Holonomic Anti-Differentiation and Feynman Amplitudes

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Abstract Computer algebra methods within the scope of the holonomic systems approach provide a versatile toolbox to integration problems in the context of Feynman diagrams. This is demonstrated with the aid of several benchmark problems, ranging from hypergeometric series evaluations to Bessel integrals of sunrise diagrams.

1 Introduction

Recent interest in the mathematical structure of Feynman diagrams has been inspired by the persistent accuracy of high-energy experiments at LHC. Still in the 1970's it was pointed out that Feynman diagrams can be understood as a special class of functions satisfying some system of differential equations. Later, it was shown in [1] within analytical regularization, that any regularized Feynman integral satisfies some holonomic system of linear differential equations. In dimension regularization, this statement was later presented by a few groups [2, 3, 4].

It was a popular idea to explore the holonomic systems approach, as originally formulated by Zeilberger [5], for the reduction of Feynman diagrams to the set of so-called master integrals [6, 7, 8]. Unfortunately, this idea was not followed up, due to the complexity of the problem. Nevertheless, we claim that the holonomic approach can be quite useful for solving other problems, related to Feynman diagrams. One goal of this paper is to substantiate this claim with the aid of a well-chosen set of problems, which we are going to tackle with the HolonomicFunctions package [9, 10].

This work was initiated at the WPC workshop "Anti-Differentiation and the Calculation of Feynman Amplitudes", that took place in October 2020 at DESY Zeuthen. The material presented here reflects the outcome of several discussions during this meeting. Specifically, we have to give Mikhail Kalmykov credit for compiling the collection of challenge problems.

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2 The Holonomic Systems Approach

Before we start, we give a very brief introduction to the main mathematical tool that is used in this paper, that is the holonomic systems approach [5]. For more details and background on the methods employed here, we refer to the survey articles [11, 12].

In order to write mixed difference-differential equations in a concise way, we employ the following usual operator notation: let D_x denote the partial derivative operator with respect to x (x is then called a continuous variable) and S_n the forward shift operator with respect to n (n is then called a discrete variable); they act on a function f by

$$D_x f = \frac{\partial f}{\partial x} \quad \text{and} \quad S_n f = f\big|_{n \to n+1}.$$

They allow us to write linear homogeneous difference-differential equations in terms of operators, e.g.,

$$\frac{\partial}{\partial x} f(k, n+1, x, y) + n \frac{\partial}{\partial y} f(k, n, x, y) + x f(k+1, n, x, y) - f(k, n, x, y) = 0$$

turns into

$$\big(D_x S_n + n D_y + x S_k - 1\big) f(k, n, x, y) = 0;$$

in other words, such equations are represented by polynomials in the operator symbols D_x, S_n, etc., with coefficients in some field $\mathbb{K}$ which is typically some rational function field in the variables x, n, etc., and possibly in some additional parameters. Note that in general the polynomial ring $\mathbb{K}\langle D_x, S_n, \dots \rangle$ is not commutative (this fact is indicated by the angle brackets): the coefficients from the field $\mathbb{K}$ do not commute with the polynomial variables D_x, S_n, etc. For instance, multiplication with $a(x, n) \in \mathbb{K}$ is subject to the rules

$$D_x \cdot a(x, n) = a(x, n) \cdot D_x + \frac{\partial}{\partial x} a(x, n) \quad \text{and} \quad S_n \cdot a(x, n) = a(x, n+1) \cdot S_n.$$

Such non-commutative rings of operators are called Ore algebras, and we typically denote them by $\mathbb{O}$; concise definitions and specifications of the properties of such algebras can be found, for instance, in [9].

We define the annihilator (with respect to some Ore algebra $\mathbb{O}$) of a function f by:

$$\operatorname{ann}_{\mathbb{O}}(f) := \{P \in \mathbb{O} \mid P(f) = 0\}.$$

It can easily be seen that $\operatorname{ann}_{\mathbb{O}}(f)$ is a left ideal in $\mathbb{O}$. Every left ideal $I \subseteq \operatorname{ann}_{\mathbb{O}}(f)$ is called an annihilating ideal for f.

Definition 1 Let $\mathbb{O} = \mathbb{K}\langle \dots \rangle$ be an Ore algebra. A function f is called ∂-finite w.r.t. $\mathbb{O}$ if $\mathbb{O}/\operatorname{ann}_{\mathbb{O}}(f)$ is a finite-dimensional $\mathbb{K}$-vector space. The dimension of this vector space is called the (holonomic) rank of f w.r.t. $\mathbb{O}$.

In the holonomic systems approach, the representing data structures of functions are (generators of) annihilating ideals (plus initial values). When working with (left) ideals, we use (left) Gröbner bases [13, 14] which are an important tool for executing certain operations (e.g., the ideal membership test) in an algorithmic way.

Without proof we state the following theorem about closure properties of ∂-finite functions; its proof can be found in [9, Chap. 2.3]. We remark that all of them are algorithmically executable, and the algorithms work with the above mentioned data structure.

Theorem 1 *Let $\mathbb{O}$ be an Ore algebra and let f and g be ∂-finite w.r.t. $\mathbb{O}$ of rank r and s, respectively. Then*

(i) $f + g$ is ∂-finite of rank $\leqslant r + s$.

(ii) $f \cdot g$ is ∂-finite of rank $\leqslant rs$.

