A SHAPE LEMMA FOR IDEALS OF DIFFERENTIAL OPERATORS

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Abstract. We propose a version of the classical shape lemma for zero-dimensional ideals of a commutative multivariate polynomial ring to the noncommutative setting of zero-dimensional ideals in an algebra of differential operators.

1. Introduction

In the classical theory of Gröbner bases for commutative polynomial rings [3, 7, 1, 8, 4], the shape lemma makes a statement about the form of the Gröbner basis with respect to a lexicographic term order of an ideal of dimension zero. It was proposed by Gianni and Mora [9], and it is almost obvious.

Consider an ideal $I \subseteq K[x, y]$ in a commutative polynomial ring over a perfect field K. The ideal has dimension zero if and only if the corresponding algebraic set

$$V(I) = \{\, (\xi, \eta) \in \bar{K}^2 \mid \forall\, p \in I : p(\xi, \eta) = 0 \,\}$$

is finite. Here, $\bar{K}$ denotes the algebraic closure of K.

The finitely many points in $V(I)$ have only finitely many distinct x-coordinates, and if p is a generator of the elimination ideal $I \cap K[x]$, then the roots of p are precisely these x-coordinates. The shape lemma says that usually there is another polynomial $q \in K[x]$ with $\deg(q) < \deg(p)$ such that I is generated by $\{y - q, p\}$. This q is then the interpolating polynomial of the points in $V(I)$.

There may be no ideal basis of the required form if $V(I)$ contains two distinct points with the same x-coordinate. The ideal is said to be *in normal position* (w.r.t. x) if this is not the case, i.e., if any two distinct elements of $V(I)$ have distinct x-coordinates. If K is sufficiently large, then every ideal I of dimension zero can be brought into normal position by applying a change of variables.

Theorem 1. *(cf. Prop. 3.7.22 in* [12]*). Let $P = K[x_1, \dots, x_n]$, let $I \subseteq P$ be an ideal of dimension zero, let $t = \dim_K P/I$, and suppose that $|K| > \binom{t}{2}$. Then there are constants $c_1, \dots, c_{n-1} \in K$ such that mapping x_n to $x_n - c_1x_1 - c_2x_2 - \dots - c_{n-1}x_{n-1}$ (and x_i to x_i for every $i < n$) transforms I into an ideal in normal position w.r.t. x_n.*

A basis of the required form may also fail to exist if I is not a radical ideal. Recall that for a radical ideal I, we have $\dim_K K[x, y]/I = |V(I)|$. Also recall that if p is a generator of $I \cap K[x]$, then $[1], [x], \dots, [x^{\deg p - 1}]$ are linearly independent over K and $[1], [x], \dots, [x^{\deg p - 1}], [x^{\deg p}]$ are linearly dependent. Therefore, the following result is quite natural.

Theorem 2. *(cf. Thm. 3.7.23 in* [12]*). Let $P = K[x_1, \dots, x_n]$ and let $I \subseteq P$ be a radical ideal of dimension zero. Let p be a generator of $I \cap K[x_n]$. Then the following conditions are equivalent:*

1. *I is in normal position w.r.t. x_n*
2. *$\deg p = \dim_K P/I$*
3. *$K[x_n]/\langle p\rangle$ and P/I are isomorphic as K-algebras.*

Finally, the shape lemma can be stated as follows.

Theorem 3. *(Shape Lemma; cf. Thm. 3.7.25 in* [12]*) Let $P = K[x_1, \dots, x_n]$ and let $I \subseteq P$ be a radical ideal of dimension zero that is in normal position w.r.t. x_n. Let p be a generator of $I \cap K[x_n]$. Then there are polynomials $q_1, \dots, q_{n-1} \in K[x_n]$ with $\deg(q_i) < \deg(p)$ for all i such that $\{x_1 - q_1, \dots, x_{n-1} - q_{n-1}, p\}$ is a basis of I.*

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Here and elsewhere, by a "basis" of an ideal we understand just a set of generators, not necessarily minimial or independent in any sense.

The purpose of this note is to extend these well-known facts from commutative polynomial rings to rings of differential operators. This is motivated by recent developments in the area of symbolic integration for so-called D-finite functions [10]. Given such a function $f(x,y)$, the goal is to evaluate a definite integral

$$F(x) = \int_{\Omega} f(x,y)\,dy.$$

More precisely, given an ideal of annihilating operators for $f(x,y)$, we want to compute an ideal of annihilating operators for the integral $F(x)$. A general approach to this problem is known as creative telescoping [14, 15, 11, 6] and has been subject of intensive research during the past decades. There are several algorithms for creative telescoping, some of which assume that the ideal of operators for $f(x,y)$ has a basis of the form $\{D_y - M, L\}$, where M and L are operators in D_x only. Thanks to the shape lemma, this is a fair assumption.

