Legendre pairs of lengths $\ell \equiv 0 \pmod 5$ and $\ell = 85$, $\ell = 87$ cases

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Abstract

By exploiting two different types of balance and a new conjecture which applies to $\ell \equiv 0 \pmod 5$ cases, we find Legendre pairs of lengths 85 and 87. As a consequence, we decrease the number of integers in the range ≤ 200 for which the existence question of Legendre pairs remains unsolved from 12 to 10.

1 Introduction

Compression of complementary vectors [1] has proved to be a valuable tool for discovering several previously unknown complementary vectors of various kinds in the past decade. Compression-based search algorithms are based on a two-step process. In the first step, several candidate compression vectors are computed. The second step searches for decompressions for the candidate compressed vectors from the first step. In this paper we shorten the first step in the case of Legendre pairs of lengths $\ell \equiv 0 \pmod 5$ by significantly decreasing the number of candidate compression vectors. This is achieved by exploiting a plausible conjecture supported by numerical evidence. In particular, we provide the first known examples of Legendre pairs of length 85, which had been the smallest open

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length. We also find Legendre pairs of the next previously open length of $\ell = 87$ by taking advantage of a balanced compression hypothesis.

In Section 2, we develop Conjecture 1 which applies to $\ell \equiv 0 \pmod 5$ cases. We then show how this conjecture can be used to prune the search space. In Section 3, we provide evidence for Conjecture 1 by confirming that it is valid for all $\ell \leq 105$ such that $\ell \equiv 0 \pmod 5$. In Section 4, we exploit a concept of balance along with Conjecture 1 in order to find Legendre pairs of lengths $\ell \equiv 0 \pmod 5$. As an application, we find the first examples of Legendre pairs of length 85. In Section 5, we show how to exploit a different concept of balance applied to Legendre pairs of composite order. As an application, we find the first examples of Legendre pairs of length 87. In Section 6, we provide the current list of ten ℓ values less than 200 for which the Legendre pair existence problem remains unsolved.

2 Legendre pairs of lengths $\ell \equiv 0 \pmod 5$

The *periodic autocorrelation function* (PAF) of a vector $\mathbf{a}$ indexed by $\mathbb{Z}_\ell$ is defined as $\mathrm{PAF}_{\mathbf{a}}(j) = \sum_{i=0}^{\ell-1} a_i a_{i-j}$. For $w = e^{2\pi i/\ell}$, define the *discrete Fourier transform* $\mathrm{DFT}_{\mathbf{a}}(j) = \sum_{r=0}^{\ell-1} w^{jr} a_r$ and the *power spectral density* $\mathrm{PSD}_{\mathbf{a}}(j) = |\,\mathrm{DFT}_{\mathbf{a}}(j)|^2$. We say a pair of row vectors $\mathbf{a}, \mathbf{b} \in \{-1, 1\}^\ell$ form a *Legendre pair* of length ℓ if

$$\sum_{i=0}^{\ell-1} a_i = \sum_{i=0}^{\ell-1} b_i = \pm 1,$$
$$\mathrm{PSD}_{\mathbf{a}}(j) + \mathrm{PSD}_{\mathbf{b}}(j) = 2\ell + 2 \quad \forall\, j \in \mathbb{Z}_\ell \setminus \{0\}.$$

Throughout the paper, WLOG, we assume that $\sum_{i=0}^{\ell-1} a_i = \sum_{i=0}^{\ell-1} b_i = 1$. Also, let $e_2(\mathbf{a}) = \sum_{i<j} a_i a_j$ be the *elementary symmetric polynomial of degree* 2 and $p_2(\mathbf{a}) = \sum_i a_i^2$ be the *power sum symmetric polynomial of degree* 2. If $\mathbf{a} = [a_1, \ldots, a_\ell]$ and $\ell = nm$ for some positive integers n, m, then we define $A_j = \sum_{i=0}^{m-1} a_{ni+j}$. The vector $\mathcal{A}_m = [A_1, \ldots, A_n]$ is called the *m-compression* of $\mathbf{a}$. For a pair of vectors $(\mathbf{a}, \mathbf{b})$ of length $\ell \equiv 0 \pmod 5$ set $m = \ell/5$. In this case,

