DETERMINANT EVALUATIONS INSPIRED BY DI FRANCESCO'S DETERMINANT FOR TWENTY-VERTEX CONFIGURATIONS

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Dedicated to the memory of Marko Petkovšek,
who had a keen interest in computer algebra and determinants

Abstract. In his work on the twenty vertex model, Di Francesco [*Electron. J. Combin.* **28**(4) (2021), Paper No. 4.38] found a determinant formula for the number of configurations in a specific such model, and he conjectured a closed form product formula for the evaluation of this determinant. We prove this conjecture here. Moreover, we actually generalize this determinant evaluation to a one-parameter family of determinant evaluations, and we present many more determinant evaluations of similar type — some proved, some left open as conjectures.

1. Introduction

In [4, 3], Di Francesco and Guitter undertook an enumerative study of configurations in the twenty vertex model and set it in relation to an analogous enumerative study of domino tilings of certain regions. Particular such regions that Di Francesco considered in [3] were coined by him "Aztec triangles". He found that certain twenty vertex configurations were equinumerous with domino tilings of such "Aztec triangles". Moreover, he established a determinantal formula for these common numbers, and he observed that this determinant had apparently an evaluation given by a closed form product.[1]

Conjecture 1 (Di Francesco [3, Conj. 8.1 + Th. 8.2])**.** *For all positive integers n, we have*

$$\det_{0\le i,j\le n-1}\left(2^i\binom{i+2j+1}{2j+1}-\binom{i-1}{2j+1}\right)=2\prod_{i=1}^{n}\frac{2^{i-1}\,(4i-2)!}{(n+2i-1)!},\tag{1.1}$$

where the binomial coefficient is defined by

$$\binom{\alpha}{p}=\begin{cases}\frac{\alpha(\alpha-1)\cdots(\alpha-p+1)}{p!}, & \text{if } p\ge 0,\\ 0, & \text{if } p<0.\end{cases}\tag{1.2}$$

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[1] There is a small subtlety that needs to be pointed out here: our definition of binomial coefficients is not the same as Di Francesco's. To be precise, his convention is to put $\binom{\alpha}{p}=0$ for $-1\le\alpha<p$. Thus, according to this convention, all the entries in row 0 of the matrix of which the determinant is taken in (1.1) would equal 1, while, with our convention (1.2), they are all equal to 2. Consequently, our right-hand side in (1.1) has an additional factor 2 compared to [3, Eq. (8.1)].

This caught the attention of the second author. Since he prefers parameters in determinants in order to facilitate their evaluation, he searched for a parametric generalization of (1.1). He successfully found such a generalization, and in addition a companion identity.

Conjecture 2 (CK_2)**.** *For all positive integers n, we have*

$$\det_{0\le i,j\le n-1}\left(2^i\binom{x+i+2j+1}{2j+1}+\binom{x-i+2j+1}{2j+1}\right) \\ = 2^{\binom n2+1}\prod_{i=0}^{n-1}\frac{i!}{(2i+1)!}\prod_{i=0}^{\lfloor n/2\rfloor}(x+4i+1)_{n-2i}\prod_{i=0}^{\lfloor (n-1)/2\rfloor}(x-2i+3n)_{n-2i-1}, \quad (1.3)$$

and

$$\det_{0\le i,j\le n-1}\left(2^i\binom{x+i+2j}{2j}+\binom{x-i+2j}{2j}\right) \\ = 2^{\binom n2+1}\prod_{i=0}^{n-1}\frac{i!}{(2i)!}\prod_{i=0}^{\lfloor (n-1)/2\rfloor}(x+4i+3)_{n-2i-1}\prod_{i=0}^{\lfloor (n-2)/2\rfloor}(x-2i+3n-1)_{n-2i-2}, \quad (1.4)$$

with the Pochhammer symbol $(\alpha)_p$ defined by $\alpha(\alpha+1)\cdots(\alpha+p-1)$ for $p\ge 1$ and $(\alpha)_0:=1$.

Indeed, the special case $x=0$ of (1.3) is equivalent with (1.1). However, this did not help: neither was the second author able to prove Conjecture 1 nor was he able to prove Conjecture 2 at the time.

In an unrelated development, during the 9th International Conference on "Lattice Path Combinatorics and Applications" that took place June 21–25, 2021 at the CIRM in Luminy, the first author[2] received the following e-mail from Doron Zeilberger. More precisely, the e-mail arrived on June 23 at 13:49 (European time) and had the subject "challenge":

```
Dear Christoph,

Philippe Di Francesco just gave a great talk at the Lattice
path conference mentioning, inter alia, a certain
conjectured determinant.  It is
Conj. 8.1 (combined with Th. 8.2) in
https://arxiv.org/pdf/2102.02920.pdf

I am curious if you can prove it by the
Koutschan-Zeilberger-Aek holonomic ansatz method.  If you
can do it before Friday, June 25, 2021, 17:00 Paris time, I
will mention it in my talk in that conference.

Best wishes
         Doron
```

[2]who did not participate in the conference; the second and third author did (via `zoom`).

It turned out that the conjectured determinant evaluation in question — namely (1.1) — was indeed routinely provable by the so-called *holonomic Ansatz*. Consequently, the challenge was met and the result was announced by Zeilberger in his talk on the last day of the conference.

Obviously, the second author immediately contacted the first. However, the determinant evaluations in Conjecture 2 could not be proved by the holonomic Ansatz.

On the other hand, the third author initiated a "hunt" for further determinant evaluations of similar kind. As a result, together we came up with many variations of the determinant identities in Conjectures 1 and 2, some of which could be proved by the holonomic Ansatz while others resisted to this method. On the other hand, at least in one case a different — non-algorithmic — method led to success.

Again in an unrelated development, at the Workshop on "Enumerative Combinatorics" in Oberwolfach in December 2022, Sylvie Corteel asked the second author how to enumerate certain tableaux that were a mixture of symplectic and supersymmetric tableaux; she and her student Frederick Huang had observed that their number seemed to be given by a nice product formula. The second author asked for a bit of time, and answered on the next day: "Calculer un déterminant !" That was actually not new for Corteel and Huang, they already knew that ... In any case, a month later this determinant (and a related one) was indeed evaluated; see [2].[3] The second author noticed that, up to a simple parameter transformation, the result of that determinant evaluation seemed to be the same as the right-hand side of (1.3) (and the result of the related determinant evaluation seemed to be the same as the right-hand side of (1.4)). It did not take for long to rigorously relate this determinant to the one on the left-hand side of (1.3) (and the related determinant to the one on the left-hand side of (1.4)). Thus, also Conjecture 2 became a theorem.

The purpose of this paper is to collect all these results and conjectures, together with our proofs (in case we found one). More precisely, in the next section we review Zeilberger's holonomic Ansatz. Then follows a "warmup" section, in which we prove a variation of (1.3) in which the terms $2j$ in the binomial coefficients get "replaced" by j and the power 2^i is replaced by an arbitrary power a^i. We actually provide two proofs: one using the holonomic Ansatz, the other using constant term calculus. Here, in this simple case, we are able to display the results of the intermediate calculations when using the holonomic Ansatz, while, due to their size, this is not possible anymore for the subsequent applications of the holonomic Ansatz in this paper.[4] In this sense, this proof also serves pedagogical purposes.

Section 4 is devoted to the earlier mentioned computer proof of Conjecture 1 due to the first author that was announced by Zeilberger during the 9th Lattice Path Conference. The proof of Conjecture 2 is the subject of Section 5. As indicated above, the idea of the proof is to relate the two determinants in (1.3) and (1.4) to two determinants that had been evaluated in [2].

The subsequent sections discuss variations of these determinant evaluations. We begin in Section 6 with determinants of the kind as in Conjecture 2 where we allow more general shifts at several places; see the definition of $D_{\alpha,\beta,\gamma,\delta}(n)$ at the beginning

[3]As it turned out, the tableau enumeration problem was not so unrelated: the actual goal of Corteel and Huang was to count the number of domino tilings of regions that generalized Di Francesco's Aztec triangles; the "super-symplectic" tableaux were in bijection with these domino tilings.

[4]Instead, we provide details of these calculations in the accompanying electronic material [12].

of the section. A computer search led to the discovery of many (more) corresponding determinant evaluations; see Theorem 10.

In Section 7, we consider variations of the determinants in Conjecture 2 in which $2j$ gets "replaced" by $3j$ and the power 2^i is replaced by 3^i, again allowing more general shifts. Also in this case, we found many corresponding determinant evaluations; see Theorem 12. Moreover, it seems that there are three one-parameter families of closed-form determinant evaluations of this type; see Conjecture 13.

Section 8 is dedicated to variations of the determinants in Conjecture 2 in which the power 2^i gets "replaced" by 4^i, with the $2j$ in the binomial coefficients being retained, again allowing shifts. The "sporadic" determinant evaluations that we found are listed (and proved) in Theorem 14. There is also a one-parameter family of such evaluations; see Theorem 15. Moreover, we discovered a second one-parameter family. However, in that case the result of the determinant evaluation does not factor completely. Our result in Theorem 16 identifies all factors but one, apparently, irreducible factor. While we failed to find an explicit formula for that factor, we found a recurrence that it seems to satisfy; see Conjecture 17. Since our (non-algorithmic) proof of Theorem 16 is somewhat lengthy, it is given separately in Section 9.

Section 10 contains yet further variations of the determinants in Conjecture 2: here, the power 2^i remains untouched, but $2j$ gets "replaced" by $4j$, and we allow more general shifts. We present our corresponding findings in Conjecture 21, Proposition 22, and Conjecture 23. Again, we are confident that the holonomic Ansatz is able to prove all these results. However, at this point in time the capacity of the available computers is not sufficient to actually carry out the necessary computations.

In the final section, Section 11, we list several problems left open or posed by this work.

2. **The Holonomic Ansatz**

The *holonomic Ansatz* [24] is a computer-algebra-based approach to find and/or prove the evaluation of a symbolic determinant $\det(A_n)$, where the dimension of the square matrix $A_n := (a_{i,j})_{0\le i,j<n}$ is given by a symbolic parameter n. The method is only applicable to non-singular matrices whose entries $a_{i,j}$ are holonomic sequences (see below) in the index variables i and j. Moreover, the entries $a_{i,j}$ must not depend on n, i.e., A_{n-1} is an upper-left submatrix of A_n.

The holonomic Ansatz works as follows: define the quantity

$$c_{n,j} := (-1)^{n-1+j}\,\frac{M_{n-1,j}}{M_{n-1,n-1}} \tag{2.1}$$

where $M_{i,j}$ denotes the (i,j)-minor of the matrix A_n (where the indexing starts at 0). In other words, $c_{n,j}$ is the $(n-1,j)$ cofactor of A_n divided by $\det(A_{n-1})$. Using Laplace expansion with respect to the last row, one can write

$$\sum_{j=0}^{n-1} a_{n-1,j}c_{n,j} = \frac{\det(A_n)}{\det(A_{n-1})}. \tag{H3}$$

Under the assumptions that (i) the bivariate sequence $c_{n,j}$ is holonomic and that (ii) its holonomic definition is known, the symbolic sum on the left-hand side of (H3) can be tackled with creative telescoping [23, 20], yielding a linear recurrence in n for the sum. If a conjectured evaluation b_n for the determinant of A_n is available, then one can prove it by verifying that b_n/b_{n-1} satisfies the obtained recurrence and by comparing a sufficient number of initial values. If in contrast such a conjecture has not been formulated, then one may succeed to find (and at the same time: prove) an evaluation of $\det(A_n)$ by solving the recurrence, thus obtaining an expression for $\det(A_n)/\det(A_{n-1})$, and by taking the product.

What can be said about the two assumptions? There is no general theorem that implies that $c_{n,j}$ is always holonomic, and in fact, there are many examples where it is not. If (i) is not satisfied, i.e., if $c_{n,j}$ is not holonomic, then the method fails (not necessarily; in some situations one may succeed to overcome the problem by applying a mild reformulation; see [14]). Concerning (ii): by a holonomic definition we mean a set of linear recurrence equations whose coefficients are polynomials in the sequence indices n and j, together with finitely many initial values, such that the entire bivariate sequence $(c_{n,j})_{1\leq n,\,0\leq j<n}$ can be produced by unrolling the recurrences and by using the initial values. The question now is how the original definition (2.1) can be converted into a holonomic definition.

Clearly, (2.1) allows one to compute the values of $c_{n,j}$ for concrete integers n and j in a certain, finite range. From these data, candidate recurrences can be constructed by the method of guessing (i.e., employing an Ansatz with undetermined coefficients; cf. [10]). It remains to prove that these recurrences, constructed from finite, and therefore incomplete data, are correct, i.e., are valid for all $n \geq 1$ and $0 \leq j < n$. For this purpose, we show that $c_{n,j}$ is the unique solution of a certain system of linear equations, and then we prove that the sequence defined by the guessed recurrences (and appropriate initial conditions) also satisfies the same system. By uniqueness, it follows that the two sequences agree, i.e., that the guessed recurrences define the desired sequence $c_{n,j}$.

Suppose that the last row of A_n is replaced by its i-th row; the resulting matrix is clearly singular, turning (H3) into

$$\sum_{j=0}^{n-1} a_{i,j}c_{n,j} = 0 \qquad (0 \leq i < n-1). \tag{H2}$$

For each $n \in \mathbb{N}$ the above equation (H2) represents a system of $n-1$ linear equations in the n "unknowns" $c_{n,0},\ldots,c_{n,n-1}$, whose coefficient matrix $(a_{i,j})_{0\leq i<n-1,0\leq j<n}$ has full rank because $\det(A_{n-1}) \neq 0$ (if the latter is not known a priori, it can be argued by induction on n). Hence the homogeneous system (H2) has a one-dimensional kernel. The solution is made unique by normalizing with respect to its last component, that is, by imposing a condition that is obvious from (2.1), namely

$$c_{n,n-1} = 1. \tag{H1}$$

Hence, (H1) and (H2) together define $c_{n,j}$ uniquely. On the other hand, given a holonomic definition of $c_{n,j}$, creative telescoping and holonomic closure properties can be applied to prove (H1) and (H2), respectively. If these proofs succeed, then it follows that the guessed recurrences are correct.

The holonomic Ansatz has already been applied in many different contexts [13, 16, 5]. Variations of it have been described in [9, 15, 14].

We conclude this introduction to the holonomic Ansatz with some remarks concerning its concrete implementation. For computing the data, i.e., the values of $c_{n,j}$, it is usually more efficient to employ their definition via (H1) and (H2), rather than computing determinants in the spirit of (2.1). We used the *Mathematica* packages `Guess.m` [10] for the guessing of the recurrences, and `HolonomicFunctions.m` [11] for the creative-telescoping proofs.

3. **A warmup exercise**

Before we dedicate ourselves to the proofs of the conjectured determinant evaluations of the introduction, we begin with a variation of the determinants appearing in Conjecture 2. The variation consists in "replacing" $2j$ in the binomial coefficients by j and the power 2^i by a^i where a is an indeterminate. It turns out that a proof of the evaluation of this latter determinant is much simpler. We provide actually two proofs: one using the holonomic Ansatz, and the other using constant term calculus. (If one wishes: a computer proof and a computer-free proof.) We will use this determinant evaluation later in the proof of Theorem 16 in Section 9.