(iii) f^2 is ∂-finite of rank $\leqslant r(r + 1)/2$.

(iv) Pf is ∂-finite of rank $\leqslant r$ for any $P \in \mathbb{O}$.

(v) $f|_{x\to A(x,y,\dots)}$ is ∂-finite of rank $\leqslant rd$ if $x, y, \dots$ are continuous variables and if the algebraic function A satisfies a polynomial equation of degree d.

(vi) $f|_{n\to A(n,k,\dots)}$ is ∂-finite of rank $\leqslant r$ if A is an integer-linear expression in the discrete variables $n, k, \dots$.

Note that in most examples the bounds on the rank are sharp.

If we want to consider integration and summation problems, then the function in question needs to be holonomic, a concept that is closely related to ∂-finiteness. The precise definition is a bit technical and therefore skipped here; the interested reader can find it, e.g., in [5, 15, 9]. All functions that appear in this paper are both ∂-finite and holonomic. The following theorem establishes the closure of holonomic functions with respect to sums and integrals; for its proof, we once again refer to [5, 9].

Theorem 2 *Let the function f be holonomic w.r.t. D_x (resp. S_n). Then also $\int_a^b f\,\mathrm{d}x$ (resp. $\sum_{n=a}^b f$) is holonomic.*

If a function is ∂-finite and holonomic then Chyzak's algorithm [16] can be used to compute an annihilating ideal for the integral (resp. sum), or a heuristic approach proposed in [17]. In either case, the treatment of integrals and summations is based on the method of creative telescoping [18]. For example, for a parametrized integral of the form $\int_{a(t)}^{b(t)} f(x,t)\,\mathrm{d}x$, one has to determine a pair (P, Q), called the telescoper and the certificate, with the properties that $P + D_x Q \in \operatorname{ann}(f)$ and that the operator P is free of x and D_x. Then, integrating the equation $(P + D_x Q)(f) = 0$ and using the fundamental theorem of calculus, yields a linear differential equation for the integral.

In our calculations we will use the software package HolonomicFunctions [10], implemented in Mathematica by the author, where all the above mentioned algorithms are available.

3 Particular Values of Hypergeometric Functions

In [19] the authors present nice evaluations of hypergeometric functions at particular values. Let ε be an arbitrary parameter, then the following holds:

$$ {}_2F_1\left(\left.\begin{matrix} 2\varepsilon, 3\varepsilon \\ \frac{1}{2}+2\varepsilon \end{matrix}\right| \frac{1}{4}\right) = \frac{\Gamma(1+\varepsilon)\,\Gamma(1+4\varepsilon)}{\Gamma(1+2\varepsilon)\,\Gamma(1+3\varepsilon)} \tag{1} $$

see also [20] for similar hypergeometric evaluations at $\frac{1}{4}$. In this section we are demonstrating the usage of computer algebra for proving identities like (1). Other software packages, specialized to the treatment of hypergeometric series, include HYP [21] and HYPERDIRE [22].

3.1 Evaluation of a ${}_2F_1$

We write down the definition of the ${}_2F_1$ hypergeometric function as an infinite sum

$$ {}_2F_1\left(\left.\begin{matrix} 2\varepsilon, 3\varepsilon \\ \frac{1}{2}+2\varepsilon \end{matrix}\right| \frac{1}{4}\right) = \sum_{k=0}^{\infty} \frac{(2\varepsilon)_k\,(3\varepsilon)_k}{\left(\frac{1}{2}+2\varepsilon\right)_k\,k!}4^{-k} $$

and denote the expression inside the sum by $f_{k,\varepsilon}$. By viewing k and ε as discrete variables, one can immediately construct two difference equations, one in k and one in ε, and both of first order, which are satisfied by $f_{k,\varepsilon}$:

$$ \begin{aligned} &2(k+1)(4\varepsilon+2k+1)f_{k+1,\varepsilon} = (2\varepsilon+k)(3\varepsilon+k)f_{k,\varepsilon}, \\ &6\varepsilon^2(2\varepsilon+1)(3\varepsilon+1)(3\varepsilon+2)(4\varepsilon+2k+1)(4\varepsilon+2k+3)f_{k,\varepsilon+1} = \\ &\quad (4\varepsilon+1)(4\varepsilon+3)(2\varepsilon+k)(2\varepsilon+k+1)(3\varepsilon+k)(3\varepsilon+k+1)(3\varepsilon+k+2)f_{k,\varepsilon}. \end{aligned} $$

Applying the creative telescoping algorithm to these recurrence equations yields a telescoper

$$ P = 3(3\varepsilon+1)(3\varepsilon+2)S_\varepsilon - 4(4\varepsilon+1)(4\varepsilon+3) $$

and a certificate Q that is given by the following rational function:

$$ \frac{(4\varepsilon+1)(4\varepsilon+3)k\big(74\varepsilon^3+3(19k+18)\varepsilon^2+(12k^2+27k+10)\varepsilon+k(k+1)(k+2)\big)}{3\varepsilon^2(2\varepsilon+1)(4\varepsilon+2k+1)}. $$