$$\begin{aligned}
\mathrm{PSD}_{\mathbf{a}}(m) &= p_2(\mathcal{A}_m) - \frac{1}{2} e_2(\mathcal{A}_m) + \frac{\sqrt{5}}{2}\Big(\mathrm{PAF}_{\mathcal{A}_m}(1) - \mathrm{PAF}_{\mathcal{A}_m}(2)\Big), \\
\mathrm{PSD}_{\mathbf{b}}(m) &= p_2(\mathcal{B}_m) - \frac{1}{2} e_2(\mathcal{B}_m) + \frac{\sqrt{5}}{2}\Big(\mathrm{PAF}_{\mathcal{B}_m}(1) - \mathrm{PAF}_{\mathcal{B}_m}(2)\Big),
\end{aligned} \tag{1}$$

where $\mathcal{A}_m = [A_1, A_2, A_3, A_4, A_5]$ and $\mathcal{B}_m = [B_1, B_2, B_3, B_4, B_5]$ are the m-compressions of $\mathbf{a}$ and $\mathbf{b}$ respectively [2]. Since $\sum_{i=0}^{\ell-1} a_i = \sum_{i=0}^{\ell-1} b_i = 1$, $A_1 + \cdots + A_5 = B_1 + \cdots + B_5 = 1$.

$$\begin{aligned} p_2(\mathcal{A}_m) - \frac{1}{2}e_2(\mathcal{A}_m) &= \frac{1}{4}\Big(5p_2(\mathcal{A}_m) - 1\Big), \\ p_2(\mathcal{B}_m) - \frac{1}{2}e_2(\mathcal{B}_m) &= \frac{1}{4}\Big(5p_2(\mathcal{B}_m) - 1\Big). \end{aligned} \tag{2}$$

In addition, if $(\mathbf{a}, \mathbf{b})$ is a Legendre pair then

$$\mathrm{PSD}_{\mathbf{a}}(m) + \mathrm{PSD}_{\mathbf{b}}(m) = 2\ell + 2.$$

Experimental evidence gathered for $\ell = 15, 25, 35, 45, 55$ indicates there are Legendre pairs of these orders such that

$$\begin{aligned} \frac{1}{4}\Big(5p_2(\mathcal{A}_m) - 1\Big) &= \ell + 1, \\ \frac{1}{4}\Big(5p_2(\mathcal{B}_m) - 1\Big) &= \ell + 1. \end{aligned} \tag{3}$$

Equations (3) simplify to

$$\begin{aligned} p_2(\mathcal{A}_m) &= 4m + 1, \\ p_2(\mathcal{B}_m) &= 4m + 1. \end{aligned} \tag{4}$$

Let $m \in \mathbb{Z}^{\geq 1}$, w be a primitive ℓth root of unity, $w' = w^m$, and $\mathbf{a}$ be a vector of length $\ell = 5m$. Then