Theorem 3. *For all non-negative integers n, we have*

$$\det_{0\le i,j\le n-1}\left(a^i\binom{x+i+j-1}{j}+\binom{x-i+j-1}{j}\right)=2(a-1)^{\binom{n}{2}}.$$

First proof. We compute the `data` for $c_{n,j}$, as defined in (2.1), for $1\le n\le 11$:

$$c_{1,0}=1, \tag{3.1}$$
$$c_{2,0}=-x, \qquad c_{2,1}=1, \tag{3.2}$$
$$c_{3,0}=\frac{ax^2+ax-x^2+x}{2(a-1)}, \qquad c_{3,1}=\frac{-ax-a+x}{a-1}, \qquad c_{3,2}=1, \tag{3.3}$$
$$\vdots$$
$$c_{11,0}=\frac{362880x+\cdots+a^9x^{10}}{3628800(a-1)^9},\ \ldots,\ c_{11,9}=\frac{-ax-9a+x}{a-1},\quad c_{11,10}=1.$$

Then we use the `Guess.m` package [10] to find plausible candidates for bivariate recurrences that $c_{n,j}$ may satisfy:

```
g = GuessMultRE[data, {c[n,j], c[n,j+1], c[n+1,j], c[n+1,j+1]},
    {n, j}, 2, StartPoint -> {1, 0}, Constraints -> (j < n)];
```

In order to have a canonical set of generators for the infinite set of such recurrences, which is a left ideal in the corresponding operator algebra, also called the *annihilator* of the sequence $c_{n,j}$, we compute a (left) Gröbner basis `annc` of the previous output:

```
OreGroebnerBasis[NormalizeCoefficients /@ ToOrePolynomial[g, c[n,j]]];
```

As a result, we obtain the following two recurrences, which, in contrast to the recurrences in later sections, are small enough to be displayed here, albeit too unhandy to process them by pencil and paper:

$$
\begin{aligned}
&(1-a)n(j-n)\big(aj^2+2ajx+aj+ax^2+ax-x^2-x\big)c_{n+1,j}\\
&-(j-n+x+2)\big(aj^3-2aj^2n+2aj^2x+aj^2-4ajnx\\
&\qquad\qquad\qquad+ajx^2+ajx-2anx^2-jx^2+jx+2nx^2\big)c_{n,j+1}\\
&+\big(a^2j^2n^2+a^2j^2nx-a^2j^2n+2a^2jn^2x+a^2jn^2+2a^2jnx^2-a^2jnx-a^2jn\\
&\quad+a^2n^2x^2+a^2n^2x+a^2nx^3-a^2nx+aj^4-4aj^3n+2aj^3x+4aj^3+3aj^2n^2\\
&\quad-9aj^2nx-9aj^2n+aj^2x^2+5aj^2x+5aj^2+6ajn^2x+3ajn^2-6ajnx^2\\
&\quad-11ajnx-5ajn+ajx^2+3ajx+2aj+2an^2x^2-2anx^3-4anx^2-j^2x^2\\
&\quad+j^2x+4jnx^2-jx^2+jx-3n^2x^2-n^2x+nx^3+4nx^2+nx\big)c_{n,j}=0,
\end{aligned}
$$

$$
\begin{aligned}
&\big(aj^2+2ajx+aj+ax^2+ax-x^2-x\big)(j-n+x+3)c_{n,j+2}\\
&+\big(a^2j^3+3a^2j^2x+3a^2j^2+3a^2jx^2+6a^2jx+2a^2j+a^2x^3+3a^2x^2+2a^2x-2aj^3\\
&\quad+2aj^2n-5aj^2x-8aj^2+4ajnx+4ajn-5ajx^2-14ajx-8aj+2anx^2+2anx\\
&\quad-2ax^3-8ax^2-6ax+2jx^2+2jx-2nx^2-2nx+x^3+5x^2+4x\big)c_{n,j+1}\\
&-(a-1)(j-n+1)\big(aj^2+2ajx+3aj+ax^2+3ax+2a-x^2-x\big)c_{n,j}=0.
\end{aligned}
$$

Next, we have to prove the identities (H1) and (H2), in order to justify that $c_{n,j}$, as defined in (2.1), agrees with the unique solution of the above recurrences, or in other words, that these guessed recurrences are correct. The command

```
DFiniteSubstitute[annc, {j -> n-1}]
```

delivers the following, second-order recurrence for $c_{n,n-1}$:

$$
\begin{aligned}
&(n+1)\big(an^2+2anx-an+ax^2-ax-x^2+x\big)c_{n+2,n+1}\\
&+\big(a^2n^3+3a^2n^2x+3a^2nx^2-a^2n+a^2x^3-a^2x-2an^3-5an^2x-5anx^2\\
&\qquad+2an-2ax^3+ax^2+ax+2nx^2-2nx+x^3-x^2\big)c_{n+1,n}\\
&-(a-1)(n+x-1)\big(an^2+2anx+an+ax^2+ax-x^2+x\big)c_{n,n-1}=0.
\end{aligned}
$$

It is easy to check that the constant solution $c_{n,n-1}=1$ is a solution to the above recurrence, which, together with the initial conditions from (3.1) and (3.2), implies (H1).

In order to prove (H2), we view $c_{n,j}$ as a trivariate sequence in n,i,j, and compute the annihilator of $\binom{x-i+j-1}{j}\cdot c_{n,j}$ via closure properties:

```
s1 = DFiniteTimes[Annihilator[Binomial[x-i+j-1,j], {S[n], S[j], S[i]}],
 OreGroebnerBasis[Append[annc, S[i]-1], OreAlgebra[S[n], S[j], S[i]]]];
```

Since we have a recursive definition of the summand, we can employ creative telescoping to find a set of recurrences that is satisfied by the sum $s^{(1)}_{n,i}=\sum_{j=0}^{n-1}\binom{x-i+j-1}{j}\cdot c_{n,j}$

```
ct1 = FindCreativeTelescoping[s1, S[j]-1];
```

and similarly for the other sum $s^{(2)}_{n,i}=\sum_{j=0}^{n-1}a^i\binom{x+i+j-1}{j}\cdot c_{n,j}$. Combining the two results via the command

```
DFinitePlus[ct1[[1]], ct2[[1]]];
```

yields recurrences for the sum $s_{n,i} = s^{(1)}_{n,i} + s^{(2)}_{n,i}$ on the left-hand side of (H2):

$$(a-1)(i+1)ns_{n+1,i} - 2i(i-n+2)s_{n,i+1} + (a+1)(i+1)(i+n-1)s_{n,i} = 0,$$

$$\begin{aligned}
&2i(i+1)(i-n+4)\big(ai^2+ai-2a-x^2-x\big)s_{n,i+3}\\
&-i\big(2a^2i^4+10a^2i^3+10a^2i^2-10a^2i-12a^2+ai^5-ai^4n-ai^4x+8ai^4+ai^3nx-3ai^3n\\
&\quad -5ai^3x+18ai^3+2ai^2nx+ai^2n-2ai^2x^2-5ai^2x+4ai^2-3ainx+3ain-8aix^2\\
&\quad +aix-19ai-6ax^2-6ax-12a-i^3x^2-i^3x+i^2nx^2+i^2nx+i^2x^3-6i^2x^2-7i^2x\\
&\quad -inx^3+inx^2+2inx+4ix^3-11ix^2-15ix-nx^3+nx+3x^3-6x^2-9x\big)s_{n,i+2}\\
&+i\big(a^2i^5-a^2i^4x+7a^2i^4+2a^2i^3n-4a^2i^3x+13a^2i^3+8a^2i^2n-a^2i^2x-3a^2i^2+2a^2in\\
&\quad +6a^2ix-18a^2i-12a^2n+ai^5-ai^4x+5ai^4-ai^3x^2-5ai^3x+5ai^3+ai^2x^3-5ai^2x^2\\
&\quad -7ai^2x-5ai^2-2ainx^2-2ainx+3aix^3-8aix^2-5aix-6ai-6anx^2+2ax^3-2x\\
&\quad -6anx-2ax-i^3x^2-i^3x+i^2x^3-3i^2x^2-4i^2x+3ix^3-2ix^2-5ix+2x^3\big)s_{n,i+1}\\
&-a(i+2)(i+n-1)(i-x+1)\big(ai^3+2ai^2-3ai-ix^2-ix-x^2-x\big)s_{n,i}=0.
\end{aligned}$$

The sequence $s_{n,i}$ is restricted to $0 \le i < n-1$, and thus the support of the above recurrences prohibits one to use them for computing $s_{2,0}, s_{3,0}, s_{3,1}, s_{4,1}, s_{4,2}, s_{5,2}$; these have to be given as initial values. Moreover, one cannot use the second recurrence for computing $s_{n,3}$ due to the factor i in its leading coefficient. This forces us to also include $s_{5,3}$ and $s_{6,3}$ into the initial conditions (note that $s_{n,3}$ for $n \ge 7$ can be computed using the first recurrence). It is not difficult to verify that all eight initial conditions are zero, and by virtue of the recurrences satisfied by $s_{n,i}$, it follows that $s_{n,i} = 0$ for all n, i with $0 \le i < n-1$.

Identity (H3) is proven in a similar way. The sum on its left-hand side is split into two sums. A recurrence for the first one is obtained by calling

```
ct1 = FindCreativeTelescoping[DFiniteTimes[
        Annihilator[Binomial[x-n+j,j], {S[n], S[j]}], annc], S[j]-1];
```

An analogous computation is done for the second sum. Combining the two results via the command

```
DFinitePlus[ct1[[1]], ct2[[1]]];
```

yields the following recurrence for the sum $s_n = \sum_{j=0}^{n-1} a_{n,j}c_{n,j}$:

$$(a-1)ns_{n+2} - (a^2n - 6an + 2a + n)s_{n+1} - 2(a-1)a(2n-1)s_n = 0.$$

It is readily checked that $(a-1)^{\binom{n}{2}-\binom{n-1}{2}} = (a-1)^{n-1}$ is a solution of this recurrence, and that the necessary initial values are correct (i.e., that the asserted determinant evaluation holds for $n \le 3$). This concludes the proof of (H3), and therefore the proof of the whole theorem. □

Second proof. We use that $\binom{N}{k} = \mathrm{CT}_z (1+z)^N z^{-k}$, where $\mathrm{CT}_z\, f(z)$ denotes the constant term in z in the Laurent series $f(z)$. Furthermore, for a Laurent aeries $f(z_0, z_1, \dots, z_{n-1})$ in $z_0, z_1, \dots, z_{n-1}$, we shall use the short notation

$$\mathrm{CT}_{\mathbf{z}}\, f(z_0, z_1, \dots, z_{n-1})$$

to denote the constant term in this Laurent series.

Using these notations, our determinant can be written as

$$\begin{aligned}
&\det_{0\le i,j\le n-1}\left(a^i\binom{x+i+j-1}{j}+\binom{x-i+j-1}{j}\right)\\
&\qquad=\mathrm{CT}_{\mathbf z}\det_{0\le i,j\le n-1}\left(a^i\frac{(1+z_j)^{x+i+j-1}}{z_j^j}+\frac{(1+z_j)^{x-i+j-1}}{z_j^j}\right)\\
&\qquad=\mathrm{CT}_{\mathbf z}\,a^{\frac12\binom n2}\left(\prod_{j=0}^{n-1}\frac{(1+z_j)^{x+j-1}}{z_j^j}\right)\det_{0\le i,j\le n-1}\left(a^{i/2}(1+z_j)^i+a^{-i/2}(1+z_j)^{-i}\right).
\end{aligned}$$

The determinant can be evaluated by means of [18, Eq. (2.5)]. Thus, we obtain

$$\begin{aligned}
&\det_{0\le i,j\le n-1}\left(a^i\binom{x+i+j-1}{j}+\binom{x-i+j-1}{j}\right)\\
&\qquad=\mathrm{CT}_{\mathbf z}\,2a^{-\frac12\binom n2}\left(\prod_{j=0}^{n-1}\frac{(1+z_j)^{x+j-n}}{z_j^j}\right)\\
&\qquad\qquad\times\left(\prod_{0\le i<j\le n-1}\left(\sqrt a(1+z_i)-\sqrt a(1+z_j)\right)\left(1-a(1+z_i)(1+z_j)\right)\right)\\
&\qquad=\mathrm{CT}_{\mathbf z}\,2\left(\prod_{j=0}^{n-1}\frac{(1+z_j)^{x+j-n}}{z_j^j}\right)\left(\prod_{0\le i<j\le n-1}(z_i-z_j)\left((1-a)-a(z_i+z_j+z_iz_j)\right)\right).
\end{aligned}$$

Since this is a constant term, we get the same value if we permute the variables $z_0,z_1,\dots,z_{n-1}$. So, let us symmetrize the last expression, meaning that we sum this expression over all possible permutations of the variables. Obviously, in order to get the same value again, we must divide the result by $n!$. This leads to

$$\begin{aligned}
&\det_{0\le i,j\le n-1}\left(a^i\binom{x+i+j-1}{j}+\binom{x-i+j-1}{j}\right)\\
&\quad=\frac{2}{n!}\,\mathrm{CT}_{\mathbf z}\left(\prod_{j=0}^{n-1}(1+z_j)^{x-n}\right)\left(\prod_{0\le i<j\le n-1}(z_i-z_j)\left((1-a)-a(z_i+z_j+z_iz_j)\right)\right)\\
&\qquad\qquad\times\det_{0\le i,j\le n-1}\left(\left(\frac{1+z_i}{z_i}\right)^j\right)
\end{aligned}$$

The determinant can be evaluated by means of the evaluation of the Vandermonde determinant, so that

$$\begin{aligned}
\det_{0\le i,j\le n-1}&\left(a^i\binom{x+i+j-1}{j}+\binom{x-i+j-1}{j}\right)\\
&=\frac{2}{n!}\operatorname{CT}_{\mathbf z}\left(\prod_{j=0}^{n-1}(1+z_j)^{x-n}\right)\left(\prod_{0\le i<j\le n-1}(z_i-z_j)\left((1-a)-a(z_i+z_j+z_iz_j)\right)\right)\\
&\qquad\times\left(\prod_{0\le i<j\le n-1}\left(\frac{1+z_j}{z_j}-\frac{1+z_i}{z_i}\right)\right)\\
&=\frac{2}{n!}\operatorname{CT}_{\mathbf z}\left(\prod_{j=0}^{n-1}\frac{(1+z_j)^{x-n}}{z_j^{n-1}}\right)\left(\prod_{0\le i<j\le n-1}(z_i-z_j)^2\left((1-a)-a(z_i+z_j+z_iz_j)\right)\right).
\end{aligned}$$

Now, the square of the Vandermonde product, $\prod_{0\le i<j\le n-1}(z_i-z_j)^2$, is a homogeneous polynomial of degree $n(n-1)$. Moreover, it is not very difficult to see that the coefficient of $(z_0z_1\cdots z_{n-1})^{n-1}$ in it equals $(-1)^{\binom n2}n!$. This implies that

$$\begin{aligned}
\det_{0\le i,j\le n-1}&\left(a^i\binom{x+i+j-1}{j}+\binom{x-i+j-1}{j}\right)\\
&=2(-1)^{\binom n2}\operatorname{CT}_{\mathbf z}\left(\prod_{j=0}^{n-1}(1+z_j)^{x-n}\right)\left(\prod_{0\le i<j\le n-1}\left((1-a)-a(z_i+z_j+z_iz_j)\right)\right)\\
&=2(a-1)^{\binom n2},
\end{aligned}$$

as desired. □

4. Proof of Conjecture 1

Here we prove Conjecture 1 using the holonomic Ansatz.

Theorem 4. *For all positive integers n, we have*

$$\det_{0\le i,j\le n-1}\left(2^i\binom{i+2j+1}{2j+1}-\binom{i-1}{2j+1}\right)=2\prod_{i=1}^{n}\frac{2^{i-1}\,(4i-2)!}{(n+2i-1)!},\tag{4.1}$$

where the binomial coefficient is defined as in (1.2).

Proof. We apply the holonomic Ansatz, described in Section 2. Computational details can be found in the accompanying electronic material [12].