2. Differential operators

Without loss of generality, and for sake of simplicity, we will restrict the presentation of our main results to the bivariate case. The role of the field K in the commutative setting sketched in the introduction is now taken over by the field $C(x,y)$ of rational functions in x and y, with coefficients in some constant field C that we assume to have characteristic zero. Hence from now on, we write $K = C(x,y)$.

We use the symbols D_x and D_y to denote the partial derivation operators, i.e., $D_x(f) = \frac{\partial f}{\partial x}$ and $D_y(f) = \frac{\partial f}{\partial y}$. Note that $D_x(c) = D_y(c) = 0$ for all $c \in C$. Let $K[D_x, D_y]$ denote the ring of linear differential operators with rational function coefficients, i.e.,

$$K[D_x, D_y] = \left\{ \sum_{i=0}^{r} \sum_{j=0}^{s} a_{i,j}(x,y) D_x^i D_y^j \;\middle|\; r, s \in \mathbb{N}, a_{i,j} \in K \right\}.$$

Because of the product rule, we have the commutation rules $D_x \cdot x = x \cdot D_x + 1$ and $D_y \cdot y = y \cdot D_y + 1$, so the ring $K[D_x, D_y]$ is non-commutative. A linear partial differential equation can then be written as $L(f) = 0$ with $L \in K[D_x, D_y]$.

Let $C[[x]]$ and $C[[x,y]]$ denote, as usual, the rings of univariate and bivariate formal power series with coefficients in C, and let $C((x))$ and $C((x,y))$ denote their respective quotient fields. Let $L = \sum_{i=0}^{r} a_i(x) D_x^i \in C(x)[D_x]$ be a linear ordinary differential operator. An element $x_0 \in C$ is called a regular point (or ordinary point) of L if $a_r(x_0) \neq 0$ and $a_i(x_0)$ is defined for all $0 \leq i \leq r$, i.e., if no coefficient a_i has a pole at x_0. Via the change of variables $x \mapsto x - x_0$ the point x_0 can be moved to the origin. Hence, without loss of generality, assume that 0 is a regular point of L. Then the set of power series solutions

$$V(L) = \{ f \in C[[x]] \mid L(f) = 0 \}$$

forms a C-vector space of dimension r.

For a power series $f(x,y) \in C[[x,y]]$, we define the ($K[D_x, D_y]$-) annihilator of f as the set of all operators that annihilate f, that is $\{L \in K[D_x, D_y] \mid L(f) = 0\}$. It is easily verified that this set forms a (left) ideal in $K[D_x, D_y]$. The series f is called D-finite if $\dim_K(K[D_x, D_y]/I) < \infty$, where I denotes the annihilator of f. Equivalently, f is called D-finite if I is an ideal of dimension zero.

Also in the multivariate setting we can make a similar statement about the dimension of the solution space, which directly follows from Thm. 3.7 in [5].

Theorem 4. *Let I be a zero-dimensional left ideal of $K[D_x, D_y]$ and $r = \dim_K(K[D_x, D_y]/I) \in \mathbb{N}$. If $(0,0)$ is an ordinary point of I, then the set*

$$V(I) = \{ f \in C[[x,y]] \mid \forall L \in I : L(f) = 0 \}$$

is a C-vector space of dimension r.

The definition of ordinary points proposed in [5] is a bit more complicated than the definition in the univariate case. We won't need it here, so we do not reproduce it. It suffices to know that almost every point is ordinary, so if $(0,0)$ is not a ordinary point, we always have the option to get into the situation of Thm. 4 by making a change of variables.

For $f_1,\dots,f_r \in C((x,y))$, their *Wronskian* (with respect to the variable x) is denoted and defined as follows:

$$\operatorname{Wr}_x(f_1,\dots,f_r) = \det \begin{pmatrix} f_1 & f_2 & \cdots & f_r \\ D_x(f_1) & D_x(f_2) & \cdots & D_x(f_r) \\ \vdots & \vdots & \ddots & \vdots \\ D_x^{r-1}(f_1) & D_x^{r-1}(f_2) & \cdots & D_x^{r-1}(f_r) \end{pmatrix}.$$

The Wronskian $\operatorname{Wr}_x(f_1,\dots,f_r)$ is equal to zero if and only if the f_i satisfy a linear relation with coefficients that do not depend on x, e.g., if $\sum_{i=1}^r a_i f_i = 0$ with $a_i \in C((y))$ not all zero [2].