$$\begin{aligned} \mathrm{PSD}_{\mathbf{a}}(m) &= \sum_{j=0}^{\ell-1} \mathrm{PAF}_{\mathbf{a}}(j) w^{mj} = \sum_{j=0}^{\ell-1} \mathrm{PAF}_{\mathbf{a}}(j)(w')^j = \sum_{k=0}^{4}\sum_{j=0}^{\frac{\ell}{5}-1} \mathrm{PAF}_{\mathbf{a}}(5j+k)(w')^k \\ &= \sum_{j=0}^{\frac{\ell}{5}-1} \mathrm{PAF}_{\mathbf{a}}(5j) + \sum_{k=1}^{2} 2\cos\left(\frac{2\pi k}{5}\right) \sum_{j=0}^{\frac{\ell}{5}-1} \mathrm{PAF}_{\mathbf{a}}(5j+k) \\ &= \sum_{j=0}^{\frac{\ell}{5}-1} \mathrm{PAF}_{\mathbf{a}}(5j) + \sum_{j=0}^{\frac{\ell}{5}-1}\sum_{k=1}^{2} \left(\frac{-1 + (-1)^{k+1}\sqrt{5}}{2}\right) \mathrm{PAF}_{\mathbf{a}}(5j+k) \\ &= \sum_{j=0}^{\frac{\ell}{5}-1} \mathrm{PAF}_{\mathbf{a}}(5j) - \sum_{j=0}^{\frac{\ell}{5}-1}\sum_{k=1}^{2} \frac{\mathrm{PAF}_{\mathbf{a}}(5j+k)}{2} + \frac{\sqrt{5}}{2}\left(\sum_{j=0}^{\frac{\ell}{5}-1}\sum_{k=1}^{2} (-1)^{k+1}\mathrm{PAF}_{\mathbf{a}}(5j+k)\right). \end{aligned}$$

Now, the first proposition follows from the above discussion.

Proposition 1. *For a pair of vectors* $(\mathbf{a}, \mathbf{b})$ *of length* $\ell = 5m$*, the Legendre pair constraint*

$$\mathrm{PSD}_{\mathbf{a}}(m) + \mathrm{PSD}_{\mathbf{b}}(m) = 2\ell + 2$$

is satisfied if and only if

$$\sum_{j=0}^{\frac{\ell}{5}-1}\Big(\mathrm{PAF}_{\mathbf{a}}(5j) + \mathrm{PAF}_{\mathbf{b}}(5j)\Big) - \sum_{j=0}^{\frac{\ell}{5}-1}\sum_{k=1}^{2}\frac{\mathrm{PAF}_{\mathbf{a}}(5j+k) + \mathrm{PAF}_{\mathbf{b}}(5j+k)}{2} = 2\ell + 2,$$

and

$$\sum_{j=0}^{\frac{\ell}{5}-1}\sum_{k=1}^{2}(-1)^{k+1}\Big(\mathrm{PAF}_{\mathbf{a}}(5j+k) + \mathrm{PAF}_{\mathbf{b}}(5j+k)\Big) = 0.$$

The next proposition follows directly from equations (1) and the fact that a Legendre pair $(\mathbf{a}, \mathbf{b})$ of length $\ell = 5m$ satisfies $\mathrm{PSD}_{\mathbf{a}}(m) + \mathrm{PSD}_{\mathbf{b}}(m) = 2\ell + 2$.

Proposition 2. *Let* $(\mathbf{a}, \mathbf{b})$ *be a Legendre pair of length* $\ell = 5m$*. Then there exists an integer* $x \geq 0$ *and* $n_1, n_2 \in \mathbb{Z}$ *with* $n_1 + n_2 = 2\ell + 2$ *such that*

$$\mathrm{PSD}_{\mathbf{a}}(m) = n_1 + \sqrt{5}x,$$
$$\mathrm{PSD}_{\mathbf{b}}(m) = n_2 - \sqrt{5}x.$$

Exhaustive searches for $\ell = 5, 15, 25$ reveal that all Legendre pairs of these lengths have $n_1 = n_2 = \ell + 1$, where n_1, n_2 are as in Proposition 2. Non-exhaustive searches for larger odd values of ℓ which are multiples of 5 reveal the same pattern, i.e. the standard relationship $\mathrm{PSD}_{\mathbf{a}}(m) + \mathrm{PSD}_{\mathbf{b}}(m) = 2\ell + 2$ is materialized (often, but not always) in the same manner. These observations give rise to the following conjecture.

Conjecture 1. *For every odd positive integer* $\ell \equiv 0 \pmod 5$*, there exists at least one Legendre pair* $(\mathbf{a}, \mathbf{b})$ *of length* $\ell = 5m$*, such that*

$$\begin{aligned}\mathrm{PSD}_{\mathbf{a}}(m) &= \ell + 1 + \sqrt{5}x,\\ \mathrm{PSD}_{\mathbf{b}}(m) &= \ell + 1 - \sqrt{5}x\end{aligned} \tag{5}$$

for some integer $x \geq 0$.