We are able to guess three recurrence relations for the quantities $c_{n,j}$, as defined in (2.1), whose shape suggests that they indeed form a holonomic sequence. The recurrences are too big to be displayed here (they would require approximately one page), so we give only their supports instead:

$$\{c_{n,j+2},c_{n+1,j},c_{n,j+1},c_{n,j}\},\quad \{c_{n+1,j+1},c_{n+1,j},c_{n,j+1},c_{n,j}\},\quad \{c_{n+2,j},c_{n+1,j},c_{n,j+1},c_{n,j}\}.$$

When translated into operator notation — S_n denoting the forward shift operator $n\mapsto n+1$ — their supports can be written more compactly as

$$\{S_j^2,S_n,S_j,1\},\quad \{S_nS_j,S_n,S_j,1\},\quad \{S_n^2,S_n,S_j,1\}.\tag{4.2}$$

The corresponding operators form a (left) Gröbner basis, which is a useful property, as we will see later. During the guessing process, we have taken care that the final operators will have this property. Also for later use, we denote by $\mathfrak{I}$ the annihilator ideal they generate.

We want to show that the guessed recurrences (represented by $\mathfrak{I}$) produce the correct values of $c_{n,j}$ for all j with $0 \leq j < n$. For this purpose, we introduce another sequence $\tilde{c}_{n,j}$ that is defined via $\mathfrak{I}$, and we show that it actually agrees with the sequence $c_{n,j}$. The latter will be done by verifying that (H1) and (H2) hold when $c_{n,j}$ is replaced by $\tilde{c}_{n,j}$.

From the leading monomials S_j^2, S_nS_j, S_n^2 in (4.2) one can deduce, using the theory of Gröbner bases, that the holonomic rank of $\mathfrak{I}$ is three. Stated differently, the three irreducible monomials 1, S_j, S_n necessitate to specify initial values $\tilde{c}_{1,0}$, $\tilde{c}_{1,1}$, $\tilde{c}_{2,0}$ in order to fix a particular solution of the annihilator $\mathfrak{I}$. Hence, we define $\tilde{c}_{n,j}$ to be the unique solution of $\mathfrak{I}$ whose three initial values agree with $c_{n,j}$.

From this definition of $\tilde{c}_{n,j}$ one can derive algorithmically a (univariate) recurrence for the almost-diagonal sequence $\tilde{c}_{n,n-1}$. This recurrence has order 3, which is equal to the holonomic rank of $\mathfrak{I}$, as expected. The corresponding operator has the right factor $S_n - 1$, and more precisely, it can be written in the form

$$\begin{aligned}\big(9(n+4)(2n+5)(3n+2)(3n+4)(3n+5)(3n+7)p_1(n)S_n^2 \\ + 12(3n+2)(3n+4)(4n+3)(4n+5)p_2(n)S_n \\ - 16n(2n+1)(4n-1)(4n+1)(4n+3)(4n+5)p_1(n+1)\big) \cdot (S_n - 1),\end{aligned}$$

where $p_1(n)$ and $p_2(n)$ are irreducible polynomials of degree 9 and 11, respectively. It follows that any constant sequence is a solution of this recurrence. Together with the initial conditions $\tilde{c}_{1,0} = \tilde{c}_{2,1} = \tilde{c}_{3,2} = 1$, which are easy to check, this proves that $\tilde{c}_{n,n-1} = 1$ holds for all $n \geq 1$.

The proof of the summation identity (H2) is achieved by the method of creative telescoping, which delivers a set of recurrence equations (in n and i) that are satisfied by the sum. For reasons of efficiency, we split the sum in (H2) into two sums as follows:

$$\sum_{j=0}^{n-1} a_{i,j}\tilde{c}_{n,j} = \sum_{j=0}^{n-1} 2^i \binom{i+2j+1}{2j+1}\tilde{c}_{n,j} - \sum_{j=0}^{n-1} \binom{i-1}{2j+1}\tilde{c}_{n,j}.$$

For each of the two sums, we obtain an annihilator ideal that is generated by four operators whose supports are as follows:

$$\begin{aligned}&\{S_i^3, S_n^2, S_nS_i, S_i^2, S_n, S_i, 1\}, \quad \{S_i^2S_n, S_n^2, S_nS_i, S_i^2, S_n, S_i, 1\}, \\ &\{S_iS_n^2, S_n^2, S_nS_i, S_i^2, S_n, S_i, 1\}, \quad \{S_n^3, S_n^2, S_nS_i, S_i^2, S_n, S_i, 1\}.\end{aligned}$$

Actually, the two sums are annihilated by the very same operators, hence these operators constitute an annihilator for the left-hand side of (H2). The leading terms of the operators have the form:

$$\begin{aligned}&12(i-1)i(i+1)(3n+1)(3n+4)(4n-1)(4n+1)(i-n+3)(i-n+4)q_1(i,n)S_i^3, \\ &-9i(3n-1)(3n+1)(3n+4)q_2(i,n)S_nS_i^2, \\ &-18(i-1)i(n+1)(2n+3)(3n-1)(3n+1)^2(3n+2)(3n+4)(i+2n+5)q_3(i,n)S_n^2S_i, \\ &-54(n+1)(n+2)(2n+3)(2n+5)(3n-1)(3n+1)^2(i-2n-6)(i-2n-5)q_4(i,n)S_n^3,\end{aligned}$$

where q_1, q_2, q_3, q_4 are (not necessarily irreducible) polynomials in n and i. It remains to check a finite set of initial values. The shape of this set is determined by the support displayed above, by the condition $i < n-1$, and by the zeros of the leading coefficients of the operators. More precisely we have to verify that $\sum_{j=0}^{n-1} a_{i,j}\tilde{c}_{n,j} = 0$ for

$$\begin{aligned}(i,n) \in \{&(0,2),(0,3),(0,4),(0,5),(0,6),(1,3),(1,4),(1,5),(2,4),\\ &(1,6),(1,7),(2,5),(2,6),(2,7),(2,8),(3,5),(4,6)\}\end{aligned}$$

(where the points in the first line are determined by the support, and the second line is determined by the zeros of the leading coefficients). This verification is successful, and hence it follows that $\tilde{c}_{n,j} = c_{n,j}$ for all j with $0 \le j < n$, which allows us to use $\mathfrak{I}$ as a holonomic definition of $c_{n,j}$.

In order to derive a recurrence for the left-hand side of (H3) we split the sum into two sums, as before:

$$\sum_{j=0}^{n-1} a_{n,j}c_{n,j} = \sum_{j=0}^{n-1} 2^n \binom{n+2j+1}{2j+1} c_{n,j} - \sum_{j=0}^{n-1} \binom{n-1}{2j+1} c_{n,j}.$$

Then we compute, for each of the two sums, a recurrence by creative telescoping. In both cases, the output is a recurrence of order 6 with polynomial coefficients of degree approximately 52. Actually one finds that both sums satisfy the same order-6 recurrence, and hence so does their sum. One now has to verify that b_n/b_{n-1} satisfies this order-6 recurrence, where b_n denotes the right-hand side of (4.1). We have

$$\frac{b_n}{b_{n-1}} = \frac{(4n-2)!}{(3n-1)!\left(\frac{n+1}{2}\right)_{n-1}}.$$

Note that this expression is hypergeometric in $n/2$ and hence satisfies a second-order recurrence whose operator has support $\{S_n^2, 1\}$. Right-dividing the operator of the order-6 recurrence, call it P, by this second-order operator yields 0, hence P annihilates b_n/b_{n-1}. The leading term of the operator P is

$$\begin{aligned}4374&(n+7)(2n+9)(2n+11)(3n-1)(3n+1)(3n+2)(3n+4)(3n+5)(3n+7)\\ &\times (3n+8)(3n+10)(3n+11)(3n+13)^2(3n+14)(3n+16)(3n+17)p(n)S_n^6,\end{aligned}$$

where $p(n)$ is an irreducible polynomial of degree 35. Obviously this leading coefficient does not vanish for any positive integer n, hence it suffices to verify

$$\frac{\det_{0\le i,j\le n-1}(a_{i,j})}{\det_{0\le i,j\le n-2}(a_{i,j})} = \frac{b_n}{b_{n-1}}$$

for $n = 2, \dots, 7$. On both sides, one calculates the values 4, 15, 832/15, 204, 9728/13, 16445/6, respectively. By virtue of the recurrence P, the asserted identity (4.1) holds for all integers $n \ge 1$. □

5. **Proof of Conjecture 2**

In this section, we present our proofs of (1.3) and (1.4). As was mentioned in the introduction, it turned out that the capacity of today's computers is not sufficient for the holonomic Ansatz to produce proofs of theses two identities, although it very likely applies. Instead, the starting point for our proofs is determinant evaluations that have been established in [2]. In their statements, there appears the *Delannoy number* $D(i,j)$, which by definition is the number of paths from $(0,0)$ to (i,j) consisting of right-steps $(1,0)$, up-steps $(0,1)$, and diagonal steps $(1,1)$. Their generating function is given by (cf. [1, Ex. 21 in Ch. I])

$$D(i,j) = \left\langle u^i v^j \right\rangle \frac{1}{1-u-v-uv}, \tag{5.1}$$

where $\langle u^i v^j \rangle\, g(u,v)$ denotes the coefficient of $u^i v^j$ in the formal power series (in the variables u and v) $g(u,v)$. The following result is [2, Th. 5.1, in combination with Eqs. (4.3)–(4.5) and paragraph above and including Eq. (4.6)]

Theorem 5. *For all positive integers k and n, we have*

$$\begin{aligned} D_1(k;n) &:= \det_{1\le i,j\le k} \left(D(2j-i, i+n-k-1)\right) \\ &= \prod_{i\ge 0} \left(\prod_{s=-2k+4i+1}^{-k+2i} (2n+s) \prod_{s=k-2i}^{2k-4i-2} (2n+s) \right) \Big/ \prod_{i=1}^{k-1} (2i+1)^{k-i}\ , \end{aligned} \tag{5.2}$$

We are now prepared for the proof of (1.3), which we restate below with a modified, but equivalent, right-hand side.

Theorem 6. *For all positive integers n, we have*

$$\begin{aligned} D_2(n;x) &:= \det_{0\le i,j\le n-1} \left(2^i \binom{x+i+2j+1}{2j+1} + \binom{x-i+2j+1}{2j+1} \right) \\ &= 2^{\binom n2 +1} \prod_{i=0}^{n-1} \frac{i!}{(2i+1)!} \prod_{i=0}^{\lfloor n/2 \rfloor} (x+4i+1)_{n-2i} \prod_{i=0}^{\lfloor (n-1)/2 \rfloor} (x-2i+3n)_{n-2i-1}, \\ &= 2 \prod_{i=1}^{n} \frac{2^{2i-2}\, \Gamma(i)\, \Gamma(2i+x)\, \Gamma(4i+x-1)\, \Gamma\left(\frac{3i+x-2}{2}\right)}{\Gamma(2i)\, \Gamma(3i+x)\, \Gamma(3i+x-2)\, \Gamma\left(\frac{i+x}{2}\right)}, \end{aligned} \tag{5.3}$$

where the binomial coefficients have to be interpreted according to (1.2).

The theorem will, up to some routine manipulations, immediately follow from the relation below.

Lemma 7. *For all positive integers k, we have*

$$D_1(k; y+k) = \tfrac12 D_2(k; 2y). \tag{5.4}$$

Proof. We follow — and extend — Di Francesco's arguments in [3, Proofs of Ths 3.3, 4.3, and 8.2]. His idea is to work with determinants of the form $\det A(n)$ where $A(n) = (a_{i,j})_{0\le i,j\le n-1}$, with the entries $a_{i,j}$ given by a two-variable generating function,

$$a(u,v) = \sum_{i,j\ge 0} a_{i,j} u^i v^j.$$

The determinant will be unchanged if the matrix is multiplied (from the right or from the left) by a triangular matrix with 1s on the diagonal. It is easy to see that multiplication of $a(u,v)$ by a power series in u or by a power series in v with constant coefficient 1 will result in the multiplication of $A(n)$ by such a triangular matrix, and thus the determinant of the new matrix is still the same. The same property holds if in $a(u,v)$

we replace u by a power series in u with zero constant coefficient and coefficient of u equal to 1. Di Francesco argues with the help of complex integrals, but this is not necessary.

We start with expressing $D_1(n;k)$ in the above form. By shifting the row and column indices i and j by 1, we have

$$D_1(k;n) = \det_{0\le i,j\le k-1}\left(D(2j-i+1, i+n-k)\right).$$

By (5.1) (with the roles of u and v interchanged), we have

$$D(2j-i+1, n-k+i) = \left\langle u^{n-k+i} v^{2j-i+1}\right\rangle \frac{1}{1-u-v-uv}.$$

By replacing u by uv, we see that

$$D(2j-i+1, n-k+i) = \left\langle u^{n-k+i} v^{n-k+2j+1}\right\rangle \frac{1}{1-v-uv-uv^2}.$$

From here on, we write N for $n-k$ for short, so that

$$D(2j-i+1, N+i) = \left\langle u^{N+i} v^{N+2j+1}\right\rangle \frac{1}{1-v-uv-uv^2}.$$

If we denote the coefficient of $u^i v^j$ in $1/(1-v-uv-uv^2)$ by $\alpha_{i,j}$, then

$$\begin{aligned}\sum_{i,j\ge 0}\alpha_{i,j}u^i v^j &= \frac{1}{1-v-uv-uv^2}\\ &= \sum_{s\ge 0}\left(\frac{uv(1+v)}{1-v}\right)^s \frac{1}{1-v}.\end{aligned}$$

Consequently,

$$\begin{aligned}\sum_{i,j\ge 0}\alpha_{i+N,j+N}u^i v^j &= (uv)^{-N}\sum_{s\ge N}\left(\frac{uv(1+v)}{1-v}\right)^s \frac{1}{1-v}\\ &= \left(\frac{1+v}{1-v}\right)^N \frac{1}{1-v-uv-uv^2}.\end{aligned}$$

We have shown that

$$D(2j-i+1, N+i) = \left\langle u^i v^{2j+1}\right\rangle \left(\frac{1+v}{1-v}\right)^N \frac{1}{1-v-uv-uv^2}.$$

By 2-section (in v) of the series on the right-hand side, we finally obtain

$$\begin{aligned} D(2j-i+1,N+i) &= \left\langle u^i v^j \right\rangle \frac{1}{2\sqrt{v}} \left(\left(\frac{1+\sqrt{v}}{1-\sqrt{v}}\right)^N \frac{1}{1-\sqrt{v}-u\sqrt{v}-uv} \right. \\ &\qquad \left. - \left(\frac{1-\sqrt{v}}{1+\sqrt{v}}\right)^N \frac{1}{1+\sqrt{v}+u\sqrt{v}-uv} \right) \\ &= \left\langle u^i v^j \right\rangle \frac{1}{2\sqrt{v}(1-v-4uv-u^2v+u^2v^2)} \\ &\quad \cdot \left(\left(\frac{1+\sqrt{v}}{1-\sqrt{v}}\right)^N \left(1-uv+\sqrt{v}(1+u)\right) - \left(\frac{1-\sqrt{v}}{1+\sqrt{v}}\right)^N \left(1-uv-\sqrt{v}(1+u)\right) \right). \end{aligned} \tag{5.5}$$

We have reached our first intermediate goal to express the determinant $D_1(k;n)$ in the form $\det B(n)$, where $B(n) = (b_{i,j})_{0\le i,j\le n-1}$ with $b(u,v) = \sum_{i,j\ge 0} b_{i,j} u^i v^j$ the double series on the right-hand side of (5.5). (Recall that $N = n-k$.)