They satisfy the telescopic relation $(P+(S_k-1)Q)(f_{k,\varepsilon})=0$, a fact that can be easily verified by applying the operator $P+(S_k-1)Q$ to $f_{k,\varepsilon}$ and by subsequent simplification (which is straightforward, but tedious by hand). Summing this relation for k from 0 to ∞ yields

$$\sum_{k=0}^{\infty} P(f_{k,\varepsilon}) + \underbrace{\lim_{k\to\infty} Q f_{k,\varepsilon} - Q f_{k,\varepsilon}\big|_{k=0}}_{=0},$$

which reveals that the ${}_2F_1$ function from Equation (1), let us denote it by $F(\varepsilon)$, satisfies the following recurrence equation:

$$3(1+3\varepsilon)(2+3\varepsilon)F(\varepsilon+1) = 4(1+4\varepsilon)(3+4\varepsilon)F(\varepsilon).$$

Plugging in the right-hand side of (1) into the above recurrence, and simplifying it, reveals that also the closed form, the quotient of Gamma functions, satisfies the same recurrence. By comparing a single initial value ($\varepsilon = 0$), we establish the identity: for $\varepsilon = 0$ the infinite sum reduces to a finite one since only the first summand (which equals 1) survives, thanks to the definition of the Pochhammer symbol. Similarly, all Gamma functions on the right-hand side evaluate to 1 when ε is sent to 0.

3.2 Evaluations of ${}_3F_2$ Hypergeometric Functions

In an analogous fashion, one can prove identities like

$$\frac{1}{(1-\varepsilon)(1+2\varepsilon)}\, {}_3F_2\left(\begin{matrix}1, 1+\varepsilon, 1+2\varepsilon\\ \frac{3}{2}+\varepsilon, 2-\varepsilon\end{matrix}\,\middle|\,\frac{1}{4}\right) = \frac{1}{3\varepsilon^2}\left(\frac{\Gamma(1+2\varepsilon)\,\Gamma(1-\varepsilon)}{\Gamma(1+\varepsilon)} - 1\right). \quad (2)$$

This one is a consequence of Equation (1), but can also be proven directly with the holonomic approach. The summand here (after expanding the definition of ${}_3F_2$) is

$$f_{k,\varepsilon} = \frac{(\varepsilon+1)_k\,(2\varepsilon+1)_k}{(1-\varepsilon)(2\varepsilon+1)\,4^k\,(2-\varepsilon)_k\,\left(\varepsilon+\frac{3}{2}\right)_k}$$

Again, creative telescoping yields

$$P = (\varepsilon+1)^2 S_\varepsilon + 2\varepsilon(2\varepsilon+1),$$
$$Q = \frac{(\varepsilon-k-1)(10\varepsilon^2+5\varepsilon k+9\varepsilon+k^2+3k+2)}{3\varepsilon}$$

with $(P + (S_k - 1)Q)(f_{k,\varepsilon}) = 0$. In contrast to the previous example, one gets an inhomogeneous contribution

$$Q f_{k,\varepsilon}\big|_{k=0} = \frac{(\varepsilon-1)(10\varepsilon^2+9\varepsilon+2)}{3\varepsilon} \cdot \frac{1}{(1-\varepsilon)(1+2\varepsilon)} = -\frac{2+5\varepsilon}{3\varepsilon},$$

which gives rise to the inhomogeneous recurrence equation

$$(\varepsilon+1)^2 F(\varepsilon+1) + 2\varepsilon(2\varepsilon+1)F(\varepsilon) = -\frac{5\varepsilon+2}{3\varepsilon}.$$

It is easy to check that also the right-hand side of Equation (2) satisfies this recurrence.

It is well known, that the two-loop massless propagator diagram posesses a large class of symmetries under exchange of indices (see [23, 24, 25]). Recently, the following relations between Clausen's hypergeometric function of arguments $z = \pm 1$ was proven (see Eq. (5) in [26]):

$$\begin{aligned}
&{}_3F_2\left(\begin{matrix}1, B, 2A\\ 1+B, 2-A\end{matrix}\,\middle|\,-1\right) + \frac{B}{1+A-B}\cdot {}_3F_2\left(\begin{matrix}1, 2A, 1+A-B\\ 2-A, 2+A-B\end{matrix}\,\middle|\,-1\right)\\
&\quad = B\frac{\Gamma(2-A)\,\Gamma(B+A-1)\,\Gamma(B-A)\,\Gamma(1+A-B)}{\Gamma(2A)\,\Gamma(1+B-2A)}\\
&\qquad - \frac{1-A}{B+A-1}\cdot {}_3F_2\left(\begin{matrix}1, B, 2A\\ 1+B, A+B\end{matrix}\,\middle|\,1\right),
\end{aligned}$$

where A and B are arbitrary numbers. Also such type of identites can be treated, by applying creative telescoping to the expression

$$\frac{(-1)^k(2A)_k(B)_k}{(2-A)_k(B+1)_k} + \frac{(-1)^k B\,(2A)_k(A-B+1)_k}{(A-B+1)\,(2-A)_k(A-B+2)_k}$$

in order to obtain a set of recurrences in A and B for the left-hand side (of holonomic rank 3), and then by analogously computing an annihilator for the right-hand side, which turns out to consist of exactly the same recurrences.

