Exact lower bounds for monochromatic Schur triples and generalizations

Christoph Koutschan and Elaine Wong

Dedicated to Peter Paule, our academic father and grandfather. Peter, we wish you many more happy, healthy, and productive years.

Abstract We derive exact and sharp lower bounds for the number of monochromatic generalized Schur triples $(x, y, x + ay)$ whose entries are from the set $\{1, \dots, n\}$, subject to a coloring with two different colors. Previously, only asymptotic formulas for such bounds were known, and only for $a \in \mathbb{N}$. Using symbolic computation techniques, these results are extended here to arbitrary $a \in \mathbb{R}$. Furthermore, we give exact formulas for the minimum number of monochromatic Schur triples for $a = 1, 2, 3, 4$, and briefly discuss the case $0 < a < 1$.

1 Introduction and historical background

Let $\mathbb{N}$ denote the set of positive integers. A triple $(x, y, z) \in \mathbb{N}^3$ is called a Schur triple if its entries satisfy the equation $x + y = z$. The set $\{1, \dots, n\}$ of all positive integers up to n will be denoted by $[n]$. A coloring of $[n]$ is a map $\chi\colon [n] \to C$ for some finite set C of colors. For example, a map $\chi\colon [n] \to \{\text{red}, \text{blue}\}$ is a 2-coloring. We say that a Schur triple is monochromatic (with respect to a given coloring) if all of its entries have been assigned the same color; we will abbreviate "monochromatic Schur triple" by MST.

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With these notations, one can ask questions like: given $n \in \mathbb{N}$ and a coloring χ of $[n]$, how many MSTs are there in $[n]^3$? Let us denote this number as follows:

$$\mathcal{M}(n,\chi) := \left|\left\{(x,y,z) \in [n]^3 : z = x + y \wedge \chi(x) = \chi(y) = \chi(z)\right\}\right|. \tag{1}$$

For our purposes, two Schur triples $(x, y, x + y)$ and $(y, x, x + y)$ are considered distinct if $x \neq y$. We emphasize this convention since sometimes in the literature these two triples are counted only once, which is equivalent to imposing the extra condition $x \leq y$. For example, there are four monochromatic Schur triples on $[6] = \{1, \ldots, 6\}$ when 2 and 4 are colored red and 1, 3, 5, 6 are colored blue, namely $(1, 5, 6)$, $(2, 2, 4)$, $(3, 3, 6)$, and $(5, 1, 6)$. We will use a short-hand notation for 2-colorings, namely as words on the alphabet $\{R, B\}$: the i-th letter is R if the integer i is colored red and B if it is blue. So the above 2-coloring would be denoted by $BRBRBB$. We will also make use of the power notation for words, e.g., $R^2B^3 = RRBBB$.

The namesake of the triples in this work refers to Issai Schur [11], who in 1917 studied a modular version of Fermat's last theorem (first formulated and proved by Leonard Dickson). In order to give a simpler proof of the theorem, Schur introduced a *Hilfssatz* confirming the existence of a least positive integer $n = n(m)$ such that for any m-coloring of $[n]$ an MST exists (this is nowadays known as Schur's theorem). In 1927, Van der Waerden [15] generalized this result to monochromatic arithmetic progressions of any length k. Then in 1928, Ramsey proved his eponymous theorem, showing the existence of a least positive integer n such that every edge-coloring of a complete graph on n vertices, with the colors red and blue, admits either a complete red subgraph or a complete blue subgraph. However, a real increase in the popularity of these kinds of Ramsey-theoretic problems came with the rediscovery of Ramsey's theorem in a 1935 paper of Erdős and Szekeres [4], which ultimately led to a simpler proof of Schur's theorem, indicating their close connections. For the curious reader, this rich history is beautifully depicted in a book by Landman and Robertson [8].

We now arrive at a point of more than just questions of existence. In 1959, Alan Goodman [5] studied the *minimum* number of monochromatic triangles under a 2-edge coloring of a complete graph on n vertices. Then in 1996, Graham, Rödl, and Ruciński [6] found it natural to extend the problem of "determining the minimum number under any 2-coloring" to Schur triples. In fact, Graham offered a prize of 100 USD for an answer to such a question; it has subsequently been successfully answered many times over, in an asymptotic sense. In order to give some more context to this problem, we first introduce some additional notation.

