p_0(n), \ldots, p_r(n)$, not all zero, and $q(n,k) \in \mathbb{K}(n,k)$ such that $g(n,k) := q(n,k) f(n,k)$ satisfies

$$\sum_{i=0}^{r} p_i(n) f(n+i,k) = g(n,k+1) - g(n,k).$$

Univariate D-finite Functions

Definition: A function $f(x)$ is called D-finite ("differentiably finite") if it satisfies a (nontrivial) linear ordinary differential equation with polynomial coefficients:

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- good data structure in symbolic computation:
- finitely many initial values $\rightsquigarrow$ finite amount of data
- operations (closure properties) can be executed algorithmically

Many Functions are D-Finite

ArcCsc, KelvinBei, HypergeometricPFQ, ExpIntegralE, ArcTanh, HankelH2, AngerJ, JacobiP, ChebyshevT, AiryBi, AiryAi, Sinc, CosIntegral, ArcSech, SphericalBesselY, Sin, WhittakerW, SphericalHankelH2, HermiteH, ExpIntegralEi, Beta, AiryBiPrime, SphericalBesselJ, ParabolicCylinderD, Erfc, EllipticK, Cos, Hypergeometric2F1, Erf, KelvinKer, BetaRegularized, HypergeometricPFQRegularized, Log, BesselY, Cosh, ArcSinh, CoshIntegral, ArcTan, ArcCoth, LegendreP, LaguerreL, EllipticE, SinhIntegral, Sinh, SphericalHankelH1, ArcSin, AiryAiPrime, EllipticThetaPrime, Root, AppellF1, FresnelC, LegendreQ, ChebyshevU, GammaRegularized, Erfi, BesselI, HypergeometricU, KelvinKei, Exp, ArcCot, Hypergeometric2F1Regularized, ArcSec, Hypergeometric0F1, EllipticPi, GegenbauerC, ArcCos, WeberE, FresnelS, EllipticF, ArcCosh, HankelH1, Sqrt, BesselK, BesselJ, Hypergeometric1F1Regularized, StruveL, KelvinBer, StruveH, WhittakerM, ArcCsch, Hypergeometric1F1, SinIntegral, . . .

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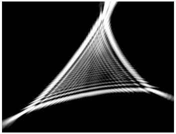

Airy function

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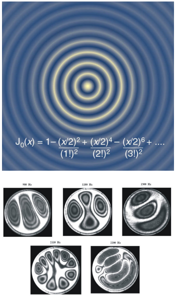

Bessel function

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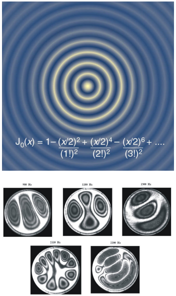

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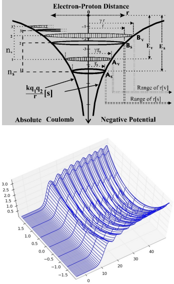

Coulomb function

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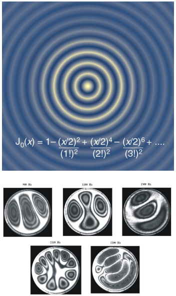

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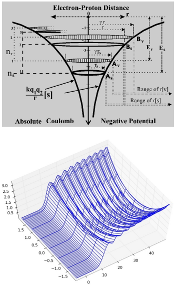

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Special Functions

- arise in mathematical analysis and in real-world phenomena
- are solutions to certain differential equations
- cannot be expressed in terms of the usual elementary functions ($\sqrt{\ }$, exp, log, sin, cos, ...)

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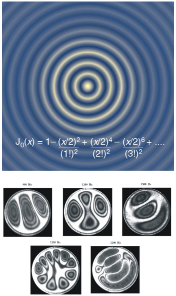

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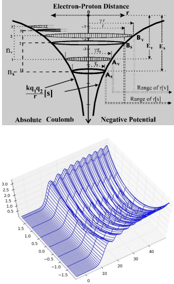

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Theorem: If $f(x)$ and $g(x)$ are D-finite functions, then also the following functions are D-finite:

(i) $f(x) \pm g(x)$

(ii) $f(x) \cdot g(x)$

(iii) $\int f(x)\,\mathrm{d}x$

(iv) $\frac{\mathrm{d}}{\mathrm{d}x} f(x)$

(v) $f(h(x))$, where $h(x)$ is an algebraic function.