Conjecture 1 says there exists a Legendre pair $(\mathbf{a}, \mathbf{b})$ such that $n_1 = n_2 = \ell + 1$ in Proposition 2. That is, the constant $2\ell+2$ can be distributed in a "balanced" manner between $\mathrm{PSD}_{\mathbf{a}}(m)$ and $\mathrm{PSD}_{\mathbf{b}}(m)$. Table 1 provides computational evidence for Conjecture 1.

The following proposition provides valid constraints on the m-compressed vectors $(\mathcal{A}_m, \mathcal{B}_m)$ of a Legendre pair $(\mathbf{a}, \mathbf{b})$ of length $\ell = 5m$.

m	$\ell = 5m$	x
1	5	1
3	15	$0, 2, 4$
5	25	1
7	35	8
9	45	3
11	55	12
13	65	7
15	75	24
17	85	18

Table 1: Computationally verified cases for Conjecture 1 with their corresponding x values

Proposition 3. *Let $\ell = 5m$ be an odd integer. Then a pair of m-compressed vectors $(\mathcal{A}_m, \mathcal{B}_m)$ of a Legendre pair $(\mathbf{a}, \mathbf{b})$ of length $\ell = 5m$, must satisfy*

$$\mathrm{PAF}_{\mathcal{A}_m}(1) - \mathrm{PAF}_{\mathcal{A}_m}(2) = -\Big(\mathrm{PAF}_{\mathcal{B}_m}(1) - \mathrm{PAF}_{\mathcal{B}_m}(2)\Big) = x,$$

where x is as in Proposition 2.

Proof. This result follows directly from equations (1) and the fact that a Legendre pair $(\mathbf{a}, \mathbf{b})$ of length $\ell = 5m$ satisfies $\mathrm{PSD}_{\mathbf{a}}(m) + \mathrm{PSD}_{\mathbf{b}}(m) = 2\ell + 2$. ∎

The following are some observations pertaining to Conjecture 1.

1. For a fixed value of ℓ, the value of x in Conjecture 1 is not necessarily unique.
2. The value of the integer x is determined by the m-compressed vectors $(\mathcal{A}_m, \mathcal{B}_m)$.
3. There exist Legendre pairs of length $\ell = 5m$ which do not satisfy Conjecture 1.

The following proposition provides valid constraints on the m-compressed vectors $(\mathcal{A}_m, \mathcal{B}_m)$ of a Legendre pair $(\mathbf{a}, \mathbf{b})$ of length $\ell = 5m$ satisfying equations (5).

Proposition 4. *A pair of m-compressed vectors $(\mathcal{A}_m, \mathcal{B}_m)$ of a Legendre pair $(\mathbf{a}, \mathbf{b})$ of length $\ell = 5m$ satisfying equations (5) must satisfy*

$$p_2(\mathcal{A}_m) = \mathrm{PAF}_{\mathcal{A}_m}(0) = p_2(\mathcal{B}_m) = \mathrm{PAF}_{\mathcal{B}_m}(0) = 4m + 1.$$

Proof. This result follows directly from equations (2) and (4). ∎

By Proposition 4

$$\begin{aligned} A_1^2 + A_2^2 + A_3^2 + A_4^2 + A_5^2 = 4m + 1, \\ B_1^2 + B_2^2 + B_3^2 + B_4^2 + B_5^2 = 4m + 1. \end{aligned} \tag{6}$$

In addition, we are only interested in all-odd solutions, as A_i, B_i are odd numbers for $i = 1, \ldots, 5$. Proposition 4 significantly reduces the search space by drastically truncating the number of possible m-compressed vectors $(\mathcal{A}_m, \mathcal{B}_m)$ which could give rise to a Legendre pair of order $\ell = 5m$. We illustrate this fact in the next section.