3.3 Finding More ${}_2F_1$ Identities

Not only can we apply the holonomic systems approach to prove identities like (1) or (2) or to evaluate the hypergeometric functions appearing there (i.e., without knowing the right-hand sides), but the holonomic approach also allows one to find, almost automatically, many more, similar identities. We exemplify this with Equation (1), i.e., we seek identities of the form

$$ {}_2F_1\left(\begin{matrix}a+2\varepsilon, b+3\varepsilon\\ c+2\varepsilon\end{matrix}\,\middle|\,x\right) = H(a,b,c,\varepsilon,x), \tag{3}$$

where H stands for some hypergeometric expression with respect to ε: the shift-quotient $H(a,b,c,\varepsilon+1,x)/H(a,b,c,\varepsilon,x)$ should be a rational function, when regarded as a function in ε. In practice, it will be the case the H is hypergeometric-hyperexponential in all parameters a, b, c, ε, x, which means that it can be expressed in closed form in terms of powers, Gamma functions, and the like.

In the algebraic language, the problem is to identify conditions on the parameters a, b, c, x such that the telescoper of the summand

$$s_{k,\varepsilon} = s_{k,\varepsilon}(a,b,c,x) = \frac{(a+2\varepsilon)_k\,(b+3\varepsilon)_k}{(c+2\varepsilon)_k}\frac{x^k}{k!}$$

is a first-order operator in S_ε. Following the approach proposed in [17], one can construct an ansatz for the telescopic operator $P + (S_k - 1)Q$ using the following specification:

$$P = P(\varepsilon, S_\varepsilon) = p_1 S_\varepsilon + p_0, \quad Q = Q(k,\varepsilon) = \frac{1}{c+2\varepsilon+k} \cdot \sum_{i=0}^{4} q_i k^i$$

Note that Q need not depend on S_k or S_ε, because the input $s_{k,\varepsilon}$ is hypergeometric. All unknowns here, namely the seven symbols $p_0, p_1, q_0, \ldots, q_4$, are assumed to be rational functions in $\mathbb{Q}(a,b,c,\varepsilon)$ and should not depend on k. Applying the telescopic operator to the summand $s_{k,\varepsilon}$, and by subsequently dividing by $s_{k,\varepsilon}$ yields

$$p_1 \cdot \frac{s_{k,\varepsilon+1}}{s_{k,\varepsilon}} + p_0 + Q(k+1,\varepsilon) \cdot \frac{s_{k+1,\varepsilon}}{s_{k,\varepsilon}} - Q(k,\varepsilon) = 0$$

where

$$\begin{aligned}
\frac{s_{k,\varepsilon+1}}{s_{k,\varepsilon}} &= \frac{(a+2\varepsilon+k)_2\,(b+3\varepsilon+k)_3\,(c+2\varepsilon)_2}{(a+2\varepsilon)_2\,(b+3\varepsilon)_3\,(c+2\varepsilon+k)_2},\\
\frac{s_{k+1,\varepsilon}}{s_{k,\varepsilon}} &= \frac{x(a+2\varepsilon+k)(b+3\varepsilon+k)}{(k+1)(c+2\varepsilon+k)}.
\end{aligned}$$

By clearing denominators, i.e., multiplying by $(c+2\varepsilon+k)(c+2\varepsilon+k+1)$, this identity of rational functions is turned into a polynomial equation of degree 6 in k. Coefficient comparison with respect to the variable k ensures that the parameters of the ansatz will not depend on k, as required, and will lead to a linear system of seven equations for the seven unknowns $p_0, p_1, q_0, \ldots, q_4$.

Since we are seeking a nontrivial solution of this system, we are interested in the cases where the system matrix is singular. We note that this matrix, although with 7×7 being small in dimension, has a nontrivial size in terms of byte count (totalling to about 1 MB), due to the appearance of the parameters a, b, c, ε, x. The determinant of the matrix is given (in fully factored form) by

$$\begin{aligned}
&x(c-a)(a-c-1)(a+2\varepsilon-1)(a+2\varepsilon)^6(a+2\varepsilon+1)^6(b+3\varepsilon-1)(b+3\varepsilon)^6\\
&\times (b+3\varepsilon+1)^6(b+3\varepsilon+2)^6(c+2\varepsilon)(c+2\varepsilon+1)(b-c+\varepsilon-1)(b-c+\varepsilon)\\
&\times \Big((x+2)(4x-1)\varepsilon^2 + (4ax^2+2ax+8bx-2b-7cx+c+2x^2+9x-2)\varepsilon\\
&\qquad + a(a+1)x^2 + x(2a+b+2)(b-c+1) + c(c-b-1)\Big).
\end{aligned}$$

The first three factors of the determinant correspond to trivial or well-known evaluations of the hypergeometric function:

$$
\begin{aligned}
x = 0: &\quad {}_2F_1\left(\begin{matrix} a+2\varepsilon, b+3\varepsilon \\ c+2\varepsilon \end{matrix} \middle| 0\right) = 1,\\
c = a: &\quad {}_2F_1\left(\begin{matrix} a+2\varepsilon, b+3\varepsilon \\ a+2\varepsilon \end{matrix} \middle| x\right) = \frac{1}{(1-x)^{b+3\varepsilon}},\\
c = a-1: &\quad {}_2F_1\left(\begin{matrix} a+2\varepsilon, b+3\varepsilon \\ a+2\varepsilon-1 \end{matrix} \middle| x\right) = \frac{a(1-x)+bx+\varepsilon(x+2)+x-1}{(a+2\varepsilon-1)(1-x)^{b+3\varepsilon+1}}.
\end{aligned}
$$