For later use, we state the following lemma.

Lemma 5. *If L is an extension field of K and I is an ideal in $L[D_x,D_y]$ which has a basis in $K[D_x,D_y]$, then also the elimination ideal $I \cap L[D_x]$ has a basis in $K[D_x]$.*

Proof. Let $P_1,\dots,P_n \in K[D_x,D_y]$ be a basis of I, and let M be an element in the elimination ideal $I \cap K[D_x]$. Then there exist $Q_1,\dots,Q_n \in L[D_x,D_y]$ such that $M = Q_1P_1 + \cdots + Q_nP_n$.

Clearly, L can be viewed as a K-vector space, of potentially infinite dimension. In any case, there exists a finite-dimensional K-subspace V of L that contains all the coefficients of the Q_i (note that each Q_i has only finitely many coefficients in L). Now let $B_1,\dots,B_d$ be a K-basis of V, which means that there are $Q_{i,j} \in K[D_x,D_y]$ such that $Q_i = Q_{i,1}B_1 + \cdots + Q_{i,d}B_d$ for all i. Hence we can write

$$M = \sum_{i=1}^n \Big(\sum_{j=1}^d Q_{i,j}B_j\Big)P_i = \sum_{j=1}^d \Big(\sum_{i=1}^n Q_{i,j}P_i\Big)B_j. \tag{1}$$

Since the B_j are linearly independent over K, it follows that for each j, the quantity $\sum_{i=1}^n Q_{i,j}P_i$ is free of D_y, because M is free of D_y and because there cannot be a cancellation on the right-hand side of (1). Therefore, the coefficients $\sum_{i=1}^n Q_{i,j}P_i$ are in $K[D_x]$, which proves the claim. $\square$

For readers familiar with the theory of Gröbner bases, we offer the following alternative proof: from a given basis of I with elements in $K[D_x,D_y]$, we obtain a basis of $I \cap L[D_x]$ by computing a Gröbner basis with respect to an elimination order. Since Buchberger's algorithm never extends the ground field, the resulting basis must be a subset of $K[D_x]$.

3. The Shape Lemma

For an ideal $I \subseteq K[D_x,D_y]$ of dimension zero, consider the quotient $K[D_x,D_y]/I$ as a $K[D_x]$-module. Since its dimension as K-vector space is finite, this module must be cyclic [13, Prop. 2.9]. If $M \in K[D_x,D_y]$ is such that $[M]$ is a generator of the module, then there is an $L \in K[D_x]$ such that $L \cdot [M] = [LM] = [1]$. Therefore, evaluating an integral

$$F(x) = \int_\Omega f(x,y)\,dy$$

for a function $f(x,y)$ whose ideal of annihilating operators is I is the same as evaluating the integral

$$F(x) = \int_\Omega L \cdot g(x,y)\,dy$$

where $g(x,y)$ is defined as $M \cdot f(x,y)$. The choice of M implies that the annihilating ideal J of $g(x,y)$ has a basis of the form $\{D_y - Q, P\}$ for two operators P,Q in $K[D_x]$.

Transforming I to J is known as gauge transform and can be considered as a satisfactory solution to our problem: every ideal $I \subseteq K[D_x,D_y]$ of dimension zero can brought to an ideal J to which the shape lemma applies by means of a gauge transform.

We shall propose an alternative approach here. Rather than applying a gauge transform, which amounts to applying an operator to the integrand, our question is whether we can also obtain an ideal basis of the required form by applying a linear change of variables, i.e., using

$$F(x) = \int_\Omega f(x,y)\,dy = \int_{\tilde\Omega} f(x,y+cx)\,dy$$

for some constant c (and an appropriately adjusted integration range). It turns out that this perspective leads to a shape lemma for differential operators that matches more closely the situation in the commutative case.

Note that $L(x, y, D_x, D_y) \in K[D_x, D_y]$ is an annihilating operator of $f(x, y + cx)$ if and only if $L(x, y - cx, D_x + cD_y, D_y)$ is an annihilating operator of $f(x, y)$. In particular, the ideal of annihilating operators of $f(x, y)$ has dimension zero if and only this is the case for the ideal of annihilating operators of $f(x, y+cx)$.