We start by wondering about what we can say about the number of MSTs on $[n]$ if we do not prescribe a particular coloring. It is not difficult to calculate that there are exactly $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1) = \binom{n+1}{2}$ Schur triples on $[n]$. Trivially, this yields an upper bound for the number of MSTs, which can be achieved by coloring all numbers with the same color. This is the reason why it is more natural (and more interesting!) to ask for a lower bound for $\mathcal{M}(n,\chi)$, that is: for given $n \in \mathbb{N}$, what is the "best" lower bound for the number of MSTs regardless of the choice of coloring? Of course, 0 is a trivial such lower bound, but we are aiming for something sharp, in the sense that for each n there exists a coloring for which this bound is actually attained.

Differently stated, we are looking for the minimal number of monochromatic Schur triples among all possible colorings of $[n]$:

$$\mathcal{M}(n) := \min_{\chi\colon [n]\to\{R,B\}} \mathcal{M}(n,\chi). \tag{2}$$

For example, for $n = 6$, one cannot avoid the occurrence of monochromatic Schur triples, but there exists a 2-coloring for which only a single such triple occurs, namely the triple $(1,1,2)$ for the coloring $RRBBBR$. Hence, $\mathcal{M}(6) = \mathcal{M}(6, RRBBBR) = 1$.

As mentioned before, this problem was only studied from an asymptotic point of view: Robertson and Zeilberger [9] was first to give the lower bound $\frac{1}{22}n^2 + O(n)$ as $n \to \infty$ (and consequently won Graham's cash prize), where it has to be noted that they count only Schur triples $(x, y, x + y)$ with the condition $x \leq y$ imposed. This lower bound was independently confirmed by Datskovsky [3], Schoen [10], and Thanatipanonda [13]. Schoen also provided a proof of an "optimal" coloring of $[n]$ that would give such a minimum number, and such a coloring is what we assume later in this paper. The asymptotic lower bounds for the generalized Schur triples case $(x, y, x + ay)$ for $a \geq 2$ is $\frac{1}{2a(a^2+2a+3)}n^2 + O(n)$ as $n \to \infty$, without the requirement of $x \leq y$. This was conjectured by Thanatipanonda [13] and Butler, Costello, and Graham [1], and subsequently proven in 2017 by Thanatipanonda and Wong [14].

In this paper, we take a slightly different approach by using known computer algebra techniques and creative simplifications to develop exact formulas for the minimum number of such triples (in both the Schur triples case and the generalized Schur triples case) and give an analysis of the transitional behavior between the cases. Thus, in order to keep some consistency for comparison, we will remove the assumption of $x \leq y$ when counting MSTs. In this way, we can explain why the behavior of the minimum number of triples jumps when moving from the case $a = 1$ to the case $a \geq 2$ (note that the above asymptotic formula does not specialize to the expected prefactor $\frac{1}{11}$ when $a = 1$ is substituted).

The overall plan is to systematically exploit the full force of symbolic computation and perform a complete analysis of determining the minimum number of monochromatic triples $(x, y, x + ay)$ in both the discrete context ($a \in \mathbb{N}$) and the continuous context ($a \in \mathbb{R}^+$). This requires three courses of a mathematical meal. We serve an appetizer in Section 2, showing how to derive an exact formula for the minimum in the classic Schur triple case (corresponding to $a = 1$ in the general equation). This sets us up for the main course in Section 3, where we perform a full analysis for $a > 0$, illustrating that a global minimum can always be found. Interesting transitional behaviors occur at many locations for $a \in (0, 1)$ and one key transition occurs at $a \approx 1.17$. Admittedly, this course may be a bit difficult to swallow, and we hope that the reader will not suffer from indigestion. For dessert, we follow the procedure described in Section 2, and illustrate how it can systematically produce (ostensibly, an infinite number) of exact formulas for the minimum number of generalized Schur triples. Accordingly, in Section 4, we leave the reader with exact formulas for the minimum number of generalized Schur triples for $a = 2, 3, 4$, and $a = \frac{1}{2}$, with the hope that s/he will leave satisfied.