Proof idea:

(i) linear algebra, see next slide

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(vi) In particular, every algebraic function $h(x)$ is D-finite.

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$\longrightarrow$ Software demo

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Hence, D-finiteness can be stated as follows:

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- operations (closure properties) can be executed algorithmically

Many Sequences are P-Recursive

Multinomial, KelvinBei, HypergeometricPFQ, HarmonicNumber, HankelH2, CatalanNumber, AngerJ, JacobiP, ChebyshevT, SphericalBesselY, WhittakerW, Gamma, Subfactorial, BesselJ, Pochhammer, SphericalHankelH2, Fibonacci, HermiteH, Beta, SphericalBesselJ, Tribonacci, StruveL, ParabolicCylinderD, Hypergeometric2F1, BesselK, BetaRegularized, KelvinKer, PolyGamma, HypergeometricPFQRegularized, SchröderNumber, SphericalHankelH1, LegendreP, LaguerreL, DelannoyNumber, BetaRegularized, AppellF1, LegendreQ, Binomial, ChebyshevU, GammaRegularized, BesselI, HypergeometricU, KelvinKei, Factorial, Hypergeometric2F1Regularized, GegenbauerC, KelvinBer, WeberE, HankelH1, Hypergeometric1F1Regularized, StruveH, WhittakerM, Hypergeometric0F1, Factorial2, Hypergeometric1F1, LucasL, MotzkinNumber, BesselY, ...

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(iv) a_{cn+d}, where $c, d \in \mathbb{Z}$.

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Theorem: A sequence $(a_n)_{n\in\mathbb{N}}$ is P-recursive iff its generating function $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is D-finite.

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- q-special functions: q-Bessel functions, q-Legendre polynomials, q-Gegenbauer polynomials, etc.

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Examples: Bessel functions, orthogonal polynomials such as the Legendre polynomials $P_n(x)$, etc.

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Definition: f is called D-finite if there is a finite set of basis functions of the form

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with $i_1,\dots,i_s,j_1,\dots,j_r\in\mathbb{N}$ such that any shifted partial derivative of f (of the above form) can be expressed as a $\mathbb{K}(x_1,\dots,x_s,n_1,\dots,n_r)$-linear combination of the basis functions.

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$$\frac{\mathrm{d}^{i_1}}{\mathrm{d}x_1^{i_1}} \cdots \frac{\mathrm{d}^{i_s}}{\mathrm{d}x_s^{i_s}} f_{n_1+j_1,\ldots,n_r+j_r}(x_1,\ldots,x_s)$$

with $i_1,\ldots,i_s,j_1,\ldots,j_r \in \mathbb{N}$ such that any shifted partial derivative of f (of the above form) can be expressed as a $\mathbb{K}(x_1,\ldots,x_s,n_1,\ldots,n_r)$-linear combination of the basis functions.

Again, finitely many initial conditions suffice to specify / fix f.

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General Ore operator:

$$\partial_v \cdot a = \sigma(a) \cdot \partial_v + \delta(a)$$

where σ is an automorphism and δ is a σ-derivation, i.e.,

$$\delta(ab) = \sigma(a)\delta(b) + \delta(a)b.$$

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Definition: Such operators form an **Ore algebra**

$$\mathbb{O} = \mathbb{K}(x, y, \dots)\langle \partial_x, \partial_y, \dots \rangle,$$

i.e., multivariate polynomials in the ∂'s with coefficients being rational functions in $x, y, \dots$, where $\mathbb{K}$ is a field ($\mathrm{char}(\mathbb{K}) = 0$).

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Definition: We define the **annihilator** of a function f to be the set

$$\mathrm{Ann}_{\mathbb{O}}\, f := \left\{ P \in \mathbb{O} \;\middle|\; P \cdot f = 0 \right\}$$

(it is a **left ideal** in the ring $\mathbb{O}$).