We shall show (Thm. 11 below) that every zero-dimensional left ideal of $K[D_x, D_y]$ can be brought to normal position by a change of variables $y \leftarrow y + cx$. For the notion of being in normal position, we propose the following definition.

Definition 6. *Let $I \subseteq K[D_x, D_y]$ be an ideal of dimension zero, so that $r = \dim_K K[D_x, D_y]/I$ is finite. The ideal I is called* in normal position *(w.r.t. D_x) if for every choice of C-linearly independent solutions $f_1, \ldots, f_r$ we have $\mathrm{Wr}_x(f_1, \ldots, f_r) \neq 0$.*

Example 7. *For the ideal $I = \langle (D_x - 1)(D_x - 2), D_y \rangle$ we have $r = 2$. The solution space of I is generated by $\exp(x)$ and $\exp(2x)$. We have $\mathrm{Wr}_x(\exp(x), \exp(2x)) = \exp(3x)$. Therefore, I is in normal position w.r.t. D_x. However, as $D_y(\exp(x)) = D_y(\exp(2x)) = 0$, we also have $\mathrm{Wr}_y(\exp(x), \exp(2x)) = 0$, so I is <u>not</u> in normal position w.r.t. D_y.*

With this notion of being in normal position, we can state the following result.

Theorem 8. *(Shape Lemma; differential analog of Thms. 2 and 3) Let $I \subseteq K[D_x, D_y]$ be an ideal of dimension zero. Let P be a generator of $I \cap K[D_x]$. Then the following conditions are equivalent:*

1. *I is in normal position w.r.t. D_x*
2. $\mathrm{ord}(P) = \dim_K K[D_x, D_y]/I$
3. *$K[D_x]/\langle P\rangle$ and $K[D_x, D_y]/I$ are isomorphic as $K[D_x]$-modules.*
4. *There is a $Q \in K[D_x]$ with $\mathrm{ord}(Q) < \mathrm{ord}(P)$ such that $\{D_y - Q, P\}$ is a basis of I.*

Proof. Let $r = \dim_K K[D_x, D_y]/I$.

$1 \Rightarrow 2$ To show that $\mathrm{ord}(P) = r$, suppose that $\mathrm{ord}(P) < r$ and let $f_1, \ldots, f_r$ be some C-linearly independent solutions of I. By Thm. 4, we may assume that such solutions exist. As no more than $\mathrm{ord}(P)$ solutions of P can be linearly independent over $C[[y]]$, it follows that $f_1, \ldots, f_r$ are linearly dependent over $C[[y]]$. This implies $\mathrm{Wr}_x(f_1, \ldots, f_r) = 0$, in contradiction to the assumption that I is in normal position.

$2 \Rightarrow 1$ Let $f_1, \ldots, f_r$ be C-linearly independent solutions of I. We have to show that they are also linearly independent over $C((y))$. Suppose otherwise. Then we may assume that f_r is a $C((y))$-linear combination of $f_1, \ldots, f_{r-1}$. The operator

$$Q = \begin{vmatrix} 1 & f_1 & \cdots & f_{r-1} \\ D_x & f_1' & \cdots & f_{r-1}' \\ \vdots & \vdots & & \vdots \\ D_x^{r-1} & f_1^{(r-1)} & \cdots & f_{r-1}^{(r-1)} \end{vmatrix} \in C((x, y))[D_x]$$

has the solutions $f_1, \ldots, f_{r-1}$ and f_r. It must therefore belong to the ideal generated by I in the larger ring $C((x, y))[D_x, D_y]$, for if it didn't, then $\dim_{C((x,y))} C((x, y))[D_x, D_y]/(\langle I\rangle + \langle Q\rangle) < r$, which is impossible when the solution space has C-dimension r.

By Lemma 5, P is also a generator of the elimination ideal $\langle I\rangle \cap C((x, y))[D_x]$, where $\langle I\rangle$ denotes the ideal generated by I in $C((x, y))[D_x, D_y]$. By assumption we have $\mathrm{ord}(P) = r > \mathrm{ord}(Q)$. This is a contradiction.

$2 \Rightarrow 3$ Consider the function $\phi\colon K[D_x]/\langle P\rangle \to K[D_x, D_y]/I$ defined by $\phi([L]_{\langle P\rangle}) := [L]_I$. This function is well-defined because $\langle P\rangle \subseteq I$. The function is obviously a morphism of $K[D_x]$-modules, and it is injective, because if $L \in K[D_x]$ is such that $[L]_I = [0]_I$, then $L \in I$, so $L \in I \cap K[D_x] = \langle P\rangle$, so $[L]_{\langle P\rangle} = 0$. Being a morphism of $K[D_x]$-modules, ϕ is in particular a morphism of K-vector spaces. Therefore, since $\dim_K K[D_x]/\langle P\rangle = r = \dim_K K[D_x, D_y]/I$ by assumption, injectivity implies bijectivity, and therefore ϕ is an isomorphism.