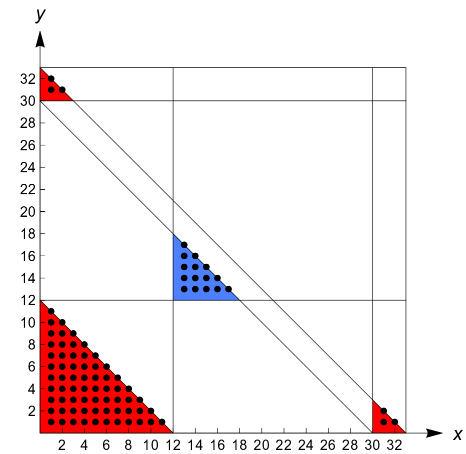

Fig. 1 All $\mathcal{M}(33) = 87$ monochromatic Schur triples for $s = 12$ and $t = 30$ with corresponding coloring $R^{12}B^{18}R^3$; each triple $(x, y, x + y)$ is represented by a dot at position (x, y). The vertical lines are given by $x = s$, $x = t$, and $x = n$, the horizonal ones by $y = s$, $y = t$, and $y = n$. The three diagonal lines visualize the equations $x + y = s$, $x + y = t$, and $x + y = n$.

For the reader's convenience, all computations and diagrams are in the Mathematica notebook [7] that accompanies this paper, freely available at the first author's website.

2 Exact lower bound for monochromatic Schur triples

It has been shown previously [9, 10] that for fixed n the number $\mathcal{M}(n, \chi)$ is minimized when χ consists of three blocks of numbers with the same color ("runs"), i.e., when χ is of the form $R^s B^{t-s} R^{n-t}$, where s and t are approximately $\frac{4}{11}n$ and $\frac{10}{11}n$, respectively. In this section, we derive exact expressions for the optimal choice of s and t, as well as for the corresponding minimum $\mathcal{M}(n)$.

Lemma 1 *Let $n, s, t \in \mathbb{N}$ be such that $1 \le s \le t \le n$. Moreover, assume that the inequalities $t \ge 2s$ and $s \ge n - t$ hold. Then the number of monochromatic Schur triples on $[n]$ under the coloring $R^s B^{t-s} R^{n-t}$, denoted by $\mathcal{M}(n, s, t)$, is exactly*

$$\mathcal{M}(n, s, t) = \frac{s(s-1)}{2} + \frac{(t-2s)(t-2s-1)}{2} + (n-t)(n-t-1). \tag{3}$$

Proof In Figure 1 the situation is depicted for $n = 33$, $s = 12$, and $t = 30$. One sees that the dots representing the MSTs are arranged in four regions of right triangular shape. The triangles arise as follows:

1. The dots in the lower left corner correspond to red MSTs all of whose components are taken from the first block of red numbers; hence there are $s-1$ dots in the first row of this triangle.
2. The triangle in the center contains all blue MSTs, whose first two components (x, y) satisfy the inequalities $x > s$, $y > s$, and $x + y \leq t$. Note that such MSTs only exist if $t \geq 2s + 2$ (for $t = 2s + 1$ and $t = 2s$ the second term in (3) vanishes and the formula is still correct). The number of dots on each side is therefore $t - 2s - 1$.
3. The two triangles in the upper left and lower right corners correspond to red MSTs, whose first two entries belong to different blocks of red numbers. By symmetry they have the same shape and they have $n - t - 1$ dots on their sides. Here we use the condition $s \geq n - t$, because otherwise these two regions would no longer be triangles and we would be counting different things beyond the scope of our assumptions.

Adding up the contributions from these three cases, one obtains the claimed formula. □

The optimal values for s and t are easily derived using the techniques of multivariable calculus, once the form $R^s B^{t-s} R^{n-t}$ is assumed: by letting n go to infinity and by scaling the square $[0, n]^2 \subset \mathbb{R}^2$ to the unit square $[0, 1]^2$, we see that the portion of pairs $(x, y) \in [n]^2$ for which $(x, y, x + y)$ is an MST among all pairs in $[n]^2$ equals the area of a certain region in the unit square; for example, see the shaded regions in Figure 1. In this limit process, the integers s and t turn into real numbers satisfying $0 \leq s \leq t \leq 1$. According to (3) the area of the shaded region in Figure 1 is given by the formula

$$A(s,t) = \frac{s^2}{2} + \frac{(t-2s)^2}{2} + 2 \cdot \frac{(1-t)^2}{2} = \frac{5s^2}{2} + \frac{3t^2}{2} - 2st - 2t + 1.$$

Equating the gradient

$$\left(\frac{\partial A}{\partial s}, \frac{\partial A}{\partial t}\right) = (5s - 2t, 3t - 2s - 2)$$

to zero, one immediately gets the location of the minimum $(s, t) = \left(\frac{4}{11}, \frac{10}{11}\right)$.