Now we transform our determinant by multiplying $b(u,v)$ by $(1-v)^{-N}$. (The latter is indeed a power series in v with constant coefficient equal to 1.[5]) Thus we see that $D_1(k;n) = \det C(n)$, where $C(n) = (c_{i,j})_{0\le i,j\le n-1}$ with

$$\begin{aligned} c(u,v) = \sum_{i,j\ge 0} c_{i,j} u^i v^j &= \frac{1}{2\sqrt{v}(1-v-4uv-u^2v+u^2v^2)} \\ &\quad \times \left((1-\sqrt{v})^{-2N} \left(1-uv+\sqrt{v}(1+u)\right) - (1+\sqrt{v})^{-2N} \left(1-uv-\sqrt{v}(1+u)\right) \right). \end{aligned}$$

We transform this series (and thus the corresponding matrix) by performing the substitution $u \mapsto \frac{u}{(1-u)(1-2u)}$, followed by multiplication by $\frac{1-2u^2}{(1-u)(1-2u)}$.[6] As a result, we obtain that $D_1(k;n) = \det D(n)$, where $D(n) = (d_{i,j})_{0\le i,j\le n-1}$ with

$$\begin{aligned} d(u,v) = \sum_{i,j\ge 0} d_{i,j} u^i v^j &= \frac{1-2u^2}{2\sqrt{v}\big((1-2u)^2-v\big)\big((1-u)^2-u^2v\big)} \\ &\quad \times \Big((1-\sqrt{v})^{-2N} \left((1-u)(1-2u)-uv+\sqrt{v}(1-2u+2u^2)\right) \\ &\qquad -(1+\sqrt{v})^{-2N} \left((1-u)(1-2u)-uv-\sqrt{v}(1-2u+2u^2)\right)\Big). \end{aligned} \tag{5.6}$$

We turn our attention to the determinant in (5.3). We have

$$\sum_{j\ge 0} \binom{x+i+j}{j} v^j = (1-v)^{-x-i-1} \quad \text{and} \quad \sum_{j\ge 0} \binom{x-i+j}{j} v^j = (1-v)^{-x+i-1}, \tag{5.7}$$

---

[5]This step was not necessary in [3] since there $N = n-k = 0$.

[6]Di Francesco does this transformation in two steps. First, he does the substitution $u \mapsto u\frac{1+u}{1-u}$ and he multiplies the resulting generating function by $\frac{1+2u-u^2}{1-u}$. (At this point, he has shown the equality of the domino tilings partition function with the 20-vertex partition function.) Subsequently, he does the substitution $u \mapsto \frac{u}{1-u}$, which leads him to the determinant in (5.3) with $x = 0$. (A subtlety is that he does not arrive exactly at the determinant in (5.3) but rather at the matrix that arises from ours by dividing all entries in the 0-th row by 2 because his binomial coefficient $\binom{i-1}{2j+1}$ must be interpreted as 0 for $i = 0$ — as opposed to our convention concerning the binomial coefficient.) We have combined these two steps here into one.

and therefore, again by a 2-section,

$$\sum_{j\ge0}\binom{x+i+2j+1}{2j+1}v^j=\frac{1}{2\sqrt{v}}\left((1-\sqrt{v})^{-x-i-1}-(1+\sqrt{v})^{-x-i-1}\right) \tag{5.8}$$

and

$$\sum_{j\ge0}\binom{x-i+2j+1}{2j+1}v^j=\frac{1}{2\sqrt{v}}\left((1-\sqrt{v})^{-x+i-1}-(1+\sqrt{v})^{-x+i-1}\right). \tag{5.9}$$

Consequently, we have $D_2(n;x)=\det E(n)$, where $E(n)=(e_{i,j})_{0\le i,j\le n-1}$ with

$$\begin{aligned} e(u,v)=\sum_{i,j\ge0}e_{i,j}u^iv^j&=\frac{1}{2\sqrt{v}}\Bigg((1-\sqrt{v})^{-x-1}\frac{1}{1-\frac{2u}{1-\sqrt{v}}}-(1+\sqrt{v})^{-x-1}\frac{1}{1-\frac{2u}{1+\sqrt{v}}}\\ &\qquad+(1-\sqrt{v})^{-x-1}\frac{1}{1-u(1-\sqrt{v})}-(1+\sqrt{v})^{-x-1}\frac{1}{1-u(1+\sqrt{v})}\Bigg)\\ &=\frac{1}{2\sqrt{v}}\Bigg((1-\sqrt{v})^{-x}\frac{1-2u+\sqrt{v}}{(1-2u)^2-v}-(1+\sqrt{v})^{-x}\frac{1-2u-\sqrt{v}}{(1-2u)^2-v}\\ &\qquad+(1-\sqrt{v})^{-x-1}\frac{1-u-u\sqrt{v}}{(1-u)^2-u^2v}-(1+\sqrt{v})^{-x-1}\frac{1-u+u\sqrt{v}}{(1-u)^2-u^2v}\Bigg). \end{aligned} \tag{5.10}$$

In order to explain the factor $\frac{1}{2}$ in (5.4), we now want to divide all entries in row 0 of the current matrix $E(n)$ by 2. In terms of generating functions, we achieve this by subtracting *half* of the generating function for the entries in row 0,

$$\frac{1}{2}\sum_{j\ge0}e_{0,j}v^j=\frac{1}{2\sqrt{v}}\left((1-\sqrt{v})^{-x-1}-(1+\sqrt{v})^{-x-1}\right),$$

from $e(u,v)$. We are led to the conclusion that $\frac{1}{2}D_2(n;x)=\det F(n)$, where $F(n)=(f_{i,j})_{0\le i,j\le n-1}$ with

$$\begin{aligned} f(u,v)=\sum_{i,j\ge0}f_{i,j}u^iv^j&=\frac{1}{2\sqrt{v}}\Bigg((1-\sqrt{v})^{-x}\frac{1-2u+\sqrt{v}}{(1-2u)^2-v}-(1+\sqrt{v})^{-x}\frac{1-2u-\sqrt{v}}{(1-2u)^2-v}\\ &\qquad+(1-\sqrt{v})^{-x-1}\frac{u(1-u-u\sqrt{v})(1-\sqrt{v})}{(1-u)^2-u^2v}\\ &\qquad\qquad-(1+\sqrt{v})^{-x-1}\frac{u(1-u+u\sqrt{v})(1+\sqrt{v})}{(1-u)^2-u^2v}\Bigg)\\ &=\frac{1}{2\sqrt{v}}\Bigg((1-\sqrt{v})^{-x}\frac{1-2u+\sqrt{v}}{(1-2u)^2-v}-(1+\sqrt{v})^{-x}\frac{1-2u-\sqrt{v}}{(1-2u)^2-v}\\ &\qquad+(1-\sqrt{v})^{-x}\frac{u(1-u-u\sqrt{v})}{(1-u)^2-u^2v}-(1+\sqrt{v})^{-x}\frac{u(1-u+u\sqrt{v})}{(1-u)^2-u^2v}\Bigg). \end{aligned}$$

One can now readily verify that $f(u,v)$ with $x=2N$ is equal to $d(u,v)$ as given in (5.6). In view of $N=n-k$, this establishes the relationship (5.4). This completes the proof of the theorem. $\square$

Now we prove (1.4), restated again with a modified, but equivalent, right-hand side.

Theorem 8. *For all positive integers n, we have*

$$\begin{aligned} D_3(n;x) &:= \det_{0\le i,j\le n-1}\left(2^i\binom{x+i+2j}{2j}+\binom{x-i+2j}{2j}\right)\\ &= 2^{\binom{n}{2}+1}\prod_{i=0}^{n-1}\frac{i!}{(2i)!}\prod_{i=0}^{\lfloor (n-1)/2\rfloor}(x+4i+3)_{n-2i-1}\prod_{i=0}^{\lfloor (n-2)/2\rfloor}(x-2i+3n-1)_{n-2i-2},\\ &= 2\prod_{i=1}^{n}\frac{2^{2i-2}\,\Gamma(i)\,\Gamma(2i+x)\,\Gamma(4i+x-3)\,\Gamma\big(\frac{3i+x-1}{2}\big)}{\Gamma(2i-1)\,\Gamma(3i+x-1)\,\Gamma(3i+x-2)\,\Gamma\big(\frac{i+x+1}{2}\big)}, \end{aligned}\tag{5.11}$$

where the binomial coefficients have to be interpreted according to (1.2).

Again, the theorem will, up to some routine manipulations, immediately follow from a relation between the above determinant and the earlier determinant $D_1(k;n)$ defined in (5.2).

Lemma 9. *For all positive integers k, we have*

$$D_1(k-1;y+k)=\tfrac12 D_3(k;2y).\tag{5.12}$$

Proof. Here we start with $D_3(n;x)$. From 2-section of the binomial series in (5.7), we obtain that $D_3(n;x)=\det H(n)$, where $H(n)=(h_{i,j})_{0\le i,j\le n-1}$ with

$$\begin{aligned} h(u,v)=\sum_{i,j\ge0}h_{i,j}u^iv^j &= \frac12\Bigg((1-\sqrt v)^{-x-1}\frac{1}{1-\frac{2u}{1-\sqrt v}}+(1+\sqrt v)^{-x-1}\frac{1}{1-\frac{2u}{1+\sqrt v}}\\ &\qquad +(1-\sqrt v)^{-x-1}\frac{1}{1-u(1-\sqrt v)}+(1+\sqrt v)^{-x-1}\frac{1}{1-u(1+\sqrt v)}\Bigg)\\ &= \frac12\Bigg((1-\sqrt v)^{-x}\frac{1-2u+\sqrt v}{(1-2u)^2-v}+(1+\sqrt v)^{-x}\frac{1-2u-\sqrt v}{(1-2u)^2-v}\\ &\qquad +(1-\sqrt v)^{-x-1}\frac{1-u-u\sqrt v}{(1-u)^2-u^2v}+(1+\sqrt v)^{-x-1}\frac{1-u+u\sqrt v}{(1-u)^2-u^2v}\Bigg). \end{aligned}$$

It should be noted that the only differences with (5.10) are that, here, the prefactor is $\frac12$ instead of $\frac{1}{2\sqrt v}$, and that there are plus-signs in front of the terms involving $(1+\sqrt v)^{-x}$. Hence, if we proceed from here as in the proof of Lemma 7 — that is, we divide the 0-th row of $H(n)$ by 2, and then do the transformations described in the proof of Lemma 7 "in reverse" — then we obtain

$$\frac12 D_3(k;2y)=\det_{0\le i,j\le k-1}\big(D(2j-i,y+i)\big).$$

Here we see that all entries in column 0 of the last matrix are zero except for the entry in row 0 which is equal to $D(0,y)=1$. By expanding the determinant of this matrix along the first column, we see that

$$\frac12 D_3(k;2y)=\det_{0\le i,j\le k-2}\big(D(2j-i+1,y+i+1)\big)=D_1(k-1;y+k).$$

This is exactly (5.12). □

6. Variations on the theme, I

There exist numerous variations of Theorem 4 in which the exponent in the exponential 2^i is shifted. In this section, we report our corresponding findings. Let

$$D_{\alpha,\beta,\gamma,\delta}(n) := \det_{0\le i,j\le n-1}\left(2^{i+\beta}\binom{i+2j+\gamma}{2j+\alpha}+\binom{-i+2j+\delta}{2j+\alpha}\right).$$

Note that, in this notation, the determinants from Theorems 6 and 8 read

$$D_2(n;x) = D_{1,0,x+1,x+1}(n), \tag{6.1}$$

$$D_3(n;x) = D_{0,0,x,x}(n), \tag{6.2}$$

respectively.

In an automated search in the parameter space $-6 \le \alpha, \beta \le 9$ and $-9 \le \gamma, \delta \le 9$, we have identified 26 cases of determinants that factor completely, and which are not special instances of $D_2(n;x)$ or $D_3(n;x)$. All of these 26 cases can be proven automatically by the holonomic Ansatz, but some of them can also easily be related to each other.

Theorem 10. *The following determinant evaluations hold for all $n \ge 1$:*

$$D_{-2,0,-1,-1}(n) = -2\prod_{i=2}^{n}\frac{8(2i-3)(2i-1)\,\Gamma\big(4i-5\big)\,\Gamma\big(\frac{i+1}{2}\big)}{i\,\Gamma\big(3i-2\big)\,\Gamma\big(\frac{3i-3}{2}\big)}, \tag{6.3}$$

$$D_{0,2,3,-1}(n) = \prod_{i=1}^{n}\frac{3(2i-1)\,\Gamma\big(4i+3\big)\,\Gamma\big(\frac{i+1}{2}\big)}{4(i+2)\,\Gamma\big(3i+1\big)\,\Gamma\big(\frac{3i+5}{2}\big)}, \tag{6.4}$$

$$D_{1,1,0,-2}(n) = -2\prod_{i=1}^{n}\frac{(2i-1)\,\Gamma\big(4i-3\big)\,\Gamma\big(\frac{i}{2}\big)}{2\,\Gamma\big(3i-2\big)\,\Gamma\big(\frac{3i}{2}\big)}, \tag{6.5}$$

$$D_{1,1,1,-1}(n) = \prod_{i=1}^{n}\frac{\Gamma\big(4i-1\big)\,\Gamma\big(\frac{i+1}{2}\big)}{\Gamma\big(3i\big)\,\Gamma\big(\frac{3i-1}{2}\big)}, \tag{6.6}$$

$$D_{2,1,2,0}(n) = \prod_{i=1}^{n}\frac{\Gamma\big(4i\big)\,\Gamma\big(\frac{i+2}{2}\big)}{\Gamma\big(3i\big)\,\Gamma\big(\frac{3i+2}{2}\big)}, \tag{6.7}$$

$$D_{0,1,1,-1}(n) = 3\prod_{i=2}^{n}\frac{\Gamma\big(4i\big)\,\Gamma\big(\frac{i-1}{2}\big)}{\Gamma\big(3i+1\big)\,\Gamma\big(\frac{3i-3}{2}\big)}. \tag{6.8}$$

Moreover, some related determinants can be expressed in terms of these; the following identities hold (at least) for all $n \geq 4$:

$$D_{2,1,2,0}(n) = \frac{1}{8}\, D_{1,1,-1,-3}(n+1) = \frac{1}{40}\, D_{0,1,-4,-6}(n+2) = -\frac{1}{24576}\, D_{1,2,-4,-8}(n+2), \tag{6.9}$$

$$\begin{aligned} D_{1,1,1,-1}(n) &= D_{2,1,1,-1}(n) = \frac{1}{3}\, D_{0,1,-2,-4}(n+1) = -\frac{1}{32}\, D_{1,1,-2,-4}(n+1) \\ &= -\frac{1}{224}\, D_{1,2,-2,-6}(n+1) = -\frac{1}{168}\, D_{0,1,-5,-7}(n+2) \\ &= -\frac{1}{3696}\, D_{0,2,-5,-9}(n+2) = -\frac{1}{337920}\, D_{1,2,-5,-9}(n+2), \end{aligned} \tag{6.10}$$

$$D_{1,1,0,-2}(n) = \frac{1}{5}\, D_{0,1,-3,-5}(n+1) = \frac{1}{1008}\, D_{1,2,-3,-7}(n+1), \tag{6.11}$$

$$D_{-2,1,0,-2}(n) = D_{0,2,3,-1}(n-1), \tag{6.12}$$

$$D_{2,1,1,-1}(n) = D_{4,2,4,0}(n-1), \tag{6.13}$$

$$D_{1,1,-2,-4}(n) = -\frac{16}{5}\, D_{3,2,1,-3}(n-1) = \frac{64}{3}\, D_{5,3,4,-2}(n-2) = -128\, D_{7,4,7,-1}(n-3), \tag{6.14}$$

$$D_{1,1,-1,-3}(n) = -4\, D_{3,2,2,-2}(n-1) = 16\, D_{5,3,5,-1}(n-2), \tag{6.15}$$

$$D_{1,1,0,-2}(n) = -2\, D_{3,2,3,-1}(n-1). \tag{6.16}$$

Proof. Identities (6.3)–(6.11) can be proven by the holonomic Ansatz. More precisely, we prove a closed-form evaluation for each of the mentioned determinants, similar to those in (6.3)–(6.8), but find that some of these are related to each other. In order to make these relations explicit, and in order to save some space, we display in (6.9)–(6.11) only the relations, not the closed forms themselves. The detailed proofs can be found in the accompanying electronic material [12], some computational data are given in Table 1.