$3 \Rightarrow 2$ clear.

$2 \Rightarrow 4$ By assumption, the elements $[1], [D_x], \ldots, [D_x^{r-1}]$ of $K[D_x, D_y]/I$ are K-linearly independent and therefore form a vector space basis of $K[D_x, D_y]/I$. Therefore, the element $[D_y]$ of $K[D_x, D_y]/I$ can be expressed as a K-linear combination of $[1], [D_x], \ldots, [D_x^{r-1}]$. This implies the existence of a Q.

$4 \Rightarrow 2$ By repeated addition of suitable multiples of basis elements, it can be seen that every element of $K[D_x, D_y]$ is equivalent modulo I to an element of the form $q_0 + q_1 D_x + \cdots + q_{r-1} D_x^{r-1}$. Therefore, the elements $[1], \ldots, [D_x^{r-1}]$ generate $K[D_x, D_y]/I$ as a K-vector space. This implies $\dim_K K[D_x, D_y]/I \leq r$.

At the same time, the dimension cannot be smaller than r, because if $[1], \dots, [D_x^{r-1}]$ were K-linearly dependent, then $I \cap K[D_x]$ would contain an element of order less than $\operatorname{ord}(P)$, which is impossible by the choice of P. □

Again, readers familiar with the theory of Gröbner bases will have no difficulty finding shorter arguments for some of the implications.

The similarity of Thm. 8 to the corresponding theorems for commutative polynomial rings is evident, but there are some subtle differences as well. One difference is that Thms. 2 and 3 require the ideal to be radical, while no such assumption is needed for Thm. 8.

However, it turns out that in order to also generalize Thm. 1 to differential operators, we do need to introduce a restriction. Note that Thm. 1 becomes wrong for non-radical ideals if we interpret their solutions as points with multiplicities. Indeed, in this sense, a non-radical ideal is never in normal position, and no linear change of variables will suffice to turn a non-radical ideal into a radical ideal.

Ideals of differential operators cannot have multiple solutions (cf. Thm. 4). Instead, it seems appropriate to adopt the following definition.

Definition 9. *An ideal $I \subseteq K[D_x, D_y]$ of dimension zero is called* radical *if $\dim_K K \otimes_C V(I) = \dim_C V(I)$, i.e., if any C-vector space basis of the solution space $V(I)$ is K-linearly independent.*

Example 10. (1) *The ideal $\langle (D_x - 1)^2, D_y \rangle$ has the C-linearly independent solutions $\exp(x)$ and $x\exp(x)$. As these are not linearly independent over K, the ideal is not radical.*
(2) *The solution space of the ideal $\langle (D_x - 1)(D_x - 2), D_y \rangle$ has the basis $\{\exp(x), \exp(2x)\}$, as $\exp(x)$ and $\exp(2x)$ are linearly independent over $K = C(x)$, the ideal is radical.*

Observe the difference between Defs. 9 and 6. In both cases we require the absence of linear relations, but with respect to different coefficient domains. For normal position, the coefficients must be free of x but can depend in an arbitrary way on y, and for being radical, the coefficients must be rational functions in x and y.

Theorem 11. *(Differential analog of Thm. 1) Let $I \subseteq K[D_x, D_y]$ be radical and of dimension zero. Then there is a constant $c \in C$ such that the ideal J obtained from I by applying the linear change of variables $y \leftarrow y + cx$ is in normal position w.r.t. D_x.*

Proof. We show that whenever $f_1(x, y), \dots, f_r(x, y)$ are such that $\operatorname{Wr}_x\big(f_i(x, y+cx)\big)_{i=1}^r = 0$ for all $c \in C$, then $f_1, \dots, f_r$ are K-linearly dependent.