Lemma 2 *For fixed $n \in \mathbb{N}$, the integers s_0 and t_0 that minimize the function $\mathcal{M}(n, s, t)$ are given by*

$$s_0 = \left\lfloor \frac{4n+2}{11} \right\rfloor \qquad \textit{and} \qquad t_0 = \left\lfloor \frac{10n}{11} \right\rfloor.$$

Proof Strictly speaking, we prove the minimality of the function $\mathcal{M}(n, s, t)$ under the additional assumption $t \geq 2s \wedge s \geq n - t$ from Lemma 1. The fact that this is also the global minimum for all $1 \leq s \leq t \leq n$ follows as a special case from the more general discussion as described in the proof of Lemma 4.

The statement is proven by case distinction into 11 cases, according to the remainder n modulo 11. Here we show details for the case $n = 11k + 5$, and the remaining cases can be similarly verified with a computer; for these cases we refer the reader to the accompanying electronic material [7].

By setting $n = 11k + 5$ we can eliminate the floors from the definitions of s_0 and t_0; we obtain $s_0 = \lfloor \frac{1}{11}(4n+2) \rfloor = 4k + 2$ and $t_0 = \lfloor \frac{10}{11} n \rfloor = 10k + 4$. Our goal is to show that among all integers $i, j \in \mathbb{Z}$ the expression $\mathcal{M}(n, s_0 + i, t_0 + j)$ is minimal for $i = j = 0$. Using (3) one gets

$$\mathcal{M}(11k+5, 4k+2+i, 10k+4+j) = \frac{1}{2}\big(2 + 5i + 5i^2 - 3j - 4ij + 3j^2 + 12k + 22k^2\big).$$

The stated goal is equivalent to showing that the polynomial

$$p(i, j) = 5i + 5i^2 - 3j - 4ij + 3j^2$$

is nonnegative for all $(i, j) \in \mathbb{Z}^2$. Such a task can, in principle, be routinely executed by cylindrical algebraic decomposition (CAD) [2]. In this method, the variables i and j are treated as real variables, which causes some problems in the present application. The reason is that $p(i, j) \geq 0$ does not hold for all $i, j \in \mathbb{R}$. The situation is depicted in Figure 2, where the ellipse represents the zero set of $p(i, j)$ and its inside those values (i, j) for which the polynomial $p(i, j)$ is negative. To our relief, we see that no integer lattice points lie inside the ellipse, since such points would be counterexamples to our claim.

Our strategy now is the following: we prove that $p(i, j) \geq 0$ for all integer points that are close to $(0, 0)$, e.g., for all (i, j) with $-2 \leq i \leq 2$ and $-2 \leq j \leq 2$. These points are shown in Figure 2, with the respective value of $p(i, j)$ attached to them. In particular, we see that the minimum $p(i, j) = 0$ is attained several times, namely on the three points that lie exactly on the boundary of the ellipse.

Then we invoke cylindrical algebraic decomposition on the formula

$$\forall i, j \in \mathbb{R}\colon (-2 \leq i \leq 2 \wedge -2 \leq j \leq 2) \vee p(i, j) \geq 0, \tag{4}$$

which states that if the point (i, j) lies outside the square that we have already considered, then $p(i, j) \geq 0$ holds. Calling the Mathematica command `CylindricalDecomposition` with input (4), we immediately get `True`. □

We are now ready to state the main theorem of this section, which is an exact formula for the minimal number of MSTs for any 2-coloring of $[n]$. Apart from the asymptotic results mentioned in Section 1, there is only one paper [10] where a similar result is stated, but only for the case $n = 22k$ and for Schur triples $(x, y, x + y)$ with $x \leq y$. In contrast, we consider all $x, y \in [n]$ and our formula holds for all $n \in \mathbb{N}$.

Theorem 1 *The minimal number of monochromatic Schur triples that can be attained under any* 2*-coloring of* $[n]$ *is*

$$\mathcal{M}(n) = \left\lfloor \frac{n^2 - 4n + 6}{11} \right\rfloor.$$

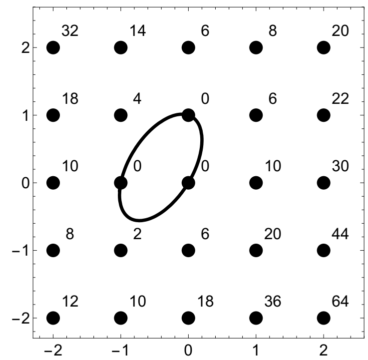

Fig. 2 Zero set of the polynomial $p(i, j)$ from Lemma 2 and its values at integer lattice points $(i, j) \in \mathbb{Z}^2$.