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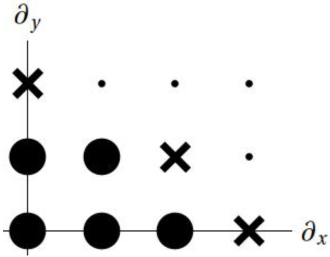

"monomials under the staircase" $(\dim = 5)$ = "holonomic rank"

Example: Ebisu's 2F1 Evaluations

For $\boldsymbol{\alpha} = (a, b, c)$ and a shift vector $\boldsymbol{\beta} \in \mathbb{Z}^3$ compute a relation

$${}_2F_1(\boldsymbol{\alpha} + \boldsymbol{\beta}; z) = R_{\boldsymbol{\beta}}(\boldsymbol{\alpha}, z) \cdot {}_2F_1(\boldsymbol{\alpha}; z) + Q_{\boldsymbol{\beta}}(\boldsymbol{\alpha}, z) \cdot {}_2F_1'(\boldsymbol{\alpha}; z)$$

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Ebisu compiled a list of hundreds of such special ${}_2F_1$ evaluations.

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Example: Relativistic Coulomb Integrals

Consider the radial wave functions F and G of the form

$$\begin{pmatrix} F(r) \\ G(r) \end{pmatrix} = E(r) \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} \begin{pmatrix} L_{n-1}^{(2\nu)}(2a\beta r) \\ L_n^{(2\nu)}(2a\beta r) \end{pmatrix}$$

where $$E(r) = a^2 \beta^{3/2} \sqrt{\frac{n!}{\gamma\, \Gamma(n+2\nu)}}\, (2a\beta r)^{\nu-1} e^{-a\beta r}$$

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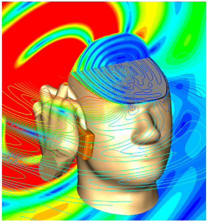
$$\int_0^\infty r^{p+2}\big(F(r)^2 \pm G(r)^2\big)\,\mathrm{d}r, \quad \int_0^\infty r^{p+2}\big(F(r)G(r)\big)\,\mathrm{d}r.$$

Task: Compute recurrences w.r.t. p for these integrals.

Find Certain Operators in Annihilator Ideals

Application: In simulations of the propagation of electromagnetic waves the following basis functions (2D case) are defined:



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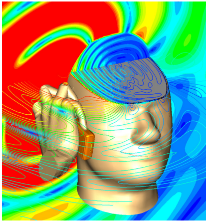

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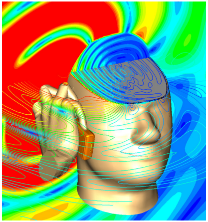

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Ansatz: One needs a relation of the form

$$\sum_{(k,l)\in A} a_{k,l}(i,j) \frac{\mathrm{d}}{\mathrm{d}x} \varphi_{i+k,j+l}(x,y) = \sum_{(m,n)\in B} b_{m,n}(i,j) \varphi_{i+m,j+n}(x,y)$$

that is free of x and y (and similarly for $\frac{\mathrm{d}}{\mathrm{d}y}$).

Holonomic Functions

Definition: Let $f(x_1, \ldots, x_s)$ depend only on continuous variables. Consider the Weyl algebra

$$\mathbb{W} = \mathbb{K}[x_1, \ldots, x_s]\langle D_{x_1}, \ldots, D_{x_s}\rangle.$$

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Sequences: $a_{n_1,\ldots,n_s}$ is holonomic if its generating function

$$A(x_1, \ldots, x_s) := \sum_{n_1=0}^{\infty} \cdots \sum_{n_s=0}^{\infty} a_{n_1,\ldots,n_s} x_1^{n_1} \cdots x_s^{n_s}$$

is holonomic in the above sense.

D-Finite and Holonomic Functions

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Application: Combine the two notions:

- Use D-finiteness for computations.
- Use holonomy for justifications (existence, termination).

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4. The output is always given as an annihilating ideal, **not as a closed form**.