Another consequence of Proposition 4 is that the alphabet of the possible m-compressed vectors $(\mathcal{A}_m, \mathcal{B}_m)$ is also significantly truncated. More specifically, while the full alphabet for candidate m-compressed vectorss is

$$\{-m, -(m-2), \ldots, -1, +1, \ldots, (m-2), m\},$$

by Proposition 4, A_i is an odd integer with $|A_i| \leq \sqrt{4m-3} < 2\sqrt{m}$.

Finally, the solutions of the sums-of-squares Diophantine equations (6) which do not satisfy $A_1 + A_2 + A_3 + A_4 + A_5 = 1$ and $B_1 + B_2 + B_3 + B_4 + B_5 = 1$ are ruled out. Hence, this sometimes provides additional reductions in the number of possible candidate m-compressed vectors $(\mathcal{A}_m, \mathcal{B}_m)$.

3 Computational evidence for Conjecture 1

In this section, we verify Conjecture 1 for Legendre pairs of the eight known lengths $\ell = 5, 15, 25, 35, 45, 55, 65, 75$ and the ninth previously open length $\ell = 85$. Therefore, we find the first known example of a Legendre pair of length $\ell = 85$, which had been the smallest previously unknown length case for Legendre pairs. Conjecture 1 holds for the subsequent lengths $\ell = 95, 105$ and has practical value as it can be used to prune the search space when $\ell \equiv 0 \pmod 5$.

In Table 3, we summarize the all-odd solutions to (6) for odd ℓ such that $\ell = 5m$ and $m \in \{1, \ldots, 23\}$. In the last column, we record the all-odd solutions (up to sign changes and permutations) of the five sums of squares Diophantine equations of the fourth column. Since the linear equations $A_1 + A_2 + A_3 + A_4 + A_5 = 1$ and $B_1 + B_2 + B_3 + B_4 + B_5 = 1$ must also be satisfied, some of these all-odd solutions are ruled out, and this is indicated by boldface.

4 Legendre pairs of order $\ell = 85$ exist

We use the method in [3] to restrict the search space for Legendre pairs of order $\ell = 85$. The subgroup $H = \{1, 69\}$ of $\mathbb{Z}_{85}^{\star}$ acts on $\mathbb{Z}_{85}$ and yields 16 orbits of size 1 and 34 orbits

ℓ	m	$\mathrm{PSD}_{\mathbf{a}}(m)$ $\mathrm{PSD}_{\mathbf{b}}(m)$	$A_1^2+A_2^2+A_3^2+A_4^2+A_5^2$ $B_1^2+B_2^2+B_3^2+B_4^2+B_5^2$	All-odd solutions
5	1	$5+1\pm\sqrt{5}x$	$4\cdot 1+1=5$	[1, 1, 1, 1, 1]
15	3	$15+1\pm\sqrt{5}x$	$4\cdot 3+1=13$	[1, 1, 1, 1, 3]
25	5	$25+1\pm\sqrt{5}x$	$4\cdot 5+1=21$	[1, 1, 1, 3, 3]
35	7	$35+1\pm\sqrt{5}x$	$4\cdot 7+1=29$	[1, 1, 1, 1, 5], [1, 1, 3, 3, 3]
45	9	$45+1\pm\sqrt{5}x$	$4\cdot 9+1=37$	[1, 1, 1, 3, 5], [1, 3, 3, 3, 3]
55	11	$55+1\pm\sqrt{5}x$	$4\cdot 11+1=45$	[1, 1, 3, 3, 5], [**3, 3, 3, 3, 3**]
65	13	$65+1\pm\sqrt{5}x$	$4\cdot 13+1=53$	[**1, 1, 1, 1, 7**], [1, 1, 1, 5, 5], [1, 3, 3, 3, 5]
75	15	$75+1\pm\sqrt{5}x$	$4\cdot 15+1=61$	[1, 1, 1, 3, 7], [1, 1, 3, 5, 5], [3, 3, 3, 3, 5]
85	17	$85+1\pm\sqrt{5}x$	$4\cdot 17+1=69$	[1, 1, 3, 3, 7], [1, 3, 3, 5, 5]
95	19	$95+1\pm\sqrt{5}x$	$4\cdot 19+1=77$	[1, 1, 1, 5, 7], [**1, 1, 5, 5, 5**], [1, 3, 3, 3, 7], [3, 3, 3, 5, 5]
105	21	$105+1\pm\sqrt{5}x$	$4\cdot 21+1=85$	[**1, 1, 1, 1, 9**], [1, 1, 3, 5, 7], [1, 3, 5, 5, 5], [3, 3, 3, 3, 7]
115	23	$115+1\pm\sqrt{5}x$	$4\cdot 23+1=93$	[**1, 1, 1, 3, 9**], [1, 3, 3, 5, 7], [3, 3, 5, 5, 5]