Proof As in Lemma 2, we argue by case distinction $n = 11k + \ell$, $0 \le \ell \le 10$. Using $s_0 = \lfloor \frac{1}{11}(4n + 2)\rfloor$ and $t_0 = \lfloor \frac{10}{11} n\rfloor$ from the lemma, we obtain the following values for $\mathcal{M}(n, s_0, t_0)$:

$$
\begin{aligned}
\ell = 0&\colon \mathcal{M}(11k, 4k, 10k) &&= 11k^2 - 4k &&= \tfrac{1}{11}(n^2 - 4n)\\
\ell = 1&\colon \mathcal{M}(11k + 1, 4k, 10k) &&= 11k^2 - 2k &&= \tfrac{1}{11}(n^2 - 4n + 3)\\
\ell = 2&\colon \mathcal{M}(11k + 2, 4k, 10k + 1) &&= 11k^2 &&= \tfrac{1}{11}(n^2 - 4n + 4)\\
\ell = 3&\colon \mathcal{M}(11k + 3, 4k + 1, 10k + 2) &&= 11k^2 + 2k &&= \tfrac{1}{11}(n^2 - 4n + 3)\\
\ell = 4&\colon \mathcal{M}(11k + 4, 4k + 1, 10k + 3) &&= 11k^2 + 4k &&= \tfrac{1}{11}(n^2 - 4n)\\
\ell = 5&\colon \mathcal{M}(11k + 5, 4k + 2, 10k + 4) &&= 11k^2 + 6k + 1 &&= \tfrac{1}{11}(n^2 - 4n + 6)\\
\ell = 6&\colon \mathcal{M}(11k + 6, 4k + 2, 10k + 5) &&= 11k^2 + 8k + 1 &&= \tfrac{1}{11}(n^2 - 4n - 1)\\
\ell = 7&\colon \mathcal{M}(11k + 7, 4k + 2, 10k + 6) &&= 11k^2 + 10k + 2 &&= \tfrac{1}{11}(n^2 - 4n + 1)\\
\ell = 8&\colon \mathcal{M}(11k + 8, 4k + 3, 10k + 7) &&= 11k^2 + 12k + 3 &&= \tfrac{1}{11}(n^2 - 4n + 1)\\
\ell = 9&\colon \mathcal{M}(11k + 9, 4k + 3, 10k + 8) &&= 11k^2 + 14k + 4 &&= \tfrac{1}{11}(n^2 - 4n - 1)\\
\ell = 10&\colon \mathcal{M}(11k + 10, 4k + 3, 10k + 9) &&= 11k^2 + 16k + 6 &&= \tfrac{1}{11}(n^2 - 4n + 6)
\end{aligned}
$$

One easily observes that in each case, the result is of the form $\frac{1}{11}(n^2 - 4n) + \delta_\ell$, where $-\frac{1}{11} \le \delta_\ell \le \frac{6}{11}$ holds for all ℓ. Hence the claimed formula follows. □

The first 25 terms of the sequence $\big(\mathcal{M}(n)\big)_{n\ge 1}$ are

$$0, 0, 0, 0, 1, 1, 2, 3, 4, 6, 7, 9, 11, 13, 15, 18, 20, 23, 26, 29, 33, 36, 40, 44, 48, \ldots$$

We have added this sequence to the Online Encyclopedia of Integer Sequences [12] under the number A321195.

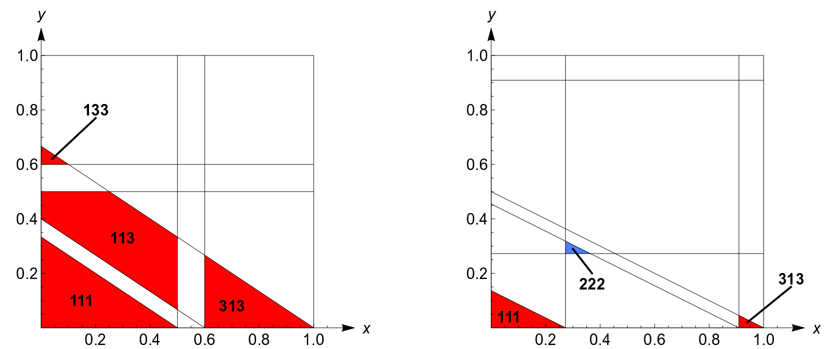

Fig. 3 Regions (in red and blue) corresponding to monochromatic generalized Schur triples for $a = \frac{3}{2}, s = \frac{1}{2}, t = \frac{3}{5}$ (left) and $a = 2, s = \frac{3}{11}, t = \frac{10}{11}$ (right); their area being measured by $A(s, t, a)$ from Lemma 3.