The Holonomic Systems Approach

Journal of Computational and Applied Mathematics 32 (1990) 321–368 321
North-Holland

A holonomic systems approach to special functions identities *

Doron ZEILBERGER
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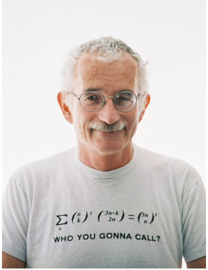
Received 14 November 1989

Abstract: We observe that many special functions are solutions of so-called holonomic systems. Bernstein's deep theory of holonomic systems is then invoked to show that any identity involving sums and integrals of products of these special functions can be verified in a finite number of steps. This is partially substantiated by an algorithm that proves terminating hypergeometric series identities, and that is given both in English and in MAPLE.



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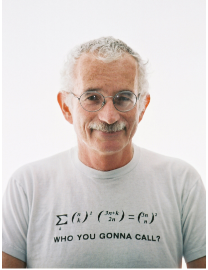

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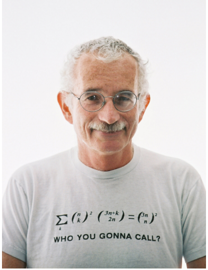

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Creative Telescoping for D-finite Sequences

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Indeed, we have $F(x) = K_\nu(x)$.

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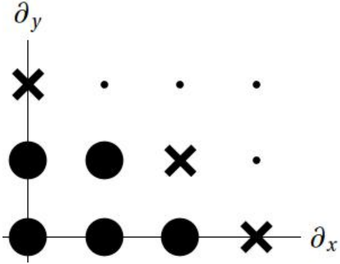

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Since $Q \in \mathbb{O}/\operatorname{Ann}_{\mathbb{O}}(f)$, we can set

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Putting things together:

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Chyzak's Algorithm

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$\longrightarrow$ There are algorithms to find **rational** solutions of such systems.

Finally: loop over the (a priori) unknown order r of the telescoper.
$\longrightarrow$ This is Chyzak's algorithm (analogously in other Ore algebras).

Creative Telescoping in Full Generality

In general, a creative telescoping operator has the form

$$P(\boldsymbol{x}, \boldsymbol{\partial_x}) + \Delta_1 Q_1(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\partial_x}, \boldsymbol{\partial_y}) + \cdots + \Delta_m Q_m(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\partial_x}, \boldsymbol{\partial_y})$$

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- The certificates **certify** the correctness of the telescoper.

Ansatz with Specific Denominators

For finding CT operators, we proposed an ansatz of the form

$$\sum_{\boldsymbol{\alpha}} p_{\boldsymbol{\alpha}}(\boldsymbol{x})\boldsymbol{\partial}_{\boldsymbol{x}}^{\boldsymbol{\alpha}} + \sum_{i=1}^{m} \Delta_i \sum_{u\in\mathfrak{U}} \frac{\sum_{\boldsymbol{\beta}} q_{i,j,\boldsymbol{\beta}}(\boldsymbol{x})\boldsymbol{y}^{\boldsymbol{\beta}}}{d_{i,j}(\boldsymbol{x},\boldsymbol{y})}\, u$$

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- implemented in `HolonomicFunctions` (Mathematica)

Application: Special Function Identities

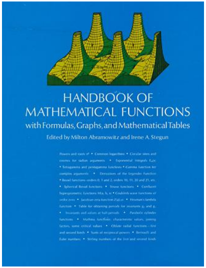
Journal of Computational and Applied Mathematics 32 (1990) 321–368 321
North-Holland

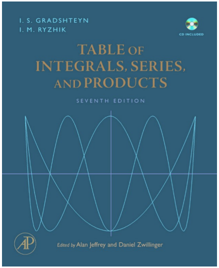
A holonomic systems approach to special functions identities *

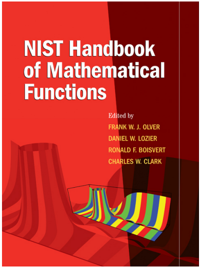
Doron ZEILBERGER
Department of Mathematics, Temple University, Philadelphia, PA 19122, USA

Received 14 November 1989

Abstract: We observe that many special functions are solutions of so-called holonomic systems. Bernstein's deep theory of holonomic systems is then invoked to show that any identity involving sums and integrals of products of these special functions can be verified in a finite number of steps. This is partially substantiated by an algorithm that proves terminating hypergeometric series identities, and that is given both in English and in MAPLE.