All remaining factors that are linear in ε do not give useful conditions: since the ε appears with a constant coefficient and since the parameters a, b, c, x are not supposed to depend on ε, these factors can never become 0. The only interesting factor is the last one, a quadratic polynomial in ε, which is zero if and only if all its three coefficients vanish. This yields three nonlinear polynomial equations in the parameters a, b, c, x. A (lexicographic) Gröbner basis of the ideal generated by these polynomials is given by the following six polynomials:

$$
\begin{aligned}
&(x+2)(4x-1),\\
&(x+2)(2a-2c+1),\\
&a^2-2ac+a+c^2-c+x,\\
&12a+8bx-2b-12cx-9c+8x+4,\\
&12ab-18ac+12a-12bc+6b+18c^2-21c+8x+4,\\
&12b^2-36bc+24b+27c^2-36c+5x+10.
\end{aligned}
$$

Thanks to their triangular shape, they allow us to determine the complete set of solutions (a, b, c, x) to our polynomial equations, parametrized by a:

$$
\begin{aligned}
&\left(a, \frac{3(a-1)}{2}, a-1, -2\right), \quad &&\left(a, \frac{3a+2}{2}, a+2, -2\right),\\
&\left(a, \frac{3a}{2}, \frac{2a+1}{2}, \frac{1}{4}\right), &&\left(a, \frac{3a-1}{2}, \frac{2a+1}{2}, \frac{1}{4}\right).
\end{aligned}
$$

Clearly, the first two families of solutions are not interesting, since the corresponding hypergeometric series are not convergent. In contrast, the two families in the second row do give us valid identities:

$$
{}_2F_1\left(\begin{matrix} a+2\varepsilon, \frac{3}{2}a+3\varepsilon \\ \frac{1}{2}(2a+1)+2\varepsilon \end{matrix} \middle| \frac{1}{4}\right) = \frac{\Gamma\left(\frac{a}{2}+\varepsilon+1\right)\,\Gamma(2a+4\varepsilon+1)}{\Gamma(a+2\varepsilon+1)\,\Gamma\left(\frac{3a}{2}+3\varepsilon+1\right)}, \tag{4}
$$

$$
{}_2F_1\left(\begin{matrix} a+2\varepsilon, \frac{1}{2}(3a-1)+3\varepsilon \\ \frac{1}{2}(2a+1)+2\varepsilon \end{matrix} \middle| \frac{1}{4}\right) = \left(\frac{4}{3}\right)^{\frac{3a}{2}+3\varepsilon} \cdot \frac{\Gamma\left(\frac{a}{2}+\varepsilon+1\right)\Gamma\left(a+2\varepsilon+\frac{1}{2}\right)}{\Gamma\left(\frac{a}{2}+\varepsilon+\frac{1}{2}\right)\Gamma\left(a+2\varepsilon+1\right)}. \tag{5}
$$

Note that Equation (4) is a generalization of the original identity (1) we started with, which is recovered for $a = 0$.

Also, we should remark that this approach is not restricted to the special form where we have 2ε and 3ε in the top parameters of the ${}_2F_1$, and 2ε in the bottom parameter, but also to other situation where the epsilon coefficients 2, 2, 3 are replaced by other integers. In this fashion one could potentially find many more similar identities.

However, we do not claim that the two identities stated above are necessarily new. There is a vast literature on special evaluations of hypergeometric functions, and it is likely that they already appear somewhere. For example, large classes of such identities were presented in [27] and [28], and in particular the latter seems to take a similar approach as the one discussed here. We nevertheless would like to point out that, although the holonomic systems approach may not be the most efficient way for *finding* new identities, it is definitely a very useful tool for *proving* them.

4 Holonomic Integration

Some of the multiloop Feynman diagrams contain the one-loop diagram or the product of one-loop diagrams insertions [29, 30, 31, 32, 33]. In particular, the L-loop bubble type diagram, can be understood as the integration of $L-1$ propagators with an external massive line. The diagrams of that type have been studied from mathematical [34, 35] as well as from practical evaluation point of view [36, 37]. In particular, it was pointed out, that both types of diagrams are expressible in terms of F_C hypergeometric functions [38].

In this section, we will be interested in the integral

$$I(a) = \int_0^\infty \frac{t^{\alpha-1}}{(t+a)^j} f(t)\,\mathrm{d}t = \int_0^\infty F(t,a)\,\mathrm{d}t. \tag{6}$$

where $j \in \mathbb{Z}$ and α is a parameter, and where the unspecified function $f(t)$ satisfies the following linear non-homogeneous differential equation with polynomial coefficients:

$$(t+1)(t+9)f''(t) + (b_2t^2 + b_1t + b_0)f'(t) + c_1(t+3)f(t) = c_2t. \tag{7}$$

where b_0, b_1, b_2, c_1, c_2 are parameters (or numerical constants). Note that such type of equation appears originally in the paper by Broadhurst-Fleischer-Tarasov [39] in the context of analytical evaluation of two-loop sunset diagrams with equal masses.

Of course, one natural question that one could ask in this context, is whether the solutions of Equation (7) can be expressed in closed form, e.g., in terms of known special functions or as hypergeometric series. However, here we want to focus on the integral (6) and ask the question: does this integral satisfy a similarly nice relation as the original function $f(t)$, that is to say: a linear differential equation, and if so, how can we find it?

From the theory of holonomy it follows immediately that this is the case: the property of $f(t)$ being holonomic transfers to the whole integral, because the kernel is just a simple combination of power functions (and hence holonomic), and because holonomicity is preserved under definite integration.