3 Asymptotic lower bound for generalized Schur triples

We now turn to generalized Schur triples, i.e., triples (x, y, z) subject to $z = x + ay$ for some parameter $a \in \mathbb{N}$, as studied by Thanatipanonda and Wong [14]. Here, we allow a to be even more general, i.e., $a \in \mathbb{R}^+$. Consequently, we have to adapt the definition of generalized Schur triples: we use the condition $z = x + \lfloor ay \rfloor$. The case $a < 0$ does not add new aspects to the analysis, as it can be transformed to the $a > 0$ case by exchanging the roles of x and z and by changing the floor function to a ceiling.

Again, we choose to use the assumption that the minimal number of monochromatic generalized Schur triples (MGSTs) occurs at a coloring in the form of three blocks $R^s B^{t-s} R^{n-t}$. We justify using this assumption with the experimental evidence of Butler, Costello, and Graham [1] (who argued for the generalized Schur triple case $a > 1$) and adapting the intuition in the argument of Schoen [10] (who only argued for the Schur triple case $a = 1$).

We would like to know for which choice of s and t (depending on n and a) the minimum occurs. Similar to the previous section, we let n go to infinity and correlate the number of MGSTs with the area of polygonal regions in the unit square. We then define a function $A(s, t, a)$ that determines this area, and minimize it. Hence, throughout this section, s and t are real numbers with $0 \leq s \leq t \leq 1$.

Figure 3 shows two situations for different choices of a, s, t. In contrast to the previous section, we do a very careful case analysis and do not impose extra conditions on s and t as in Lemma 1, at the cost of introducing a "few" more case distinctions. The full case analysis for normal Schur triples then follows by specializing to $a = 1$ in the resulting formulas.

In the process of analyzing the different cases, we encounter several conditions on a, s, t. For our referencing convenience, we distinguish these conditions here using the following abbreviations:

$$
\begin{aligned}
C_1 &\equiv 1-as \geq 0, & C_2 &\equiv 1-as-s \geq 0,\\
C_3 &\equiv 1-as-t \geq 0, & C_4 &\equiv t-as \geq 0,\\
C_5 &\equiv t-as-s \geq 0, & C_6 &\equiv 1-at \geq 0,\\
C_7 &\equiv 1-at-s \geq 0, & C_8 &\equiv 1-at-t \geq 0,\\
C_9 &\equiv 1-a \geq 0, & C_{10} &\equiv 1-a-s \geq 0,\\
C_{11} &\equiv s-a \geq 0, & C_{12} &\equiv 1-a-t \geq 0,\\
C_{13} &\equiv t-a \geq 0, & C_{14} &\equiv t-a-s \geq 0,\\
C_{15} &\equiv s-at \geq 0, & C_{16} &\equiv t-at-s \geq 0.
\end{aligned} \tag{5}
$$

In Figures 5 and 6, the lines that represent some of these conditions are depicted. They split the triangle $0 \leq s \leq t \leq 1$ into several regions, depending on the value of a.

Lemma 3 *Let $a, s, t \in \mathbb{R}$ with $a > 0$ and $0 \leq s \leq t \leq 1$. Then the area $A(s,t,a)$ of the region*

$$\left\{(x,y) \in \mathbb{R}^2 : (x,y,x+ay) \in \big([0,s] \cup (t,1]\big)^3 \vee (x,y,x+ay) \in (s,t]^3\right\}$$

is given by a piecewise defined function, where 70 case distinctions have to be made. For the sake of brevity, only the first 17 cases are listed below, since they will be the most important ones in the subsequent analysis; in fact they are sufficient to describe $A(s,t,a)$ for $a \geq 1$. We label the region corresponding to the i-th case as (R_i). They are expressed in terms of the conditions (5) (where overlines denote negations):