Table 2: All-odd solutions to equations (6)

of size 2. We search for a Legendre pair of length $\ell = 85$ which can be obtained by combining the orbits of the subgroup H. This restriction has the benefit of reducing the search space provided that such a Legendre pair exists. We choose 12 orbits of size 1 and 15 orbits of size 2 to make blocks of size $12\cdot 1 + 15\cdot 2 = 42$. Therefore, the size of the search space is $\binom{16}{12}\cdot\binom{34}{15} = 1820\cdot 1{,}855{,}967{,}520 = 3{,}377{,}860{,}886{,}400$. We find 4 Legendre pairs of length $\ell = 85$ made out of 6 different vectors via a non-exhaustive search. Their lexicographic rank encodings as subsets of size 12 out of 16 and 15 out of 34 are

$$\begin{aligned}
&\big(\{12, 1321116338\},\ \{42, 1275934280\}\big),\\
&\big(\{12, 1843909851\},\ \{42, 606586783\}\big),\\
&\big(\{42, 1275934280\},\ \{9, 1555522731\}\big),\\
&\big(\{42, 606586783\},\ \{9, 788215097\}\big).
\end{aligned}$$

See [4] for lexicographic ranking and unranking algorithms for subsets of size k. For the first Legendre pair $(\mathbf{a}, \mathbf{b})$ of length $\ell = 85$ shown above, its lexicographic rank encoding $\big(\{12, 1321116338\},\ \{42, 1275934280\}\big)$, is decoded as follows.

- In the space of $\binom{16}{12} = 1820$ 12-subsets of $\{1, \ldots, 16\}$, decode

$$\begin{aligned}
&12 \text{ as } \mathbf{a}_{\text{ones}} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 14, 15\},\\
&42 \text{ as } \mathbf{b}_{\text{ones}} = \{1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 14, 15\}.
\end{aligned}$$

- In the space of $\binom{34}{15} = 1,855,967,520$ 15-subsets of $\{1,\ldots,34\}$, decode

$$1321116338 \text{ as } \mathbf{a}_{\text{twos}} = \{3,4,5,7,10,11,22,24,25,27,28,29,30,31,34\},$$
$$1275934280 \text{ as } \mathbf{b}_{\text{twos}} = \{2,8,10,11,12,15,19,21,23,25,26,28,29,33,34\}.$$

- Enumerate the 16 orbits of size 1 in increasing order as $O_1 = \{\{5\},\{10\},\{15\},\{20\},\{25\},\{30\},\{35\},\{40\},\{45\},\{50\},\{55\},\{60\},\{65\},\{70\},\{75\},\{80\}\}$.

- Enumerate the 34 orbits of size 2 in increasing order of their smallest element as $O_2 = \{\{1,69\},\{2,53\},\{3,37\},\{4,21\},\{6,74\},\{7,58\},\{8,42\},\{9,26\},\{11,79\},\{12,63\},\{13,47\},\{14,31\},\{16,84\},\{17,68\},\{18,52\},\{19,36],\{22,73],\{23,57\},\{24,41\},\{27,78\},\{28,62\},\{29,46\},\{32,83\},\{33,67\},\{34,51\},\{38,72\},\{39,56\},\{43,77\},\{44,61\},\{48,82\},\{49,66\},\{54,71\},\{59,76\},\{64,81\}\}$.