In order to perform actual calculation, we shall first devise a holonomic description for the function f, by artificially viewing it as a bivariate function $f(t,a)$. In other words, we want to give generators of a holonomic ideal in the operator algebra $\mathbb{O} = \mathbb{K}(t,a)\langle D_t, D_a\rangle$ where $\mathbb{K} = \mathbb{Q}(\alpha, b_0, b_1, b_2, c_1, c_2, j)$. The first generator is readily obtained Equation (7), which one has to homogenize in order to get an annihilating operator. In terms of operators, this corresponds to left-multiplying the operator given by the left-hand side of (7) by an annihilating operator of its right-hand side:

$$\begin{aligned}
&(tD_t - 1) \cdot \big((t+1)(t+9)D_t^2 + (b_2t^2 + b_1t + b_0)D_t + c_1(t+3)\big)\\
&= (t^3 + 10t^2 + 9t)D_t^3 + (b_0t + b_1t^2 + b_2t^3 + t^2 - 9)D_t^2\\
&\quad + (-b_0 + b_2t^2 + c_1t^2 + 3c_1t)D_t - 3c_1.
\end{aligned}$$

The second generator is just D_a since $f(t,a)$ actually does not depend on a. From the noncommutative version of Buchberger's product criterion it follows that these two operators form a Gröbner basis, and by the definition of f, it is clear that they generate $\mathrm{ann}_{\mathbb{O}}(f)$, the annihilator of f with respect to $\mathbb{O}$.

Simple transformations convert the annihilating operators for f into operators that annihilate the whole integrand of (6), let us denote this integrand by $F(t,a)$. Algorithmically we can do it by exploiting the closure property that the product of two holonomic functions is again holonomic, but in such simple instances, one could even do it by hand. In any case, the result is as follows:

$$\begin{aligned}
&(a+t)D_a + j,\\
&t(1+t)(9+t)D_t^3 + \big(-9 + b_0t + t^2 + b_1t^2 + b_2t^3 - 3t(9+10t+t^2)R\big)D_t^2\\
&+ \big(b_2t^2 + c_1t^2 + 3c_1t - b_0 - 2(-9 + b_0t + (1+b_1)t^2 + b_2t^3)R\\
&\quad + 3t(9+10t+t^2)R^2 - 3t(9+10t+t^2)R'\big)D_t\\
&- 3c_1 - (-9 + b_0t + (1+b_1)t^2 + b_2t^3)R' - (9t + 10t^2 + t^3)R''\\
&+ (b_0 - t(b_2t + c_1(3+t))) + 3t(9+10t+t^2)R'R\\
&+ (-9 + b_0t + (1+b_1)t^2 + b_2t^3)R^2 - t(9+10t+t^2)R^3
\end{aligned}$$

where $R = R(t,a) = \frac{(a+t)(\alpha-1)-jt}{t(a+t)}$ and where R' refers to the differentiation with respect to t. In expanded form, this annihilator covers about a page.

Equipped with this holonomic description of the integrand, we can now employ the creative telescoping algorithm as implemented in the HolonomicFunctions package [10], in order to obtain two operators, namely a telescoper $P = P(a, D_a)$ and a certificate $Q = Q(t, a, D_t, D_a)$ with the property that $P + D_tQ$ is an element in $\mathrm{ann}_{\mathbb{O}}(F)$. To keep the exposition concise, we first look at the special case $\alpha = 1$. Then these two operators are given as follows:

$$P = -(a-9)(a-1)D_a^3 + (a^2b_2 - ab_1 - 2aj + b_0 + 10j)D_a^2 \\ + (2ab_2j + ac_1 - b_1j - 3c_1 - j^2 + j)D_a + j(b_2j - b_2 + c_1),$$

and

$$Q = \frac{(t+1)(t+9)(a+jt)}{(j-1)t}D_t^2 + \Big(a^2(b_0 + t(b_1 + b_2t)) + a(j(t(b_0 + t(b_1 + b_2t + 2) \\ + 20) + 18) + t(b_0 + t(b_1 + b_2t))) + jt(t(b_0 + t(b_1 + b_2t - 1) - 10) \\ + 3j(t+1)(t+9) - 9)\Big)\Big/\Big((j-1)t(a+t)\Big)D_t \\ + \Big(a^3c_1(t+3) + a^2(b_0j + jt(b_1 + b_2t) + c_1(j+2)t(t+3)) \\ + a\big(j^2(t(2b_0 + t(2b_1 + 2b_2t - 1)) + 9) + j(t+3)\big(2c_1t^2 + t - 3\big) + c_1t^2(t+3)\big) \\ + jt\big(t(t(-b_1 - b_2t + c_1(t+3) + 1) - b_0) + 2j(t(b_0 + t(b_1 + b_2t - 2) - 15) - 9) \\ + 3j^2(t+1)(t+9) - 9)\Big)\Big/\Big((j-1)t(a+t)^2\Big).$$

By denoting the result of applying the operator Q to the integrand F by $g(t,a)$, we express the above property as the equation

$$P\big(F(t,a)\big) = -\frac{\mathrm{d}}{\mathrm{d}t}g(t,a).$$

Integrating both side of this equation (almost) yields the desired relation for the integral:

$$P\big(I(a)\big) = g(0,a) - \lim_{t\to\infty} g(t,a) = g(0,a)$$

(the latter simplification under appropriate convergence assumptions on the given integral). Since the right-hand side of this (potentially) inhomogeneous differential equation is not given explicitly, but in terms of the unspecified function $f(t)$, it may be desirable to convert it to a holonomic description, i.e., into a homogeneous linear differential equation.