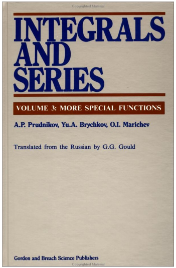
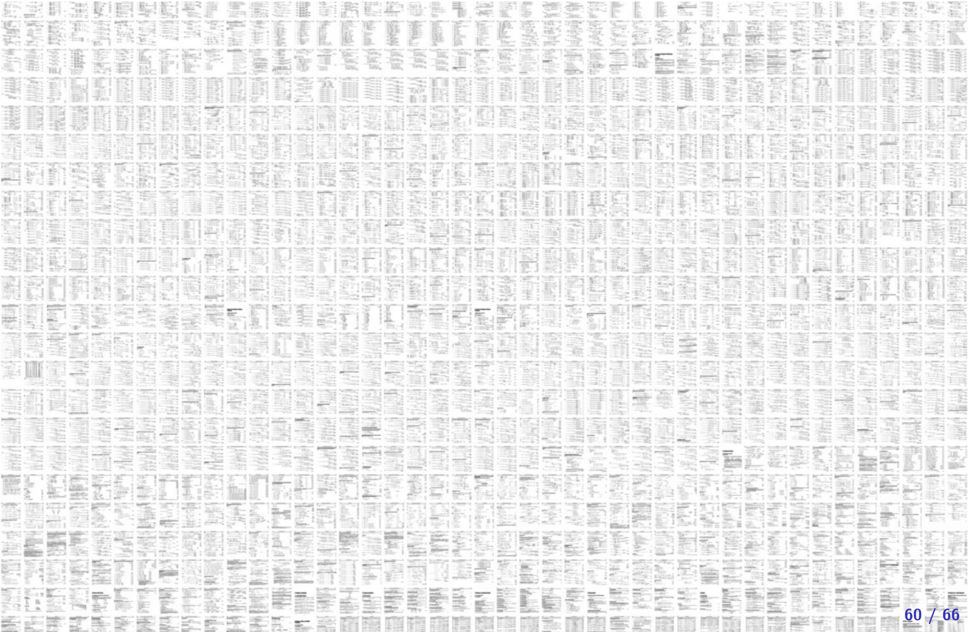


Table of Integrals by Gradshteyn and Ryzhik



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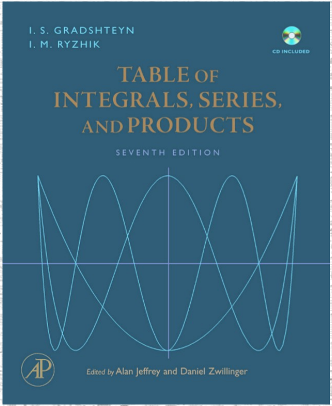