	conditions on a, s, t	$A(s,t,a)$
(R_1)	$\overline{C_1}$	$\frac{s^2-2ts+2s+t^2-2t+1}{2a}$
(R_2)	$C_3 \wedge C_4 \wedge \overline{C_6}$	$\frac{2as^2+2s^2+2as-4ats-2ts+t^2}{2a}$
(R_3)	$C_3 \wedge \overline{C_4} \wedge \overline{C_6}$	$\frac{-a^2s^2+2as^2+2s^2+2as-2ats-2ts}{2a}$
(R_4)	$\overline{C_2} \wedge C_4 \wedge \overline{C_6}$	$\frac{s^2+2as-2ats-2ts+2s+2t^2-2t}{2a}$
(R_5)	$\overline{C_2} \wedge \overline{C_4} \wedge C_6$	$\frac{-a^2s^2+s^2+2as-2ts+2s+a^2t^2+t^2-2at-2t+1}{2a}$
(R_6)	$C_1 \wedge \overline{C_2} \wedge \overline{C_4} \wedge \overline{C_6}$	$\frac{-a^2s^2+s^2+2as-2ts+2s+t^2-2t}{2a}$
(R_7)	$C_2 \wedge \overline{C_3} \wedge C_4 \wedge \overline{C_6}$	$\frac{a^2s^2+2as^2+2s^2-2ats-2ts+2t^2-2t+1}{2a}$
(R_8)	$C_2 \wedge \overline{C_3} \wedge \overline{C_4} \wedge C_6$	$\frac{2as^2+2s^2-2ts+a^2t^2+t^2-2at-2t+2}{2a}$
(R_9)	$C_2 \wedge \overline{C_3} \wedge \overline{C_4} \wedge \overline{C_6}$	$\frac{2as^2+2s^2-2ts+t^2-2t+1}{2a}$
(R_{10})	$C_3 \wedge C_4 \wedge C_6 \wedge \overline{C_7}$	$\frac{2as^2+2s^2+2as-4ats-2ts+a^2t^2+t^2-2at+1}{2a}$

(R_{11}) $C_3 \wedge \overline{C_4} \wedge C_6 \wedge \overline{C_7}$ $\frac{-a^2s^2+2as^2+2s^2+2as-2ats-2ts+a^2t^2-2at+1}{2a}$

(R_{12}) $\overline{C_4} \wedge C_8$ $\frac{(1+2a-a^2)s^2+2s(1-2at+a-t)+(at+t-1)^2}{2a}$

(R_{13}) $\overline{C_4} \wedge C_7 \wedge \overline{C_8}$ $\frac{-a^2s^2+2as^2+s^2+2as-4ats-2ts+2s}{2a}$

(R_{14}) $C_4 \wedge C_8 \wedge \overline{C_9}$ $\frac{(a^2+2a+2)t^2-2t(3as+a+s+1)+(s+1)(2as+s+1)}{2a}$

(R_{15}) $C_4 \wedge C_7 \wedge \overline{C_8} \wedge \overline{C_9}$ $\frac{2as^2+s^2+2as-6ats-2ts+2s+t^2}{2a}$

(R_{16}) $\overline{C_2} \wedge C_4 \wedge C_6 \wedge \overline{C_9}$ $\frac{s^2+2as-2ats-2ts+2s+a^2t^2+2t^2-2at-2t+1}{2a}$

(R_{17}) $C_2 \wedge \overline{C_3} \wedge C_4 \wedge C_6 \wedge \overline{C_9}$ $\frac{a^2s^2+2as^2+2s^2-2ats-2ts+a^2t^2+2t^2-2at-2t+2}{2a}$

Proof As can be seen in Figure 3, the region whose area we would like to determine is the union of several polygons. Let $I_1 = [0, s]$, $I_2 = (s, t]$, and $I_3 = (t, 1]$ denote the intervals that correspond to the different blocks of the coloring (I_1 and I_3 being red and I_2 being blue). Then $x, y \in I_1 \wedge x+ay \in I_3$ is allowed while $x, y \in I_1 \wedge x+ay \in I_2$ is not. From this point on, we will refer to the case $(x, y, x + ay) \in I_i \times I_j \times I_k$ by ijk. It is easy to see that we have to consider only seven cases: 111, 222, 113, 131, 133, 313, 333. The cases 311 and 331 are clearly impossible since $x \geq t$ contradicts $x + ay \leq s$. All other combinations of $1, 2, 3$ violate the monochromatic coloring condition.