For this purpose, one shall derive a linear differential equation for $g(0,a)$ which, thanks to holonomic closure properties, is possible even without knowing its explicit closed form. The procedure consists of two steps: (1) derive an annihilator for $Q(F)$, which is possible by the closure under operator application (the command `DFiniteOreAction` yields an output of several pages), and (2) by applying the closure property "algebraic substitution" (the corresponding command is called `DFiniteSubstitute`). As a result, one receives the following operator that annihilates $g(0,a)$:

$$a^3D_a^3 + (3a^2j + 5a^2)D_a^2 + (3aj^2 + 7aj + 2a)D_a + (j^3 + 2j^2 - j - 2).$$

Multiplying this operator from the left to the telescoper P yields an order-6 annihilating operator for the integral $I(a)$ (not printed here for space reasons).

When we want to deal with the case of general α, then the approach is completely analogous, with the difference that all expressions get more unhandy, and that the telescoper for general α is an operator of order 6.

In this way, starting from the linear differential equation (7) for the 2-loop sunset, we have obtained the differential equation for the *3-loop* bubble diagram with two masses, studied recently in [40, 41].

5 Sunrise in Terms of Bessel-K Functions

In this section, we are studying a family of integrals that correspond to sunrise Feynman diagrams. Within dimensional regularization [42] in the momentum space it is defined as

$$J^{(L)}\left(\overrightarrow{M_j^2};\overrightarrow{\alpha_j};p^2\right) = \int \prod_{j=1}^{L} \frac{\mathrm{d}^n k_j}{\left(k_j^2+M_j^2\right)^{\alpha_j}} \cdot \frac{1}{\left((p-k_1-\ldots-k_L)^2+M_{L+1}^2\right)^{\alpha_{L+1}}},$$

where α_j are positive integers and M_j^2 and p^2 are some (in general, complex) parameters and n is an (in general, non-integer) parameter of dimensional regularization. Using the coordinate representation for the Feynman propagator and performing an integration over the angle,

$$\int \exp(ipx)\,\mathrm{d}^n\hat{x} = 2\pi^{\frac{n}{2}}\left(\frac{2}{qx}\right)^{\frac{n}{2}-1} J_{\frac{n}{2}-1}(qx),$$

where $q^2=-p^2$ and where $J_\nu(z)$ denotes the Bessel function of the first kind, it is easy to get a one-fold integral representation for this type diagram [43]:

$$\begin{aligned}
J^{(L)}(\overrightarrow{M_j^2};\overrightarrow{\alpha_j};p^2) &= \int \mathrm{d}^n k\,\exp(iqx)\prod_{i=1}^{L+1}\int \frac{\mathrm{d}^n k_i\,\exp(ik_ix)}{\left(k^2+M_i^2\right)^{\alpha}}\\
&= \int k^{n-1}\,\mathrm{d}k\int \mathrm{d}\hat{k}\,\exp(iqx)\prod_{i=1}^{L+1}\int \frac{\mathrm{d}^n k_i\,\exp(ik_ix)}{\left(k^2+M_i^2\right)^{\alpha}}\\
&= 2\pi^{\frac{n}{2}}\int x^{n-1}\left(\frac{2}{qx}\right)^{\frac{n}{2}-1} J_{\frac{n}{2}-1}(qx)\prod_{i=1}^{L+1}\frac{2\pi^{\frac{n}{2}}}{\Gamma(\alpha_i)}\left(\frac{2M_i}{x}\right)^{\frac{n}{2}-\alpha_i} K_{\frac{n}{2}-\alpha}(M_ix)\,\mathrm{d}x\\
&= \frac{\left(\pi^{\frac{n}{2}}\right)^{L+2}}{2^{\alpha-\frac{n}{2}(L+2)-L-1}}\left(\frac{1}{q}\right)^{\frac{n}{2}-1}\int_0^\infty t^{\alpha-\frac{n}{2}L} J_{\frac{n}{2}-1}(tq)\prod_{j=1}^{L+1}\left(K_{\frac{n}{2}-\alpha_j}(M_jt)\frac{M_j^{\frac{n}{2}-\alpha_j}}{\Gamma(\alpha_j)}\right)\mathrm{d}t \qquad (8)
\end{aligned}$$

where $\alpha=\sum_{k=1}^{L+1}\alpha_k$, and n is the dimension of space-time, where $q^2=-p^2$, and $K_\nu(z)$ denotes the modified Bessel function of the second kind. This integral has been studied in [44, 45, 46, 47].

In the rest of this section, we will focus on the integral representation (8). Note that the integrand contains a product of $L+2$ Bessel functions (both J and K counted together). Since a Bessel function has holonomic rank 2, it follows by Theorem 1(ii) that the expression in the integral has holonomic rank at most 2^{L+2}. Unfortunately, it turns out that the bound in this instance is tight, i.e., the holonomic rank of the integrand is exactly 2^{L+2}. Since all creative telescoping algorithms are very sensitive concerning the holonomic rank of the input annihilator, this class of integrals is going to pose challenges for our package.