Consider c as an additional variable and recall that the assumption $\operatorname{Wr}_x\big(f_i(x, y+cx)\big)_{i=1}^r = 0$ implies that the $f_i(x, y + cx)$ are linearly dependent over the constant field with respect to x, i.e., $C((y, c))$-linearly dependent: thus we can assume that there exist $p_1, \dots, p_r \in C((y, c))$, not all 0, such that

$$\sum_{i=1}^{r} p_i(y, c) \cdot f_i(x, y + cx) = 0. \tag{2}$$

Each f_i has an expansion as a series in x:

$$f_i(x, y + cx) = \sum_{j=0}^{\infty} \underbrace{\frac{1}{j!} \frac{\partial^j f_i(x, y + cx)}{\partial x^j}\bigg|_{x=0}}_{=:\, f_{i,j}(y,c)} \cdot x^j.$$

Note that the series coefficients $f_{i,j}$ are polynomials in c:

$$\frac{\partial^j f_i(x, y + cx)}{\partial x^j} = \sum_{k=0}^{j} \binom{j}{k} \cdot f_i^{(j-k,k)}(x, y + cx) \cdot c^k \in C((x, y))[c],$$

and therefore $f_{i,j}(y, c) \in C((y))[c]$. It follows that Eq. (2) can be expanded as

$$\sum_{j=0}^{\infty} \left(\sum_{i=1}^{r} p_i(y, c) \cdot f_{i,j}(y, c) \right) \cdot x^j = 0$$

and therefore, for all $j \in \mathbb{N}$, $\sum_{i=1}^{r} p_i(y, c) \cdot f_{i,j}(y, c) = 0$.

Let M be the matrix $\big(f_{i,j}(y, c)\big)_{j \geq 0, 1 \leq i \leq r}$ with infinitely many rows and r columns. From the above,

$$\big(p_i(y, c)\big)_{i=1}^r \in \ker M,$$

and therefore M is rank-deficient; let $R < r$ denote the rank of M. Hence there exists an integer $m \in \mathbb{N}$ such that the rank of the $(m \times r)$-submatrix M', that is obtained by taking the first m rows of M, is also equal to R. It follows that $\ker(M') = \ker(M)$, and since $M' \in C((y))[c]^{m\times r}$ we have that $\ker(M')$ is a subspace of $C((y))(c)^r$. Therefore, the coefficients $p_i(y,c)$ can be chosen in $C((y))(c)$.

Now perform the substitution $c \to c - y/x$ in (2) to get

$$\sum_{i=1}^{r} p_i(y, c - y/x) \cdot f_i(x, cx) = 0. \tag{3}$$

Each $p_i(y, c-y/x)$ admits an expansion as a Laurent series in y

$$p_i(y, c - y/x) = \sum_{j=-k}^{\infty} q_{i,j}(c,x) y^j$$

for some $k \in \mathbb{N}$. Eq. (3) then expands as

$$\sum_{j=-k}^{\infty} \Big(\sum_{i=1}^{r} q_{i,j}(c,y) \cdot f_i(x,cx) \Big) y^j = 0$$

and therefore, for all $j \geq -k$,

$$\sum_{i=1}^{r} q_{i,j}(c,y) \cdot f_i(x,cx) = 0. \tag{4}$$

Since the p_i are not all 0, there must exist i,j with $q_{i,j} \neq 0$, and therefore for such a value of j, the left-hand side of Eq. (4) is a non-trivial linear combination.

Furthermore, observe that since the p_i are rational in their second argument, the coefficients q_i are bivariate rational functions. So finally, substituting $c \to y/x$ yields the desired dependency with coefficients in C:

$$\sum_{i=1}^{r} q_i(y/x, x) f_i(x,y) = 0. \qquad \square$$

Example 12. *The annihilator* I_1 *of* $\exp(x), y\exp(x)$ *is not radical. The annihilator* I_2 *of* $\exp(x), \exp(x+y)$ *is radical but not in normal position w.r.t.* D_x*. Setting* y *to* $y + cx$ *in* I_1 *gives the annihilator of* $\exp(x), (y+cx)\exp(x)$*, which is still not radical. However, setting* y *to* $y + cx$ *in* I_2 *gives the annihilator of* $\exp(x), \exp((1+c)x+y)$*, which is in normal* ∂_x*-position for every choice* $c \neq 0$*.*

In the case of more than two variables, Thm. 8 generalizes as the corresponding Thms. 2 and 3 from the commutative case suggest. We then have one operator D_x that plays the role of x_n and several operators $D_{y_1}, \dots, D_{y_{n-1}}$ that play the roles of $x_1, \dots, x_{n-1}$. Also Thm. 11 generalizes in a straightforward way to more variables, but in a slightly different way than suggested by Thm. 1: while we replace x_n by $x_n - c_1x_1 - \cdots - c_{n-1}x_{n-1}$ in the commutative case, we have to replace each y_i by $y_i + c_ix$, for $i = 1, \dots, n-1$.

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