- Make the block of size 42 of the indices of the positions of the -1 elements in $\mathbf{a}$, by combining 12 elements of O_1 whose indices are given by $\mathbf{a}_{\text{ones}}$ and 15 elements of O_2 whose indices are given by $\mathbf{a}_{\text{twos}}$. This yields the following $\mathbf{a}$-block of size 42: $\{5,10,15,20,25,30,35,40,45,50,70,75,3,37,4,21,6,74,8,42,12,63,13,47,29,46,33,67,34,51,39,56,43,77,44,61,48,82,49,66,64,81\}$.

- Make the block of size 42 of the indices of the positions of the -1 elements in $\mathbf{b}$, by combining 12 elements of O_1 whose indices are given by $\mathbf{b}_{\text{ones}}$ and 15 elements of O_2 whose indices are given by $\mathbf{b}_{\text{twos}}$. This yields the following $\mathbf{b}$-block of size 42: $\{5,10,15,20,25,30,35,40,50,55,70,75,2,53,9,26,12,63,13,47,14,31,18,52,24,41,28,62,32,83,34,51,38,72,43,77,44,61,59,76,64,81\}$.

- The above $\mathbf{a}$-block and $\mathbf{b}$-block (both of size 42) yield a Legendre pair for $\ell = 85$.

For the first Legendre pair $(\mathbf{a},\mathbf{b})$ of length $\ell = 85$ shown above, the 17-compressions are

$$\mathcal{A}_{17} = [1,3,3,1,-7],\ \mathcal{B}_{17} = [3,1,1,3,-7].$$

These possess the properties required by Proposition 4. That is,

$$\text{PAF}_{\mathcal{A}_{17}}(0) = \text{PAF}_{\mathcal{B}_{17}}(0) = 4 \cdot 17 + 1 = 69.$$

As a consequence, the coefficients of $\sqrt{5}$ in $\text{PSD}_{\mathbf{a}}(17)$ and $\text{PSD}_{\mathbf{b}}(17)$ cancel out. Specifically,

$$\text{PSD}_{\mathbf{a}}(17) = 85 + 1 + \sqrt{5} \cdot 18 = 126.2492236,$$

$$\text{PSD}_{\mathbf{b}}(17) = 85 + 1 - \sqrt{5} \cdot 18 = 45.75077641.$$

5 Legendre pairs of order $\ell = 87$ with balanced compressions exist

By a property of compression of Legendre pairs which can be deduced from a theorem in [1], a 3-compression of a Legendre pair of order $\ell = 87$ must contain 14 elements with absolute value equal to three and $2 \cdot 29 - 14 = 44$ elements with absolute value equal to 1. Experimental evidence for other orders which are divisible by 3 indicates that "balanced" configurations of candidate 3-compressions (which have exactly half of the total number of elements with absolute value equal to 3) are likely to yield Legendre pairs. Hence, we compute approximately six thousand candidate 3-compressions satisfying the following constraints, where only the last constraint is not necessary.

1. The vectors $\mathcal{A}_3, \mathcal{B}_3 \in \{-3, -1, +1, +3\}^{29}$ contain 14 elements with absolute value equal to 3 and 44 elements with absolute value equal to 1.
2. $\mathrm{PAF}_{\mathcal{A}_3}(s) + \mathrm{PAF}_{\mathcal{B}_3}(s) = (-2) \cdot 3 = -6$ for $s = 1, \ldots, 28$.
3. $\mathrm{PSD}_{\mathcal{A}_3}(s) + \mathrm{PSD}_{\mathcal{B}_3}(s) = 2 \cdot 87 + 2 = 176$ for $s = 1, \ldots, 28$.
4. $A_1 + \cdots + A_{29} = 1$.
5. $B_1 + \cdots + B_{29} = 1$.
6. Each of $\mathcal{A}_3$ and $\mathcal{B}_3$ contains 7 elements with absolute value equal to 3 and 22 elements with absolute value equal to 1.