For computing telescopers of holonomic integrals, we have several algorithms at hand: we mention just Chyzak's algorithm [16] and a heuristic ansatz proposed by the author [17]. The advantage of the former algorithm is that it is theoretically sound and is guaranteed to terminate and to return the minimal-order telescoper, while the latter uses several heuristics to shape the ansatz, which may result in a non-minimal telescoper and in some instances this "algorithm" even fails to terminate. The disadvantage of Chyzak's algorithm is that it is very sensitive to the holonomic rank of the input due to the uncoupling step. The other approach [17] was designed specifically to address this issue and to circumvent the costly uncoupling step. Since our examples have relatively large holonomic rank, we will use the heuristic approach, and hence the reported telescopers need not necessarily be minimal. On the other side, the usage of this algorithm enables us to get some results at all: for example, in the most simple example (first line of Table 1), we obtain a result after about 2 seconds, while the algorithm [16] was aborted after 1000 seconds without yielding any result.

As a toy example, we start with the case $L = 1$. Hence, in this case the integral depends on the six parameters $q, n, \alpha_1, \alpha_2, M_1, M_2$. Nevertheless, the HolonomicFunctions program is able to compute a telescoper within a few seconds. This telescoper is a third-order operator in D_q, but is still too long to be printed here. Instead, we display the specialized version with $M_1 = M_2 = 1$:

$$
\begin{aligned}
&-8q^3(q^2+4)D_q^3 + 4q^2\big(5nq^2 - 16\alpha_1 - 16\alpha_2 + 12n - 8\alpha_1 q^2 - 8\alpha_2 q^2 - 12q^2 - 24\big)D_q^2\\
&-2q\big(16\alpha_1^2 + 16\alpha_2^2 + 48\alpha_1 + 32\alpha_1\alpha_2 + 48\alpha_2 + 7n^2q^2 + 12n^2 - 32\alpha_1 n - 32\alpha_2 n - 24n\\
&\ + q^2(-24\alpha_1 n - 24\alpha_2 n - 30n + 16\alpha_1^2 + 16\alpha_2^2 + 48\alpha_1 + 48\alpha_2 + 48\alpha_1\alpha_2 + 28)\big)D_q\\
&\ - 32\alpha_1^2 - 32\alpha_2^2 + 32\alpha_1 - 64\alpha_1\alpha_2 + 32\alpha_2 + 3n^3q^2 + 4n^3 - 16\alpha_1 n^2 - 16\alpha_2 n^2\\
&\ - 16\alpha_1 n^2q^2 - 16\alpha_2 n^2 q^2 - 14n^2q^2 + 16\alpha_1^2 n + 16\alpha_2^2 n + 16\alpha_1 n + 32\alpha_1\alpha_2 n + 16\alpha_2 n\\
&\ + 16\alpha_1^2 nq^2 + 16\alpha_2^2 nq^2 + 48\alpha_1 nq^2 + 48\alpha_2 nq^2 + 80\alpha_1\alpha_2 nq^2 + 20nq^2 - 16n - 8q^2\\
&\ - 32\alpha_1^2 q^2 - 32\alpha_2^2 q^2 - 64\alpha_1\alpha_2^2 q^2 - 32\alpha_1 q^2 - 32\alpha_2 q^2 - 64\alpha_1^2\alpha_2 q^2 - 96\alpha_1\alpha_2 q^2.
\end{aligned}
$$

We have also looked at some "less trivial" cases of the integral (8). As the above output suggests, it will not be reasonable to print the resulting operators, but instead we tabularize our findings (see Table 1, together with some information on timings and sizes of the outputs. These data allow us to acquire an intuition on how the difficulty of the computation depends on the holonomic rank of the input, the number of parameters, etc.

Case	Rank	Order	Time	Size
$L = 1, M_j = 1$	8	3	2 s	15 KB
$L = 1$	8	3	6 s	149 KB
$L = 2, M_j = 1, \alpha_j = j$	16	4	34 s	36 KB
$L = 2, M_j = 1$	16	7	797 s	6.8 MB
$L = 2, M_1 = q, M_2 = M_3 = 1$	16	7	8003 s	4.6 MB
$L = 2, M_1 = M_2 = M_3$	16	7	ERR	–
$L = 2, \alpha_1 = 5, \alpha_2 = 11, \alpha_3 = 14$	16	7	> 36 h	–
$L = 3, M_j = 1, \alpha_j = j$	32	11	83006 s	1.4 MB
$L = 3, M_1 = q, M_2 = M_3 = M_4 = 1, \alpha_j = j$	32	11	763 s	1.0 MB

Table 1 Some benchmark computations: the first column gives the specification of an instance of integral (8) (parameters that are not mentioned are kept symbolic), "Rank" refers to the holonomic rank of the integrand, "Order" to the order (degree w.r.t. D_q) of the telescoper, "Time" to the computation time (in seconds), and "Size" to the byte size of the telescoper (not the certificate), using Mathematica's `ByteCount`. One computation didn't finish within 36 hours, one computation crashed (ERR); nevertheless, the order of the telescoper could be extracted from the log files.

Table 1 gives an impression that the computation of integrals related to sunrise Feynman diagrams is challenging but not completely hopeless for the holonomic systems approach. We plan to explore further the applicability of this approach to Feynman integrals in a forthcoming publication.

Concluding, we have shortly discussed a set of problems related to the evaluation of Feynman diagrams, where the holonomic systems approach, implemented in the package HolonomicFunctions, could be quite useful. We are looking forward to many other exciting collaborations between computer algebra and particle physics.

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