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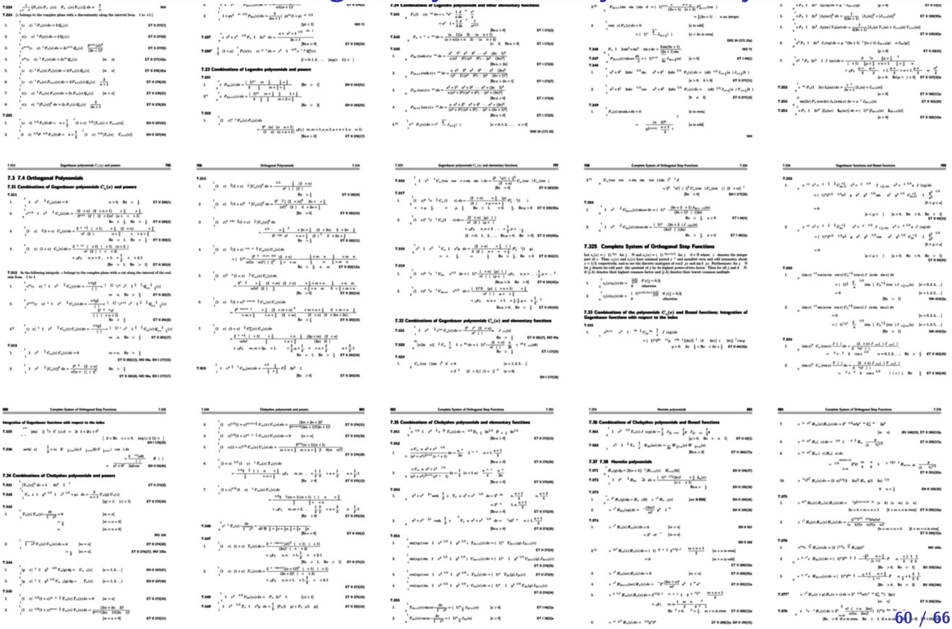

7.319

1. $$\int_0^1 (1-x)^{\mu-1} x^{\nu-1}\, C_{2n}^{\lambda}\left(\gamma x^{1/2}\right) dx = (-1)^n \frac{\Gamma(\lambda+n)\,\Gamma(\mu)\,\Gamma(\nu)}{n!\,\Gamma(\lambda)\,\Gamma(\mu+\nu)}\ {}_3F_2\left(-n, n+\lambda, \nu; \frac{1}{2}, \mu+\nu; \gamma^2\right)$$

$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ ET II 191(41)a

2. $$\int_0^1 (1-x)^{\mu-1} x^{\nu-1}\, C_{2n+1}^{\lambda}\left(\gamma x^{1/2}\right) dx = \frac{(-1)^n 2\gamma\,\Gamma(\mu)\,\Gamma(\lambda+n+1)\,\Gamma\left(\nu+\frac{1}{2}\right)}{n!\,\Gamma(\lambda)\,\Gamma\left(\mu+\nu+\frac{1}{2}\right)} \times {}_3F_2\left(-n, n+\lambda+1, \nu+\tfrac{1}{2}; \tfrac{3}{2}, \mu+\nu+\frac{1}{2}; \gamma^2\right)$$

$\left[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right]$ ET II 191(42)

7.32 Combinations of Gegenbauer polynomials $C_n^{\nu}(x)$ and elementary functions

7.321 $$\int_{-1}^{1} \left(1-x^2\right)^{\nu-\frac{1}{2}} e^{iax}\, C_n^{\nu}(x)\, dx = \frac{\pi 2^{1-\nu} i^n\, \Gamma(2\nu+n)}{n!\,\Gamma(\nu)} a^{-\nu} J_{\nu+n}(a)$$

$\left[\operatorname{Re}\nu > -\frac{1}{2}\right]$ ET II 281(7), MO 99a

7.322 $$\int_0^{2a} [x(2a-x)]^{\nu-\frac{1}{2}}\, C_n^{\nu}\left(\frac{x}{a}-1\right) e^{-bx}\, dx = (-1)^n \frac{\pi\,\Gamma(2\nu+n)}{n!\,\Gamma(\nu)} \left(\frac{a}{2b}\right)^{\nu} e^{-ab}\, I_{\nu+n}(ab)$$

$\left[\operatorname{Re}\nu > -\frac{1}{2}\right]$ ET I 171(9)

7.323

1. $$\int_0^{\pi} C_n^{\nu}(\cos\varphi)\,(\sin\varphi)^{2\nu}\, d\varphi = 0$$ $[n = 1, 2, 3, \ldots]$

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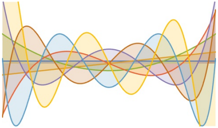

Gegenbauer polynomials $C_n^{(\alpha)}(x)$

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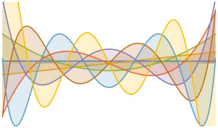

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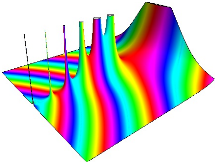