Identities (6.12)–(6.16) can easily be established by exploiting the structure of the corresponding matrices: the matrices of the determinants on the right-hand sides take the block form $\left(\begin{smallmatrix} C & 0 \\ * & A \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} C & * \\ 0 & A \end{smallmatrix}\right)$ where in each case C is a fixed matrix of dimension 1×1, or 2×2, or 3×3, and where A is the matrix of the determinant on the corresponding right-hand side. The latter follows from the fact that the transformation $(\alpha,\beta,\gamma,\delta)\mapsto(\alpha+2,\beta+1,\gamma+3,\delta+1)$ is equivalent to shifting $(i,j)\mapsto(i+1,j+1)$. ☐

By looking at (6.9)–(6.11) one is tempted to prove these relations directly, without taking the detour via the closed-form evaluations. We demonstrate with one example how this can work. Let L_n be the lower-triangular $(n\times n)$-matrix with entries $2^{i-j+1}-1$ and R_n be the $(n\times n)$-matrix with 1's on the main diagonal, -1's on the upper diagonal, and 0 elsewhere, i.e.,

$$L_n := \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 3 & 1 & 0 & 0 & \\ 7 & 3 & 1 & 0 & \\ 15 & 7 & 3 & 1 & \\ \vdots & & & & \ddots \end{pmatrix} \qquad \text{and} \qquad R_n := \begin{pmatrix} 1 & -1 & 0 & 0 & \cdots \\ 0 & 1 & -1 & 0 & \\ 0 & 0 & 1 & -1 & \\ 0 & 0 & 0 & 1 & \ddots \\ \vdots & & & & \ddots \end{pmatrix}$$

Then, for example, we claim that

$$L_n \cdot A_{2,1,2,0}(n) \cdot R_n = \begin{pmatrix} 2 & 0 \\ * & A_{5,3,5,-1}(n-1) \end{pmatrix}, \tag{6.17}$$

where $A_{\alpha,\beta,\gamma,\delta}(n)$ denotes the matrix from the definition of $D_{\alpha,\beta,\gamma,\delta}(n)$. Equation (6.17) immediately implies the identity $D_{2,1,2,0}(n) = 2\, D_{5,3,5,-1}(n-1)$, which has already been stated (implicitly; it is obtained by combining (6.9) with (6.15)). It remains to show (6.17), which boils down to proving the binomial sum identity

$$\begin{aligned} \sum_{k=0}^{i} \Bigg[(1 - 2^{i-k+1}) \left(\binom{-1+2j-k}{2j+2} - \binom{1+2j-k}{2j+4} \right) \\ + (2^{k+2} - 2^{i+3}) \left(\binom{3+2j+k}{2j+2} - \binom{5+2j+k}{2j+4} \right) \Bigg] \\ = \binom{-1-i+2j}{2j+5} + 2^{i+3} \binom{5+i+2j}{2j+5}. \end{aligned} \tag{6.18}$$

This can be achieved by observing that

$$\binom{a+k}{b} = \binom{a+k+1}{b+1} - \binom{a+k}{b+1}$$

and[7]

$$2^k \left(\binom{a+k}{b} - \binom{a+k+2}{b+2} \right) = -2^{k+1} \binom{a+k+1}{b+2} + 2^k \binom{a+k}{b+2},$$

so that all sums in (6.18) are telescoping sums. The identity can also be automatically proved by Zeilberger's algorithm [22, 20].

Similarly to (6.9)–(6.16) one can connect the determinants from Theorem 10 to the determinants $D_2(n;x)$ and $D_3(n;x)$, whose evaluations have already been proven in Theorems 6 and 8, respectively.

Corollary 11. *The following identities hold for all integers $n \geq 2$:*

$$2\, D_{1,1,1,-1}(n) = D_3(n+1;-2) = D_3(n;1) = D_2(n;0) = D_2(n-1;3), \tag{6.19}$$

$$2\, D_{2,1,2,0}(n) = D_3(n+1;-1) = D_2(n;1), \tag{6.20}$$

$$-D_{1,1,0,-2}(n) = D_3(n;0) = D_2(n-1;2). \tag{6.21}$$

7. Variations on the theme, II

In this section we present — and prove — several determinant evaluations in which, compared with Theorem 4, the power 2^i gets replaced by 3^i, and the terms $2j$ in the binomials are replaced by $3j$. As it turns out, there are even more variations of Theorem 4 associated with the modulus 3 if one also shifts the exponent in the exponential 3^i. For brevity, let us denote

$$E_{\alpha,\beta,\gamma,\delta}(n) := \det_{0 \leq i,j \leq n-1} \left(3^{i+\beta} \binom{i+3j+\gamma}{3j+\alpha} + \binom{-i+3j+\delta}{3j+\alpha} \right).$$

[7]We found this relation by means of Gosper's algorithm [8], using the implementation [19].

An automated search in the parameter space

$$\big\{(\alpha,\beta,\gamma,\delta): -6\le\alpha,\beta\le 6 \text{ and } -8\le\gamma,\delta\le 8\big\}\\ \cup\big\{(\alpha,\beta,\gamma,\delta): 6\le\alpha\le 10 \text{ and } 0\le\beta\le 10 \text{ and } -10\le\gamma,\delta\le 10\big\}$$

delivered 26 cases of determinants that factor completely. All of these 26 cases can be proven automatically by the holonomic Ansatz (see the accompanying electronic material [12]), but some of them can also easily be related to each other.

Finally, we have discovered three parametric families of determinant evaluations of this kind, in addition to the other, (seemingly) sporadic ones. The parametric families are presented in Conjecture 13 below. Here, it seems difficult to apply the holonomic Ansatz, but purely because of the computational complexity that is added by the parameter x. We are absolutely convinced that it should work in principle, since we observed that it works for specific values of x without much difficulty. We admit that we do not know a different method that would work here.

Theorem 12. *The following determinant evaluations hold for all $n\ge 1$:*

$$E_{-3,0,-1,-1}(n) = 2\prod_{i=2}^{n}\frac{2^{i+1}(2i-1)\,\Gamma\big(4i-5\big)\,\Gamma\big(\frac{i+2}{3}\big)}{i(i+1)\,\Gamma\big(3i-5\big)\,\Gamma\big(\frac{4i-1}{3}\big)}, \tag{7.1}$$

$$E_{-3,1,0,-2}(n) = -2\prod_{i=2}^{n}\frac{2^{i+1}(2i-1)\,\Gamma\big(4i-4\big)\,\Gamma\big(\frac{i}{3}\big)}{i(i+1)^2\,\Gamma\big(3i-5\big)\,\Gamma\big(\frac{4i-3}{3}\big)}, \tag{7.2}$$

$$E_{0,3,5,-1}(n) = \prod_{i=1}^{n}\frac{2^{i+1}(3i-2)(3i-1)\,\Gamma\big(4i+4\big)\,\Gamma\big(\frac{i+2}{3}\big)}{(i+1)(i+2)(i+3)(i+4)\,\Gamma\big(3i+1\big)\,\Gamma\big(\frac{4i+5}{3}\big)}, \tag{7.3}$$

$$E_{0,1,1,-1}(n) = \prod_{i=1}^{n}\frac{2^{i+1}\,\Gamma\big(4i-2\big)\,\Gamma\big(\frac{i+2}{3}\big)}{i\,\Gamma\big(3i-2\big)\,\Gamma\big(\frac{4i-1}{3}\big)}, \tag{7.4}$$

$$E_{1,1,2,0}(n) = \prod_{i=1}^{n}\frac{2^{i}\,\Gamma\big(4i\big)\,\Gamma\big(\frac{i+1}{3}\big)}{3i\,\Gamma\big(3i-1\big)\,\Gamma\big(\frac{4i+1}{3}\big)}, \tag{7.5}$$

$$E_{3,2,3,-1}(n) = \prod_{i=1}^{n}\frac{2^{i}\,\Gamma\big(4i+1\big)\,\Gamma\big(\frac{i+2}{3}\big)}{\Gamma\big(3i+1\big)\,\Gamma\big(\frac{4i+2}{3}\big)}, \tag{7.6}$$

$$E_{1,0,1,1}(n) = 2\prod_{i=1}^{n}\frac{2^{i-2}\,\Gamma\big(4i-1\big)\,\Gamma\big(\frac{i}{3}\big)}{3\,\Gamma\big(3i-1\big)\,\Gamma\big(\frac{4i}{3}\big)}, \tag{7.7}$$

$$E_{2,0,2,2}(n) = 2\prod_{i=1}^{n}\frac{2^{i-3}\,\Gamma\big(4i+1\big)\,\Gamma\big(\frac{i+2}{3}\big)}{\Gamma\big(3i+1\big)\,\Gamma\big(\frac{4i+2}{3}\big)}. \tag{7.8}$$

Moreover, some related determinants can be expressed in terms of these; the following identities hold (at least) for all $n \geq 3$:

$$E_{0,0,0,0}(n) = \frac{1}{2} E_{0,1,-1,-3}(n) = \frac{1}{5} E_{0,2,-2,-6}(n), \tag{7.9}$$

$$E_{1,0,1,1}(n) = -\frac{1}{84} E_{1,3,-2,-8}(n) = 2\, E_{4,2,4,0}(n-1) = \frac{6}{5} E_{4,3,3,-3}(n-1), \tag{7.10}$$

$$E_{2,0,2,2}(n) = 2\, E_{5,2,5,1}(n-1) = 18\, E_{8,4,8,0}(n-2) = \frac{162}{5} E_{8,5,7,-3}(n-2), \tag{7.11}$$

$$E_{-3,2,1,-3}(n) = E_{0,3,5,-1}(n-1), \tag{7.12}$$

$$E_{0,1,-1,-3}(n) = 4\, E_{3,2,3,-1}(n-1), \tag{7.13}$$

$$E_{1,1,0,-2}(n) = -2\, E_{4,2,4,0}(n-1), \tag{7.14}$$

$$E_{1,2,-1,-5}(n) = -12\, E_{4,3,3,-3}(n-1) = -180\, E_{7,4,7,-1}(n-2), \tag{7.15}$$

$$E_{2,1,1,-1}(n) = E_{5,2,5,1}(n-1), \tag{7.16}$$

$$E_{2,2,0,-4}(n) = \frac{15}{2} E_{5,3,4,-2}(n-1) = -45\, E_{8,4,8,0}(n-2), \tag{7.17}$$

$$E_{2,3,-1,-7}(n) = 36\, E_{5,4,3,-5}(n-1) = -\frac{13608}{5} E_{8,5,7,-3}(n-2). \tag{7.18}$$

Proof. Identities (7.1)–(7.11) can be proven by the holonomic Ansatz, see [12] for the details. Some computational data are given in Table 1. Identities (7.12)–(7.18) can easily be established by exploiting the block structure of the corresponding matrices: the larger matrices in each formula have a block of zeros, and the smaller matrices from the same formula in the lower right corner. □

Conjecture 13. *Let*

$$\Xi(x) := \prod_{i=2}^{x} \frac{3\,\Gamma(i)\,\Gamma(4i-3)\,\Gamma(4i-2)}{2\,\Gamma(3i-2)^2\,\Gamma(3i-1)} \qquad \text{and} \qquad \mu_m(x) := \begin{cases} 2, & \text{if } 3 \mid (x-m), \\ 1, & \text{otherwise.} \end{cases}$$

Then, for all non-negative integers x and for all $n \geq x$, we have

$$E_{0,x,-x,-3x}(n) = 2\mu_1(x)\,\Xi(x)\,(-1)^{\lfloor \frac{x}{3} \rfloor} \prod_{i=1}^{n} \frac{2^{i-1}\,\Gamma(4i-3)\,\Gamma\left(\frac{i+1}{3}\right)}{\Gamma(3i-2)\,\Gamma\left(\frac{4i-2}{3}\right)}, \tag{7.19}$$

$$E_{1,x,1-x,1-3x}(n) = 2\mu_2(x)\,\Xi(x)\,(-1)^{\lfloor \frac{x+2}{3} \rfloor} \prod_{i=1}^{n} \frac{2^{i-2}\,\Gamma(4i-1)\,\Gamma\left(\frac{i}{3}\right)}{3\,\Gamma(3i-1)\,\Gamma\left(\frac{4i}{3}\right)}, \tag{7.20}$$

$$E_{2,x,2-x,2-3x}(n) = \frac{\mu_0(x)}{n}\,\Xi(x)\,(-1)^{\lfloor \frac{x+1}{3} \rfloor} \prod_{i=2}^{n} \frac{2^{i-3}\,\Gamma(4i+1)\,\Gamma\left(\frac{i-1}{3}\right)}{9\,\Gamma(3i)\,\Gamma\left(\frac{4i+2}{3}\right)}. \tag{7.21}$$

Remark. Identity (7.19) generalizes (7.9) (their closed forms are obtained by combining (7.13) with (7.6)). Identity (7.20) generalizes some determinants given in (7.10), (7.14), and (7.15). Identity (7.21) generalizes some determinants given in (7.11), (7.16), (7.17), and (7.18).

8. **Variations on the theme, III**

In this section, we present several variations of the determinant evaluations in Section 5 in which the power 2^i gets replaced by 4^i. As in the previous sections, we start by identifying some sporadic cases, which were found in an automated search inside the parameter space $-6 \le \alpha, \beta \le 9$ and $-9 \le \gamma, \delta \le 9$, before we turn to two parametric families. We are able to prove one of them using the holonomic Ansatz; see Theorem 15. The second, Theorem 16, does not seem suitable for the application of the holonomic Ansatz. On the other hand, the application of a — non-algorithmic — method is feasible: identification of factors. Due to its length, we provide the corresponding proof separately in the next section. Still, this second result must be considered as incomplete as we are not able to identify one factor in the determinant evaluation; we are only able to provide a conjectural recurrence that this factor seems to satisfy; see Conjecture 17.

Let us introduce the following notation for the determinants in question:

$$F_{\alpha,\beta,\gamma,\delta}(n) := \det_{0 \le i,j \le n-1} \left(4^{i+\beta} \binom{i+2j+\gamma}{2j+\alpha} + \binom{-i+2j+\delta}{2j+\alpha} \right).$$

Theorem 14. *The following determinant evaluations hold for all $n \ge 1$:*

$$F_{1,0,1,1}(n) = 2 \prod_{i=1}^{n} \frac{3^{i-1}\,\Gamma\big(3i-1\big)\,\Gamma\big(\frac{i+1}{2}\big)}{\Gamma\big(2i\big)\,\Gamma\big(\frac{3i-1}{2}\big)}, \tag{8.1}$$

$$F_{1,0,2,2}(n) = 2 \prod_{i=1}^{n} \frac{3^{i-1}\,\Gamma\big(3i\big)\,\Gamma\big(\frac{i}{2}\big)}{2\,\Gamma\big(2i\big)\,\Gamma\big(\frac{3i}{2}\big)}, \tag{8.2}$$

$$F_{1,0,3,3}(n) = 2 \prod_{i=1}^{n} \frac{3^{i}\,\Gamma\big(3i-1\big)\,\Gamma\big(\frac{i+1}{2}\big)}{\Gamma\big(2i\big)\,\Gamma\big(\frac{3i-1}{2}\big)}. \tag{8.3}$$

Moreover, some related determinants can be expressed in terms of these; the following identities hold (at least) for all $n \ge 4$:

$$F_{1,0,1,1}(n) = \frac{2}{3}\, F_{1,1,-1,-3}(n) = \frac{1}{21}\, F_{1,2,-3,-7}(n), \tag{8.4}$$

$$F_{1,0,2,2}(n) = -2\, F_{1,1,0,-2}(n) = \frac{2}{7}\, F_{1,2,-2,-6}(n), \tag{8.5}$$

$$F_{1,0,3,3}(n) = 2\, F_{1,1,1,-1}(n) = \frac{2}{5}\, F_{1,2,-1,-5}(n) = \frac{1}{99}\, F_{1,3,-3,-9}(n), \tag{8.6}$$

$$F_{1,1,-1,-3}(n) = -6\, F_{3,2,2,-2}(n-1) = 24\, F_{5,3,5,-1}(n-2), \tag{8.7}$$

$$F_{1,1,0,-2}(n) = -2\, F_{3,2,3,-1}(n-1). \tag{8.8}$$

Proof. Identities (8.1)–(8.6) can be proven, quite effortlessly, by the holonomic Ansatz, see [12]. For the determinants on the right-hand sides of (8.4)–(8.6) we have established closed forms, from which the displayed relations follow. Identities (8.7)–(8.8) can easily be established by exploiting the block structure of the matrices $F_{1,1,-1,-3}(n)$ respectively $F_{3,2,2,-2}(n)$ and $F_{1,1,0,-2}(n)$, which have a block of zeros (of size $2 \times (n-2)$ respectively $1 \times (n-1)$) in their upper right corner. □

The parameters of the determinants in (8.1)–(8.3) follow an obvious pattern (in contrast to their right-hand sides). Indeed, the determinants $F_{1,0,4,4}(n), \ldots, F_{1,0,9,9}(n)$ were also found to factor nicely, and in fact one can come up with a general closed form. Note that the determinant below corresponds to $F_{1,0,x+1,x+1}(n)$.