Subsequently, we run our `C` 3-uncompression code for approximately two thousand candidate 3-compressions and discovered the two Legendre pairs of order $\ell = 87$ below.

$$\begin{aligned}
\mathbf{a} = [&-1,-1,-1,-1,1,-1,-1,-1,1,1,-1,1,-1,1,-1,1,1,-1,-1,1,1,-1,-1,1,-1,1,-1,1,1,\\
&-1,-1,1,1,1,-1,-1,-1,-1,-1,1,-1,-1,1,-1,-1,-1,1,1,1,1,-1,-1,-1,1,-1,1,1,\\
&1,-1,1,1,-1,-1,-1,-1,1,1,1,1,1,1,-1,-1,1,1,1,-1,1,1,1,1,1,-1,1,-1,1,-1],\\
\mathbf{b} = [&-1,-1,-1,-1,-1,-1,-1,-1,1,1,1,1,-1,1,1,-1,1,1,-1,1,-1,1,1,1,1,-1,-1,1,1,\\
&-1,1,1,-1,1,-1,-1,1,-1,-1,1,1,1,-1,1,-1,1,1,-1,-1,1,-1,-1,1,1,1,1,1,-1,-1,\\
&-1,1,1,1,-1,-1,1,-1,1,-1,-1,1,-1,1,-1,-1,-1,1,-1,-1,1,1,1,-1,1,-1,1,-1],\\
\mathbf{a} = [&-1,-1,-1,-1,1,-1,-1,-1,1,1,1,1,1,1,-1,1,-1,-1,-1,1,1,1,-1,1,-1,-1,-1,1,\\
&1,-1,-1,1,1,1,-1,-1,-1,1,-1,-1,1,-1,1,-1,-1,1,1,1,1,1,-1,-1,1,1,1,-1,1,1,-1,\\
&1,1,-1,-1,-1,-1,1,-1,1,1,-1,-1,-1,-1,1,1,1,-1,1,1,-1,1,-1,-1,1,1,1,1,-1],\\
\mathbf{b} = [&-1,1,-1,1,1,-1,-1,1,1,1,1,1,1,-1,1,-1,1,1,1,-1,-1,1,-1,1,-1,1,-1,1,-1,\\
&-1,-1,1,-1,-1,-1,-1,-1,-1,-1,1,-1,1,1,1,-1,1,-1,-1,1,-1,-1,1,1,1,-1,-1,\\
&1,1,-1,-1,1,-1,1,-1,-1,1,-1,1,-1,1,-1,-1,1,-1,-1,1,-1,-1,1,1,1,1,1,1,1,1,-1].
\end{aligned}$$

Both of the above Legendre pairs of length $\ell = 87$ 3-compress to

$$\mathcal{A}_3 = [-3, -1, 1, -1, 1, -3, -3, -1, 1, 1, 1, 1, -1, 1, -3, 1, 1, 1, -1, 3, 3, -1, -1, 1, -1, 1, -1, 3, 1],$$
$$\mathcal{B}_3 = [-3, -1, 1, -1, 1, -3, -3, 1, -1, 1, 1, 1, 1, -1, 3, -3, 1, 1, -1, -1, -1, 1, 1, 3, 1, 1, -1, 3, -1].$$

There are seven ± 3 elements in each of the two vectors $\mathcal{A}_3$ and $\mathcal{B}_3$. Therefore, this "balanced" configuration does yield Legendre pairs of order $\ell = 87$.

6 Conclusion

Recently, Legendre pairs of (the previously open) lengths $77, 117, 129, 133, 147$ were found in [3, 5]. In this paper, we find Legendre pairs of (the previously open) lengths $85, 87$, i.e. the only two remaining open lengths less that 100 [3, 5]. This reduces the list of integers less than 200 for which the existence of Legendre pairs problem remains open to the ten values:

$$115, 145, 159, 161, 169, 175, 177, 185, 187, 195.$$

Half of these ten values are multiples of 5, so they fit within the scope of Conjecture 1.

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