Gamma function $\Gamma(x)$

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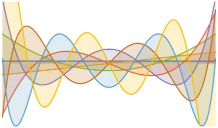

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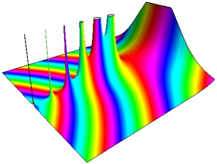

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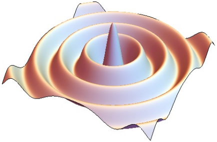

Bessel function $J_\nu(x)$

$$\int_{-1}^{1} \left(1 - x^2\right)^{\nu - \frac{1}{2}} e^{iax}\, C_n^{\nu}(x)\, dx = \frac{\pi 2^{1-\nu} i^n\, \Gamma(2\nu + n)}{n!\, \Gamma(\nu)} a^{-\nu}\, J_{\nu+n}(a)$$

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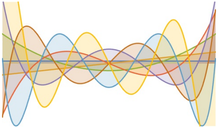

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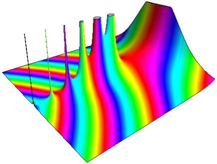

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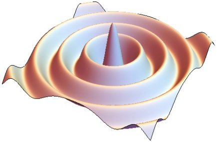

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Let's prove this identity with creative telescoping...

Von Doron Zeilberger

An Mich <christoph.koutschan@ricam.oeaw.ac.at>

Kopie (CC) Alberto Maspero <amaspero@sissa.it>, Mark van Hoeij <hoeij@m

Betreff **Challenge to your Holonomic package**

Dear Christoph,
Hope all is well.

I recently wrote a paper
front:
https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/bcmv.html
pdf:
http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimPDF/bcmvV2.pdf

where I claimed that your amazing package can routinely prove
that the unique solution of the sequence defined in
procedure DxH(p,x) is the same as the
unique sequence defined in DxR(p,x)
and similarly for CxH(p,x) and CxR(p,x)

https://sites.math.rutgers.edu/~zeilberg/tokhniot/BCMV.txt

(i) Was I right?

(ii) If it is not too much trouble, can you actually do it.

In version 1 it was not so important, since I did not claim a fully
rigorous proof to conj. (4) in the paper, but now that Mark van Hoeij
was able to solve the recurrence that would imply a rigorous proof,
just to appease the god of rigorous mathematics, can you do it?

Best wishes
Doron

A Problem from Doron Zeilberger

Let $D_p(x)$ be defined as follows:

$$D_1(x) = \frac{12(1-x)}{x^3-x}\left(\frac{1}{6}(x - x^3) - \frac{28x^2}{9} + \frac{1}{5}(x^2 - 1) - \frac{13x}{9} + \frac{101}{15}\right)$$

A Problem from Doron Zeilberger

Let $D_p(x)$ be defined as follows:

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$\longrightarrow$ Hence, the desired telescoper is $p_0 + p_1 D_x + \cdots + p_r D_x^r$.

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- Chen, Kauers, K. (2016): integration of algebraic functions
- Chen, Hoeij, Kauers, K. (2018): fuchsian D-finite functions

Reduction-Based Creative Telescoping

- Bostan, Chen, Chyzak, Li (2010): integrating rat. functions
- Chen, Singer (2012): (q-) summation of rational functions
- Bostan, Lairez, Salvy (2013, 2015): multiv. rational functions
- Bostan, Chen, Chyzak, Li, Xin (2013): hyperexp. functions
- Chen, Huang, Kauers, Li (2015, 2016): hypergeometric terms
- Bostan, Dumont, Salvy (2016): hypergeom.-hyperexp. terms
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- Bostan, Chyzak, Lairez, Salvy (2018): D-finite functions
- Chen, Du, Kauers, Wang (2023): P-recursive sequences

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- Bostan, Chyzak, Lairez, Salvy (2018): D-finite functions
- Chen, Du, Kauers, Wang (2023): P-recursive sequences
- Brochet, Salvy (2023): summation of D-finite functions

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- Chen, Du, Kauers, Wang (2023): P-recursive sequences
- Brochet, Salvy (2023): summation of D-finite functions
- Brochet (today!): multiple integrals