Theorem 15. *Let x be an indeterminate. Then, for all integers $n \geq 1$, we have:*

$$\det_{0\le i,j\le n-1}\left(4^i\binom{x+i+2j+1}{2j+1}+\binom{x-i+2j+1}{2j+1}\right)=2\prod_{i=1}^{n}\frac{2^{2i-1}\,3^{i-1}\,\Gamma(i)\,\Gamma\left(\frac{3i+x}{2}\right)}{\Gamma(2i)\,\Gamma\left(\frac{i+x}{2}\right)}$$
$$=2^{\binom{n+1}{2}+1}3^{\binom{n}{2}}\prod_{i=1}^{n}\frac{i!}{(2i)!}\prod_{i=0}^{n-1}(x+3i+1)_{n-i}. \quad (8.9)$$

Proof. The proof is analogous to the proof of Theorem 14, with the only difference that the computations are heavier, due to the additional parameter x. Since among all determinants in this paper, the ones stated in Theorem 14 require the least computational effort, their parameterized version (8.9) is still doable, while all other parameterized determinants resisted a proof via the holonomic Ansatz, due to their computational complexity (compare the data given in Table 1). □

Theorem 16. *For all positive integers n, we have*

$$\det_{0\le i,j\le n-1}\left(4^i\binom{x+i+2j+3}{2j+3}+\binom{x-i+2j+3}{2j+3}\right)$$
$$=\left(2\cdot 6^{\binom{n}{2}}\prod_{i=0}^{n-1}\frac{i!}{(2i+3)!}\right)\left((x+2)(x+3)\prod_{i=0}^{n-1}(x+3i+1)_{n-i}\right)\times \operatorname{Pol}_n(x), \quad (8.10)$$

where $\operatorname{Pol}_n(x)$ is a monic polynomial in x of degree $2n-2$.

The proof of this theorem is given in the next section.

As already mentioned, we do not know an explicit formula for the polynomials $\operatorname{Pol}_n(x)$ but, experimentally, we found a recurrence that they seem to satisfy.

Conjecture 17. *The polynomial $\operatorname{Pol}_n(x)$ in Theorem 16 is given by the recurrence*

$$\begin{aligned}&3\operatorname{Pol}_{n+3}(x)-2\left(18n^2+9nx+72n-3x^2-3x+49\right)\operatorname{Pol}_{n+2}(x)\\&\quad+\left(135n^4+108n^3x+810n^3-54n^2x^2+108n^2x+1395n^2-52nx^3-510nx^2\right.\\&\qquad\left.-1100nx+120n-9x^4-152x^3-855x^2-1780x-1020\right)\operatorname{Pol}_{n+1}(x)\\&\quad-6(n+1)(n-x-2)(n+x+2)(3n+x+3)(3n+x+5)(3n+x+7)\operatorname{Pol}_n(x)=0\end{aligned}$$

and initial values

$$\begin{aligned}\operatorname{Pol}_1(x)&=1,\\ \operatorname{Pol}_2(x)&=\frac{1}{3}\left(3x^2+31x+60\right),\\ \operatorname{Pol}_3(x)&=\frac{1}{9}\left(9x^4+234x^3+2061x^2+6956x+7680\right).\end{aligned}$$

9. **Proof of Theorem 16**

We now provide our proof of Theorem 16. Essential parts of it are based on several auxiliary results that, for the sake of better readability, are stated and proved separately in Lemmas 18–20 further below.

Proof of Theorem 16. Let us denote the matrix on the left-hand side of (8.10) of which we take the determinant by $D_n(x)$.

We proceed in several steps. First we show that the linear factors that appear on the right-hand side of (8.10) are indeed polynomial factors of $\det D_n(x)$; see Steps 1–6 below. In Step 7, we show that the degree of $\det D_n(x)$ as a polynomial in x is bounded above by $\binom{n+1}{2} + 2n$. Since the prefactor of $\mathrm{Pol}_n(x)$ on the right-hand side of (8.10) has degree $2 + \sum_{i=0}^{n-1}(n-i) = 2 + \binom{n+1}{2}$, this implies that the degree of $\mathrm{Pol}_n(x)$ is at most $2n-2$. We complete the proof by Step 8 in which we compute the coefficient of $x^{\binom{n+1}{2}+2n}$ in the determinant $\det D_n(x)$, and thus the leading coefficient of both $\det D_n(x)$ and $\mathrm{Pol}_n(x)$.

Below, we use the truth function χ which is defined by $\chi(\mathcal{A}) = 1$ if $\mathcal{A}$ is true and $\chi(\mathcal{A}) = 0$ otherwise.

We start with some simple divisibility properties of $\det D_n(x)$.

Step 1. $(x+1)$ *is a factor of* $\det D_n(x)$. This is seen by noting that $(x+1)$ is a factor of each entry in row 0 of $D_n(x)$.

Step 2. $(x+2)^{1+\chi(n\ge 2)}$ *is a factor of* $\det D_n(x)$. On the one hand, $(x+2)$ is also a factor of each entry in row 0 of $D_n(x)$. Moreover, $(x+2)$ is a factor of each entry in row 1.

Step 3. $(x+3)^{1+\chi(n\ge 3)}$ *is a factor of* $\det D_n(x)$. Similarly, also $(x+3)$ is a factor of each entry in row 0 of $D_n(x)$. On the other hand, $(x+3)$ is also a factor of each entry in row 1 and row 2, except for the entries in column 0, which are $4\binom{x+4}{3} + \binom{x+2}{3}$ and $16\binom{x+5}{3} + \binom{x+1}{3}$, respectively. One can check that 4 times the first expression minus the second yields a polynomial that is divisible by $(x+3)$. By an elementary row operation, this implies that, as soon as $n \ge 3$, another term $(x+3)$ divides the determinant $\det D_n(x)$.

Before we continue with the "general" case, we need a few preparations. By inspection of the right-hand side of (8.10), we see that it remains to show that for $4 \le \beta \le 3n-2$ the term

$$\begin{aligned}(x+\beta)^{\#\{i\ge 0: 3i+1\le\beta\le n+2i\}} &= (x+\beta)^{\lfloor(\beta-1)/3\rfloor - \max\{0,\lceil(\beta-n)/2\rceil\}+1}\\ &= (x+\beta)^{\min\{\lfloor(\beta+2)/3\rfloor,\lfloor(\beta+2)/3\rfloor-\lceil(\beta-n)/2\rceil\}}\end{aligned} \tag{9.1}$$

divides $\det D_n(x)$. We are going to do this by applying the idea of "identification of factors" as described in Section 2.4 of [18]. To be precise, in order to prove that $(x+\beta)^E$ is a factor of $\det D_n(x)$, we find E linear combinations of rows of $D_n(x)$ that vanish and that are linearly independent. (In other words, we find E linearly independent vectors in the left kernel of the matrix $D_n(x)$. That the latter is indeed sufficient to infer the claimed divisibility is argued in [17, Sec. 2].)

Our description of these linear combinations of rows of $D_n(x)$ is in terms of generating functions, in complete analogy to the calculus that we applied in Section 5. Namely, by (5.8) and (5.9) the generating function for the entries in row i of our matrix $D_n(x)$

is

$$\begin{aligned}
\sum_{j\ge0}&\left(4^i\binom{x+i+2j+3}{2j+3}+\binom{x-i+2j+3}{2j+3}\right)v^j\\
&=4^i\cdot\frac{1}{v}\left(\frac{1}{2\sqrt{v}}\left((1-\sqrt{v})^{-x-i-1}-(1+\sqrt{v})^{-x-i-1}\right)-(x+i)\right)\\
&\qquad+\frac{1}{v}\left(\frac{1}{2\sqrt{v}}\left((1-\sqrt{v})^{-x+i-1}-(1+\sqrt{v})^{-x+i-1}\right)-(x-i)\right)\\
&=\frac{1}{2v^{3/2}}\Big(4^i\left((1-\sqrt{v})^{-x-i-1}-(1+\sqrt{v})^{-x-i-1}\right)\\
&\qquad\qquad+\left((1-\sqrt{v})^{-x+i-1}-(1+\sqrt{v})^{-x+i-1}\right)\Big)-\frac{1}{v}\left(4^i(x+i)+(x-i)\right).
\end{aligned}\tag{9.2}$$

In view of this expression, Lemma 19 says that, for non-negative real numbers s and t with $t\le s$ such that all of $s+2t$, $2s+t$, $3s$, and $3t$ are integers, we have

$$\sum_{i=0}^{s+2t}(-1)^i2^{s+2t-i}\alpha(i)\left(\sum_{j=0}^{3t}\binom{3t}{j}\binom{2s+t-j}{s+2t-2j-i}\right)\cdot(\text{row } i \text{ of } D_n(-3s-1))=0,\tag{9.3}$$

where $\alpha(i)=1$ if $i>0$ and $\alpha(0)=\frac{1}{2}$. Indeed, Lemma 19 implies that, when we apply generating function calculus to prove (9.3) using (9.2), all powers of $(1-\sqrt{v})$ and $(1+\sqrt{v})$ cancel out. On the other hand, it seems that we would have to check that also the terms that result from the expressions $\frac{1}{v}\left(4^i(x+i)+(x-i)\right)$ on the right-hand side of (9.2) cancel out. That could certainly be done by computing the corresponding binomial sums. However, it comes for free: we use the generating function in (9.2) with $x=-3s-1$ and i in the range $0\le i\le 2s+t$; this implies that

$$x+i=-3s-1+i\le-3s-1+2s+t=-s+t-1\le-1$$

by one of the assumptions of Lemma 19. Therefore, with these choices of x and i, both binomial coefficients in the sum on left-hand side of (9.2) vanish for large enough j. In other words: the generating function in (9.2) is always a polynomial in v. Hence, terms involving negative powers of v must automatically cancel out.

We now discuss the divisibility of $\det D_n(x)$ by the power in (9.1) for the congruence classes of β modulo 3 separately.

Step 4. $(x+3s+1)^{\min\{s+1,s+1-\lceil(3s+1-n)/2\rceil\}}$ *is a factor of* $\det D_n(x)$ *for* $1\le s\le n-1$. The linear combinations of rows of $D_n(-3s-1)$ given in (9.3) vanish for $0\le t\le s$. They are linearly independent since the highest row number involved is $s+2t$, which is different for different t. Another restriction that must be taken into account is that we may only use actually existing rows of $D_n(x)$, meaning that we must have $s+2t\le n-1$. In summary, the number of vanishing linear combinations (9.3) of rows, or, equivalently, the number of integers t with $0\le t\le s$ and $s+2t\le n-1$, equals $\min\{s+1,\lfloor(n-1-s)/2\rfloor+1\}$, which agrees with the claimed exponent.

Step 5. $(x+3s+2)^{\min\{s+1,s+1-\lceil(3s+2-n)/2\rceil}$ *is a factor of* $\det D_n(x)$ *for* $1\le s\le n-2$. We use (9.3) with s replaced by $s+\frac{1}{3}$ and t replaced by $t+\frac{1}{3}$. The conclusion is that the linear combinations of rows of $D_n(-3s-2)$ given by (9.3) vanish for $0\le t\le s$.

They are linearly independent since the highest row number involved is $s+2t+1$, which is different for different t. Another restriction that must be taken into account is that we may only use actually existing rows of $D_n(x)$, meaning that we must have $s+2t+1 \le n-1$. In summary, the number of vanishing linear combinations (9.3) of rows, or, equivalently, the number of integers t with $0 \le t \le s$ and $s+2t+1 \le n-1$, equals $\min\{s+1, \lfloor (n-2-s)/2 \rfloor + 1\}$, which agrees with the claimed exponent.

Step 6. $(x+3s+3)^{\min\{s+1,s+1-\lceil (3s+3-n)/2 \rceil\}}$ *is a factor of* $\det D_n(x)$ *for* $1 \le s \le n-2$. We use (9.3) with s replaced by $s+\frac{2}{3}$ and t replaced by $t+\frac{2}{3}$. The conclusion is that the linear combinations of rows of $D_n(-3s-3)$ given in (9.3) vanish for $0 \le t \le s$. They are linearly independent since the highest row number involved is $s+2t+2$, which is different for different t. Another restriction that must be taken into account is that we may only use actually existing rows of $D_n(x)$, meaning that we must have $s+2t+2 \le n-1$. In summary, the number of vanishing linear combinations (9.3) of rows, or, equivalently, the number of integers t with $0 \le t \le s$ and $s+2t+2 \le n-1$, equals $\min\{s+1, \lfloor (n-3-s)/2 \rfloor + 1\}$, which agrees with the claimed exponent.

Step 7. $\det D_n(x)$ *is a polynomial in* x *of degree at most* $\binom{n+1}{2} + 2n$. To see this, we replace column j of the matrix by

$$\sum_{k=0}^{j} (-1)^{j-k} \binom{j}{k} (2k+3)!\, (-x)_{2j-2k} \cdot (\text{column } k \text{ of the matrix}).$$

Clearly, this can be achieved by elementary column operations. Thereby, the j-th column is multiplied by $(2j+3)!$, and therefore the new determinant equals $\det D_n(x)$ multiplied by

$$\prod_{i=0}^{n-1} (2i+3)!. \tag{9.4}$$

Let $M(n)$ denote the new matrix. The (i,j)-entry of $M(n)$ then is

$$\sum_{k=0}^{j} (-1)^{j-k} \binom{j}{k} (2k+3)!\, (-x)_{2j-2k} \left(4^i \binom{x+i+2k+3}{2k+3} + \binom{x-i+2k+3}{2k+3} \right).$$

Using the standard hypergeometric notation

$${}_rF_s\left[\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; z\right] = \sum_{l=0}^{\infty} \frac{(a_1)_l \cdots (a_r)_l}{l!\, (b_1)_l \cdots (b_s)_l} z^l \ ,$$

we have

$$\begin{aligned} \sum_{k=0}^{j} (-1)^{j-k} \binom{j}{k} (2k+3)!\, (-x)_{2j-2k} \binom{x+i+2k+3}{2k+3} \\ = (-1)^j (-x)_{2j}\, (x+i+1)_3 \cdot {}_3F_2\left[\begin{matrix} \frac{x}{2}+\frac{i}{2}+\frac{5}{2}, \frac{x}{2}+\frac{i}{2}+2, -j \\ \frac{x}{2}-j+1, \frac{x}{2}-j+\frac{1}{2} \end{matrix}; 1\right]. \end{aligned} \tag{9.5}$$

To this ${}_3F_2$-series we apply the transformation formula (see [7, Eq. (3.1.1)])

$${}_3F_2\left[\begin{matrix} a, b, -n \\ d, e \end{matrix}; 1\right] = \frac{(e-b)_n}{(e)_n} {}_3F_2\left[\begin{matrix} -n, b, d-a \\ d, 1+b-e-n \end{matrix}; 1\right],$$

where n is a non-negative integer. Thus, we obtain

$$\sum_{k=0}^{j}(-1)^{j-k}\binom{j}{k}(2k+3)!\,(-x)_{2j-2k}\binom{x+i+2k+3}{2k+3}$$
$$=(-1)^j\frac{(x+i+1)_3\,(-x)_{2j}\,(-j-\frac{i}{2}-\frac{3}{2})_j}{(\frac{x}{2}-j+\frac{1}{2})_j}\,{}_3F_2\left[\begin{matrix}-j,\frac{x}{2}+\frac{i}{2}+2,-j-\frac{i}{2}-\frac{3}{2}\\ \frac{x}{2}-j+1,\frac{i}{2}+\frac{5}{2}\end{matrix};1\right]$$
$$=2^{2j}\,(x+i+1)_3\,(-j-\tfrac{i}{2}-\tfrac{3}{2})_j$$
$$\times\sum_{k=0}^{j}(-1)^{j-k}\binom{j}{k}\frac{(\frac{x}{2}-j+k+1)_{j-k}\,(\frac{x}{2}+\frac{i}{2}+2)_k\,(-j-\frac{i}{2}-\frac{3}{2})_k}{(\frac{i}{2}+\frac{5}{2})_k}.$$

We see that this is a polynomial in x of degree $j+3$, with leading coefficient

$$2^{2j}\,(-j-\tfrac{i}{2}-\tfrac{3}{2})_j\sum_{k=0}^{j}(-1)^{j-k}\binom{j}{k}\frac{2^{-j}\,(-j-\frac{i}{2}-\frac{3}{2})_k}{(\frac{i}{2}+\frac{5}{2})_k}$$
$$=(-1)^j2^j\,(-j-\tfrac{i}{2}-\tfrac{3}{2})_j\,{}_2F_1\left[\begin{matrix}-j,-j-\frac{i}{2}-\frac{3}{2}\\ \frac{i}{2}+\frac{5}{2}\end{matrix};1\right].$$

The ${}_2F_1$-series can be evaluated by means of the Chu–Vandermonde summation (see [21, Eq. (1.7.7); Appendix (III.4)])

$${}_2F_1\left[\begin{matrix}a,-n\\ c\end{matrix};1\right]=\frac{(c-a)_n}{(c)_n},$$

where n is a non-negative integer. After simplification, we see that (9.5) is a polynomial in x of degree $j+3$ with leading coefficient

$$2^j(i+j+4)_j.$$

In its turn, this implies that the (i,j)-entry of $M(n)$ is a polynomial in x of degree $j+3$ with leading coefficient

$$2^j\left(4^i(i+j+4)_j+(-i+j+4)_j\right). \tag{9.6}$$

Consequently, the determinant $\det M(n)$ is a polynomial in x of degree at most

$$\sum_{j=0}^{n-1}(j+3)=\binom{n+1}{2}+2n.$$

Since $\det D_n(x)$ is a scalar multiple of $\det M(n)$, the same degree bound holds for $\det D_n(x)$.

STEP 8. *Computation of the leading coefficient of $Pol_n(x)$.* In the previous step we found that the degree of $\det D_n(x)$ as a polynomial in x is at most $\binom{n+1}{2}+2n$. We are now going to show that this is the exact degree, by computing the coefficient of $x^{\binom{n+1}{2}+2n}$ in $\det D_n(x)$, which then is at the same time the leading coefficient of $\mathrm{Pol}_n(x)$.

In order to compute this coefficient of $x^{\binom{n+1}{2}+2n}$, we should recall from the previous step, that we transformed our original determinant $\det D_n(x)$ into (cf. the sentence containing (9.4))

$$\left(\prod_{i=0}^{n-1}\frac{1}{(2i+3)!}\right)\det M(n), \tag{9.7}$$

where the (i,j)-entry of $M(n)$ is a polynomial in x of degree $j+3$ with leading coefficient given by (9.6). Hence, the leading coefficient of $\det M(n)$ (the coefficient of $x^{\binom{n+1}{2}+2n}$) equals

$$2^{\binom{n}{2}}\det_{0\le i,j\le n-1}\left(4^i(i+j+2)_j+(-i+j+2)_j\right).$$

By applying column operations, this expression can be reduced to

$$2^{\binom{n}{2}}\det_{0\le i,j\le n-1}\left(4^i i^j+(-i)^j\right).$$

By the same argument, this expression equals

$$2^{\binom{n}{2}}\det_{0\le i,j\le n-1}\left(4^i p_j(i)+p_j(-i)\right),$$

where $p_j(t)$ is any monic polynomial in t of degree j. We choose $p_j(t)=(t)_j$, so that we need to evaluate

$$\begin{aligned} 2^{\binom{n}{2}}&\det_{0\le i,j\le n-1}\left(4^i(i)_j+(-i)_j\right)\\ &=2^{\binom{n}{2}}\left(\prod_{j=0}^{n-1}j!\right)\det_{0\le i,j\le n-1}\left(4^i\binom{i+j-1}{j}+\binom{-i+j-1}{j}\right). \end{aligned} \tag{9.8}$$

If we now combine (9.7) and (9.8), and subsequently evaluate the last determinant by means of Theorem 3 with $a=4$ and $x=0$, then we obtain the first expression in parentheses on the right-hand side of (8.10) for the leading coefficient of our determinant $\det D_n(x)$.

This completes the proof of the theorem modulo Lemmas 18–20 which are stated and proved separately below. □

Lemma 18. *For all non-negative integers s, we have*

$$\sum_{i=0}^{s}(-1)^i2^i\binom{s}{i}\left((1-\sqrt{v})^{2s-i}-(1+\sqrt{v})^{2s-i}\right)=0.$$

Proof. By the binomial theorem, we get

$$\begin{aligned} \sum_{i=0}^{s}&(-1)^i2^i\binom{s}{i}\left((1-\sqrt{v})^{2s-i}-(1+\sqrt{v})^{2s-i}\right)\\ &=(1-\sqrt{v})^{2s}\left(1-\frac{2}{1-\sqrt{v}}\right)^s-(1+\sqrt{v})^{2s}\left(1-\frac{2}{1+\sqrt{v}}\right)^s\\ &=(1-\sqrt{v})^{2s}\left(\frac{-1-\sqrt{v}}{1-\sqrt{v}}\right)^s-(1+\sqrt{v})^{2s}\left(\frac{-1+\sqrt{v}}{1+\sqrt{v}}\right)^s\\ &=0, \end{aligned}$$

as desired. □

Lemma 19. *Let*

$$S(y,i):=4^i\left((1-\sqrt{v})^{y-i}-(1+\sqrt{v})^{y-i}\right)+\left((1-\sqrt{v})^{y+i}-(1+\sqrt{v})^{y+i}\right). \tag{9.9}$$

Then, for all non-negative real numbers s and t with $t\le s$ such that all of $s+2t$, $2s+t$, $3s$, and $3t$ are integers, we have

$$\sum_{i=0}^{s+2t}(-1)^i 2^{s+2t-i}\alpha(i)\left(\sum_{j=0}^{3t}\binom{3t}{j}\binom{2s+t-j}{s+2t-2j-i}\right)S(3s,i)=0, \tag{9.10}$$

where $\alpha(i)=1$ if $i>0$ and $\alpha(0)=\frac{1}{2}$.

Remarks. (1) The integrality conditions on the parameters s and t may seem a bit contrived. Indeed, these conditions are equivalent to saying that the pair (s,t) is of the form $(s_1+\frac{u}{3},t_1+\frac{u}{3})$, where all of s_1,t_1,u are non-negative integers. It is exactly in this form in which the lemma is used in Steps 4–6 of the proof of Theorem 16. On the other hand, for the proof of the lemma it is more convenient to have these conditions in this "contrived" form.

(2) The condition $t\le s$ is needed to make sure that the range of the summation index i on the left-hand side of (9.10) does not extend beyond $3s$. The latter would yield negative powers of $(1-\sqrt{v})$ and $(1+\sqrt{v})$ in the definition of $S(3s,i)$, for which the below application of Lemma 18 is not possible.

Proof of Lemma 19. By the definition of $S(y,i)$ in (9.9), the left-hand side in (9.10) equals

$$\begin{aligned}
&\sum_{i=0}^{s+2t}(-1)^i 2^{s+2t-i}\alpha(i)\left(\sum_{j=0}^{3t}\binom{3t}{j}\binom{2s+t-j}{s+2t-2j-i}\right)\\
&\qquad\cdot\left(4^i\left((1-\sqrt{v})^{3s-i}-(1+\sqrt{v})^{3s-i}\right)+\left((1-\sqrt{v})^{3s+i}-(1+\sqrt{v})^{3s+i}\right)\right)\\
&=\sum_{i=0}^{s+2t}(-1)^i 2^{s+2t+i}\alpha(i)\left(\sum_{j=0}^{3t}\binom{3t}{j}\binom{2s+t-j}{s+2t-2j-i}\right)\left((1-\sqrt{v})^{3s-i}-(1+\sqrt{v})^{3s-i}\right)\\
&\quad+\sum_{i=0}^{s+2t}(-1)^i 2^{s+2t-i}\alpha(i)\left(\sum_{j=0}^{3t}\binom{3t}{j}\binom{2s+t-j}{s+2t-2j-i}\right)\\
&\qquad\cdot\left((1-\sqrt{v})^{3s+i}-(1+\sqrt{v})^{3s+i}\right)\\
&=\sum_{i=0}^{s+2t}(-1)^i 2^{s+2t+i}\alpha(i)\left(\sum_{j=0}^{3t}\binom{3t}{j}\binom{2s+t-j}{s+2t-2j-i}\right)\left((1-\sqrt{v})^{3s-i}-(1+\sqrt{v})^{3s-i}\right)\\
&\quad+\sum_{i=-s-2t}^{0}(-1)^i 2^{s+2t+i}\alpha(i)\left(\sum_{j=0}^{3t}\binom{3t}{j}\binom{2s+t-j}{s+2t-2j+i}\right)\\
&\qquad\cdot\left((1-\sqrt{v})^{3s-i}-(1+\sqrt{v})^{3s-i}\right).
\end{aligned}$$

Now we use Lemma 20 to replace the term $-i$ in the first sum over j by $+i$. Having done this, the two sums over i can now be "concatenated" into one sum,

$$\sum_{i=-s-2t}^{s+2t}(-1)^i 2^{s+2t+i}\left(\sum_{j=0}^{3t}\binom{3t}{j}\binom{2s+t-j}{s+2t-2j+i}\right)\left((1-\sqrt{v})^{3s-i}-(1+\sqrt{v})^{3s-i}\right)$$
$$=\sum_{j=0}^{3t}\binom{3t}{j}\sum_{i=-s-2t}^{s+2t}(-1)^i 2^{s+2t+i}\binom{2s+t-j}{s+2t-2j+i}\left((1-\sqrt{v})^{3s-i}-(1+\sqrt{v})^{3s-i}\right)$$
$$=\sum_{j=0}^{3t}\binom{3t}{j}\sum_{i=0}^{2s+t-j}(-1)^i 2^{i+2j}\binom{2s+t-j}{i}\left((1-\sqrt{v})^{4s+2t-2j-i}-(1+\sqrt{v})^{4s+2t-2j-i}\right),$$

where we performed the shift of index $i\mapsto i-s-2t+2j$ to obtain the last line. For fixed j, the inner sum over i vanishes due to Lemma 18 with s replaced by $2s+t-j$. This proves the assertion of the lemma. □

Lemma 20. *For all non-negative integers s, t, and i, the sum*

$$\sum_{j=0}^{3t}\binom{3t}{j}\binom{2s+t-j}{s+2t-2j-i} \tag{9.11}$$

is invariant under the replacement $i\mapsto -i$.

Proof. We write the sum in (9.11) in terms of a complex contour integral. We have

$$\sum_{j=0}^{3t}\binom{3t}{j}\binom{2s+t-j}{s+2t-2j-i}=\sum_{j=0}^{3t}\frac{1}{(2\pi \mathbf{i})^2}\int_{C_x}\int_{C_y}\frac{(1+y)^{3t}}{y^{3t-j+1}}\frac{(1+x)^{2s+t-j}}{x^{s+2t-2j-i+1}}\,dy\,dx,$$

where C_x and C_y are contours encircling the origin once in positive direction. We assume in both cases that the contours are strictly contained in the unit disk which has the origin as centre. In the formula above, $\mathbf{i}$ stands for $\sqrt{-1}$.

We may extend the sum to all non-negative j because this only adds vanishing terms. Moreover, since we assumed that along the contours the moduli of x and y are always strictly less than 1, we may interchange integrals and sum and then evaluate the arising geometric series. The conclusion is that the sum in (9.11) is equal to

$$\frac{1}{(2\pi \mathbf{i})^2}\int_{C_x}\int_{C_y}\frac{(1+y)^{3t}}{y^{3t+1}}\frac{(1+x)^{2s+t}}{x^{s+2t-i+1}}\frac{1}{1-\frac{x^2y}{1+x}}\,dy\,dx.$$

Now we may blow up the contour C_y. We will pick up a residue at the singularity $y=\frac{1+x}{x^2}$. On the other hand, since (for fixed x) the integrand is of the order $O(y^{-2})$ as $|y|\to\infty$, the limit of the integral as the contour tends to infinity vanishes. In summary, this leads to the expression

$$\frac{1}{2\pi \mathbf{i}}\int_{C_x}\frac{\left(1+\frac{1+x}{x^2}\right)^{3t}}{\left(\frac{1+x}{x^2}\right)^{3t+1}}\frac{(1+x)^{2s+t+1}}{x^{s+2t-i+3}}\,dx=\frac{1}{2\pi \mathbf{i}}\int_{C_x}\frac{\left(1+x+x^2\right)^{3t}(1+x)^{2s-2t}}{x^{s+2t-i+1}}\,dx$$

for the sum in (9.11).

Our task is to show that the last expression is invariant under the replacement $i\mapsto -i$. Indeed, the substitution $x\mapsto\frac{1}{x}$ turns this expression into itself with $+i$ in place of $-i$. This completes the proof of the lemma. □

Remark. It would be possible, using Lemmas 18–20, to provide an alternative proof of Theorem 15. We are also convinced that proofs in a similar style of Theorems 6 and 8 are possible. In its turn, via the determinantal relations established in Lemmas 7 and 9, this would yield new proofs of the enumerative results in [2].

10. Variations on the theme, IV

We conclude with further variations of Theorem 4. Here, the power 2^i remains unchanged, but the terms $2j$ in the binomials are replaced by $4j$. Conjecture 21 contains the determinant evaluation of this type that we found experimentally which does not contain any shifts. We applied the holonomic Ansatz, and we are confident that it would go through once our computers are "strong" enough to carry out the necessary calculations. At this point in time, however, we must leave the determinant evaluation as a conjecture. Moreover, we performed again an automated search for determinant evaluations where shifts are allowed. This led to the — again conjectural — discovery of many more determinant evaluations; see Proposition 22 and Conjecture 23. The same remark applies here: we are confident that all of these could be proved by the holonomic Ansatz once our computers dispose of sufficient computational power.

Conjecture 21. *For all positive integers n, we have*

$$\det_{0\le i,j\le n-1}\left(2^i\binom{i+4j+3}{4j+3}+\binom{-i+4j+3}{4j+3}\right)$$
$$=\frac{2^{n^2-n+1}3^{2n}5^{-\frac{5}{8}n^2+\frac{5}{4}n}(2n)!\,(\frac{2}{3})_n\prod_{i=1}^{n}\frac{(6i-4)!}{(5i)!}}{\prod_{i=1}^{\lfloor (n+3)/4\rfloor}(\frac{1}{5})_{n+3-4i}\prod_{i=1}^{\lfloor (n+2)/4\rfloor}(\frac{2}{5})_{n+2-4i}\prod_{i=1}^{\lfloor (n+2)/4\rfloor}(\frac{3}{5})_{n+1-4i}\prod_{i=1}^{\lfloor (n+2)/4\rfloor}(\frac{4}{5})_{n-4i}}$$
$$\times\begin{cases}5^{3/8}, & \text{for } n=4m-3,\\ 1, & \text{for } n=4m-2,\\ 5^{-1/8}, & \text{for } n=4m-1,\\ 1, & \text{for } n=4m,\end{cases}\tag{10.1}$$

As in previous sections, we performed a systematic search for determinants of the same form. Let us denote

$$G_{\alpha,\beta,\gamma,\delta}(n):=\det_{0\le i,j\le n-1}\left(2^{i+\beta}\binom{i+4j+\gamma}{4j+\alpha}+\binom{-i+4j+\delta}{4j+\alpha}\right).$$

In the parameter space $-6\le\alpha,\beta\le 9$ and $-9\le\gamma,\delta\le 9$, we have identified 18 cases of determinants that factor completely. Unfortunately, we were not able to prove their conjectured evaluations, but at least we can state some simple relationships.

Proposition 22. *For all integers $n\ge 2$ we have the following relations:*

$$G_{0,1,-2,-4}(n)=3\,G_{4,2,3,-1}(n-1)=3\,G_{8,3,8,2}(n-2),\tag{10.2}$$
$$G_{1,1,0,-2}(n)=-2\,G_{5,2,5,1}(n-1),\tag{10.3}$$
$$G_{3,3,2,-4}(n)=-20\,G_{7,4,7,-1}(n-1).\tag{10.4}$$

Proof. These identities can easily be established by exploiting the block structure of the matrices $G_{0,1,-2,-4}(n)$ respectively $G_{4,2,3,-1}(n)$, $G_{1,1,0,-2}(n)$, $G_{3,3,2,-4}(n)$, which have a block of zeros (of size $2\times(n-2)$ respectively $1\times(n-1)$) in their upper right corner. $\square$

Conjecture 23. *The following determinant evaluations hold for all $n\geq 1$:*

$$G_{0,2,3,-1}(n)=\prod_{i=1}^{n}\frac{(2i-1)(4i-3)(4i-1)\,\Gamma\big(6i\big)\,\Gamma\big(\frac{i+3}{4}\big)}{i(i+1)(i+2)(3i-1)\,\Gamma\big(5i-1\big)\,\Gamma\big(\frac{5i+3}{4}\big)},\tag{10.5}$$

$$G_{1,3,6,0}(n)=\prod_{i=1}^{n}\frac{8(2i-1)(2i+1)^2(4i-1)(4i+1)\,\Gamma\big(6i+2\big)\,\Gamma\big(\frac{i+2}{4}\big)}{(i+1)(i+2)(i+3)(i+4)\,\Gamma\big(5i+2\big)\,\Gamma\big(\frac{5i+6}{4}\big)},\tag{10.6}$$

$$G_{1,1,0,-2}(n)=-4\prod_{i=1}^{n}\frac{(3i-2)\,\Gamma\big(6i-5\big)\,\Gamma\big(\frac{i}{4}\big)}{8\,\Gamma\big(5i-4\big)\,\Gamma\big(\frac{5i}{4}\big)},\tag{10.7}$$

$$G_{3,0,3,3}(n)=2\prod_{i=1}^{n}\frac{\Gamma\big(6i-1\big)\,\Gamma\big(\frac{i+3}{4}\big)}{\Gamma\big(5i\big)\,\Gamma\big(\frac{5i-1}{4}\big)},\tag{10.8}$$

$$G_{2,1,2,0}(n)=\prod_{i=1}^{n}\frac{\Gamma\big(6i-1\big)\,\Gamma\big(\frac{i+2}{4}\big)}{2(2i-1)\,\Gamma\big(5i-1\big)\,\Gamma\big(\frac{5i-2}{4}\big)}.\tag{10.9}$$

Moreover, the following identities are conjectured to hold for all $n\geq 3$:

$$\begin{aligned}G_{3,0,3,3}(n)&=\frac{2}{3}\,G_{0,1,-2,-4}(n+1)=-\frac{1}{672}\,G_{1,3,-2,-8}(n+1)\\&=\frac{1}{63}\,G_{5,4,3,-5}(n)=\frac{4}{1002001}\,G_{6,6,3,-9}(n)=-\frac{8}{5}\,G_{9,5,8,-2}(n-1),\end{aligned}\tag{10.10}$$

$$G_{1,1,0,-2}(n)=-\frac{1}{49}\,G_{2,3,0,-6}(n)=-\frac{2}{7}\,G_{6,4,5,-3}(n-1)=-\frac{4}{5577}\,G_{7,6,5,-7}(n-1),\tag{10.11}$$

$$G_{2,1,2,0}(n)=2\,G_{7,4,7,-1}(n-1).\tag{10.12}$$

Note that (10.8) is the same determinant as in (10.1).

11. **Open questions**

In this paper, we have proven 68 determinant evaluations that are inspired by Di Francesco's determinant for twenty-vertex configurations (Theorems 3, 4, 6, 8, 10, 12, 14, 15, 16). We found another 21 determinants that seem to have a nice closed form, which we unfortunately were not able to prove (Conjectures 13, 21, 23, and Proposition 22). Also the precise description of the polynomial factor in $F_{3,0,x+3,x+3}(n)$ is left as an open problem (Conjecture 17). Most of these determinants were found by computer search in certain ranges, and some of them were found to belong to infinite families. It would be interesting to study whether the remaining ones, which at the moment seem to be "sporadic" and unsystematic cases, can be explained and characterized, and whether there are more examples outside of our search ranges.

Another intriguing question concerns the existence of q-analogues of the presented determinant formulas. Indeed, Theorem 3 has the following q-analogue.

Theorem		3	4	6	8	10	12	13	14	15	23
$\mathfrak{I}$	hol. rank	2	3	3	3	3	4	4	3	3	5
	degree in n	2	11	14	14	10–14	7–8	7	5	5	14–19
	degree in j	4	11	12	12	7–11	8–17	9–11	5–8	9	10–15
	`ByteCount`	0.03	0.16	2.12	1.94	0.08–0.17	0.06–0.23	0.26–0.51	0.03–0.05	0.26	0.24–0.52
(H1)	order of rec.	2	3	3	3	3	4	4	3	3	5
	degree in n	3	15	25	25	12–21	18–26	21–23	6–10	11	49–59
(H2)	time 1. sum	0.006	0.67	-	-	0.18–0.74	2.26–24.6	-	0.01–0.03	2.54	-
	time 2. sum	0.009	0.71	-	-	0.19–0.83	3.36–20.8	-	0.01–0.04	7.18	-
	hol. rank	3	6	-	-	6	10	-	7	7	-
	`ByteCount`	0.03	1.32	-	-	0.71–1.62	0.47–2.51	-	0.04–0.19	2.08	-
(H3)	time 1. sum	0.005	0.77	21.7	17.0	0.47–0.92	7.39–12.2	-	0.02–0.04	0.35	-
	time 2. sum	0.011	0.6	-	65.7	0.35–0.71	3.81–8.42	-	0.01–0.03	0.97	-
	order of rec.	2	6	-	6	6	10	-	5	5	-
	degree in n	1	52	-	75	45–57	74–93	-	14–22	24	-

TABLE 1. Computational data from the proofs by holonomic Ansatz; if there is more than a single determinant in a theorem, the range of values over all instances is displayed; if in such a situation only a single value appears, it means that all instances had the same value. `ByteCount` refers to the homonymous *Mathematica* command (applied to the final annihilator, not the intermediate creative telescoping results) and the values are given in MB. All timings are given in hours.

Theorem 24. *For all non-negative integers n, we have*

$$\det_{0\le i,j\le n-1}\left(a^i\frac{(xq^{1+i};q)_j}{(q;q)_j}+\frac{(xq^{1-i};q)_j}{(q;q)_j}\right)=2\,q^{-\binom{n}{3}}(-x)^{\binom{n}{2}}\prod_{i=0}^{n-1}(aq^i;q)_i, \tag{11.1}$$

where $(\alpha;q)_p:=(1-\alpha)(1-q\alpha)\cdots(1-q^{p-1}\alpha)$ for $p\ge 1$ and $(\alpha;q)_0:=1$.

Proof. We begin in the spirit of the second proof of Theorem 3. In particular, we adopt the constant-term notation from there.

Using the q-binomial theorem (see [7, Eq. (1.3.2); Appendix (II.3)])

$$\sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}z^k=\frac{(az;q)_\infty}{(z;q)_\infty},$$

our determinant on the left-hand side of (11.1) can be written as

$$\begin{aligned}\mathrm{CT}_{\mathbf{z}}&\det_{0\le i,j\le n-1}\left(z_i^{-j}\left(a^i\frac{(xz_iq^{1+i};q)_\infty}{(z_i;q)_\infty}+\frac{(xz_iq^{1-i};q)_\infty}{(z_i;q)_\infty}\right)\right)\\&=\mathrm{CT}_{\mathbf{z}}\left(\prod_{i=0}^{n-1}\frac{z_i^{-n+1}}{(z_i;q)_\infty}\left(a^i(xz_iq^{1+i};q)_\infty+(xz_iq^{1-i};q)_\infty\right)\right)\det_{0\le i,j\le n-1}\left(z_i^{n-j-1}\right).\end{aligned}$$

The last determinant can be evaluated by means of the evaluation of the Vandermonde determinant. Thus, we obtain

$$\mathrm{CT}_{\mathbf{z}}\left(\prod_{i=0}^{n-1}\frac{(xz_iq^n;q)_\infty}{z_i^{n-1}\,(z_i;q)_\infty}\left(a^i(xz_iq^{1+i};q)_{n-i-1}+(xz_iq^{1-i};q)_{n+i-1}\right)\right)\prod_{0\le i<j\le n-1}(z_i-z_j)$$

for the determinant on the left-hand side of (11.1). Again, since this is a constant term, we get the same value if we permute the variables $z_0, z_1, \dots, z_{n-1}$. So, we symmetrize the last expression and get

$$\frac{1}{n!}\operatorname{CT}_{\mathbf{z}}\left(\prod_{i=0}^{n-1}\frac{(xz_iq^n;q)_\infty}{z_i^{n-1}\,(z_i;q)_\infty}\right)\left(\prod_{0\le i<j\le n-1}(z_i-z_j)\right)\\ \times \det_{0\le i,j\le n-1}\left(a^i(xz_jq^{1+i};q)_{n-i-1}+(xz_jq^{1-i};q)_{n+i-1}\right)$$

for our determinant. The determinant in the above expression is a polynomial in $xz_0, xz_1, \dots, xz_{n-1}$, which is skew-symmetric in these quantities. Hence, it is divisible by the Vandermonde product

$$\prod_{0\le i<j\le n-1}(xz_i-xz_j)=x^{\binom n2}\prod_{0\le i<j\le n-1}(z_i-z_j).$$

This shows that the determinant on the left-hand side of (11.1) equals

$$\frac{x^{\binom n2}}{n!}\operatorname{CT}_{\mathbf{z}}\left(\prod_{i=0}^{n-1}\frac{(xz_iq^n;q)_\infty}{z_i^{n-1}\,(z_i;q)_\infty}\right)\left(\prod_{0\le i<j\le n-1}(z_i-z_j)^2\right)f(xz_0,xz_1,\dots,xz_{n-1}),$$

where $f(xz_0, xz_1, \dots, xz_{n-1})$ is some polynomial in the given quantities.

Now, repeating arguments from the proof of Theorem 3, the square of the Vandermonde product, $\prod_{0\le i<j\le n-1}(z_i-z_j)^2$, is a homogeneous polynomial of degree $n(n-1)$. Moreover, it is not very difficult to see that the coefficient of $(z_0z_1\cdots z_{n-1})^{n-1}$ in it equals $(-1)^{\binom n2}n!$. In view of what we have found so far, this implies that the determinant on the left-hand side of (11.1) equals

$$(-x)^{\binom n2}f(0,0,\dots,0).$$

It remains to compute the constant $f(0,0,\dots,0)$. By inspection, the highest power of x in the determinant on the left-hand side of (11.1) is exactly $x^{\binom n2}$. Thus, we will obtain $(-1)^{\binom n2}f(0,0,\dots,0)$ if we take the highest coefficient of each individual entry of this determinant, that is, if we compute

$$\det_{0\le i,j\le n-1}\left(a^i\frac{(-1)^jq^{\binom j2}\,(q^{1+i})^j}{(q;q)_j}+\frac{(-1)^jq^{\binom j2}\,(q^{1-i})^j}{(q;q)_j}\right)\\ =(-1)^{\binom n2}q^{\binom n3+\binom n2}a^{\frac12\binom n2}\left(\prod_{j=0}^{n-1}\frac{1}{(q;q)_j}\right)\det_{0\le i,j\le n-1}\left(\left(a^{1/2}q^j\right)^i+\left(a^{1/2}q^j\right)^{-i}\right).$$

This determinant can be evaluated by means of [18, Eq. (2.5)]. After some simplification, one obtains the desired result. □

However, (so far?) we were not able to find q-analogues of any of the other determinant evaluations proved or conjectured in this paper.

Since Di Francesco's original determinant arose in combinatorics, we propose as a future research direction to come up with combinatorial interpretations of our "variations on the theme". In this regard, we report two compelling coincidences, where some of our *product formulas* appear in a seemingly unrelated — combinatorial — context. These may hint at where to look for such combinatorial interpretations.

Namely, in [6], Fischer and Schreier-Aigner consider the (-1)-enumeration of arrowed Gelfand–Tsetlin patterns. These are intimately related to alternating sign matrices, and thus to configurations in the six-vertex model. The main results in [6] "overlap" with two of our results. However, what the exact relationship is, is mysterious to us, as we now explain.

In Theorem 1 of [6] it is shown that a certain (-1)-enumeration of arrowed Gelfand–Tsetlin patterns is given by

$$2^n \prod_{i=1}^{n} \frac{(m-n+3i+1)_{i+1}\,(m-n+i+1)_i}{\left(\frac{m-n+i+2}{2}\right)_{i-1}\,(i)_i}.$$

It is not difficult to see that, if in this expression we replace m by $x+n-1$, then we obtain exactly the right-hand side of (1.3) multiplied by 2^{n-1}. Although Fischer and Schreier-Aigner also obtain the above formula by a determinant evaluation, the relationship with our determinant and the enumeration of domino tilings of generalized Aztec triangles (cf. the proof of Theorem 6 in Section 5) eludes us.

On the other hand, in Theorem 2 of [6] Fischer and Schreier-Aigner consider another (-1)-enumeration of arrowed Gelfand–Tsetlin patterns and find that it is given by

$$3^{\binom{n+1}{2}} \prod_{i=1}^{n} \frac{(2n+m+2-3i)_i}{(i)_i} = 3^{\binom{n+1}{2}} \prod_{i=1}^{n} \frac{(m-n+3i+2)_{n-i}}{(i)_i}.$$

Here, visibly, if in this expression we replace m by $x+n-1$, then we obtain the right-hand side of (8.9) multiplied by $2^{-\binom{n}{2}-1}3^n$. Again, Fischer and Schreier-Aigner obtain this formula by a determinant evaluation, which however does not help us to understand what this has to do with our determinant.

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