In both parts of Figure 3, case 111 corresponds to the triangle that touches the origin. The coordinates of its other two vertices are $(s, 0)$ and $(0, \frac{s}{a})$, hence its area is $\frac{1}{2} \cdot s \cdot \frac{s}{a}$. However, this is valid only for $a \geq 1$. If $a < 1$, then the point $(0, \frac{s}{a})$ is above the line $y = s$ and so the top of the triangle is cut off. As a result, one obtains a quadrilateral with vertices $(0, 0)$, $(s, 0)$, $(s - as, s)$, $(0, s)$, whose area is given by $\frac{1}{2} \cdot s \cdot (2s - as)$.

The case 222 is similar, with the difference being that the corresponding polygon disappears if $\frac{t-s}{a} < s$; in the right part of Figure 3 the polygon 222 is present while in the left part it is not. The polygons 313, 333, and 131 are characterized by comparably simple case distinctions, while 133 and 113 require a much more involved analysis. In Figure 4, we present such an analysis for 133, and refer to the accompanying electronic material [7] for 113.

What we have achieved so far is a representation of $A(s, t, a)$ as a sum of seven piecewise functions. However, what is required is a representation of $A(s, t, a)$ as a single piecewise function, since that will be needed for determining the location of the minimum.

The conditions that are used to characterize the different pieces in Figure 4 (and in the remaining cases that have not been discussed explicitly), are listed in (5). In order to combine the seven piecewise functions, we need a common refinement of the regions on which they are defined. We start with the finest possible refinement, which is obtained by considering all $2^{16} = 65536$ logical combinations of C_i and $\overline{C_i}$ for $1 \leq i \leq 16$. Using Mathematica's simplification procedures, we remove those cases that contain contradictory combinations of conditions, such as $C_1 \wedge \overline{C_2}$ for example. After this purging, we are left with a subdivision of the set

$$\{(s,t,a) : 0 \leq s \leq t \leq 1 \wedge a \geq 0\} \subset \mathbb{R}^3, \tag{6}$$

which is an infinite triangular prism, into 114 polyhedral regions. Finally, we merge regions on which $A(s,t,a)$ is defined by the same expression into a single region, yielding a representation of $A(s,t,a)$ as a piecewise function defined by 70 different expressions. Each of them is of the form $\frac{1}{a}p(s,t,a)$ where p is a polynomial in s,t,a of degree at most 2 in each of the variables. For more details, and to see the definition of $A(s,t,a)$ in its full glory, see the accompanying electronic material [7]. □

We have seen that the different domains of definition for $A(s,t,a)$ are polyhedra in $\mathbb{R}^3$ (some of which are unbounded). In Figures 5 and 6 two 2-dimensional slices of the set (6) for particular choices of a are shown. Note that in Figure 5 condition C_5 is not shown since it was eliminated in the process of merging regions on which A is defined by the same expression. Moreover, $C_9 \equiv a \leq 1$ is not visible since its plane $a = 1$ is parallel to the depicted cross section $a = 1.4$.

Lemma 4 *For $a > 0$, the minimum of the function $A(s,t,a)$ (defined in Lemma 3) on the triangle* $0 \leq s \leq t \leq 1$,

$$m(a) := \min_{0 \leq s \leq t \leq 1} A(s,t,a)$$

is given by a piecewise rational function, depending on a, according to the following case distinctions (where we also give the location (s_0,t_0) of the minimum):

	s_0	t_0	$m(a)$
$0 \leq a \leq \alpha_1$	$\frac{(a-4)a}{a^3-a-4}$	$\frac{-2a^2+4a+2}{-a^3+a+4}$	$\frac{-a^4+2a^3-2a^2+6a-4}{2(a^3-a-4)}$
$\alpha_1 \leq a \leq \alpha_2$	$\frac{a(a^2-3)}{a^4-8a-1}$	$\frac{a^3+a^2-5a-1}{a^4-8a-1}$	$\frac{a^3-2a^2+a-2}{2(a^4-8a-1)}$
$\alpha_2 \leq a \leq \alpha_3$	$\frac{-2a^3+2a+1}{-a^4+8a+3}$	$\frac{2a^3+a^2-6a-2}{a^4-8a-3}$	$\frac{a^6+a^4-12a^3+4a^2-1}{2a(a^4-8a-3)}$
$\alpha_3 \leq a \leq \alpha_4$	$\frac{-2a^2+a+1}{-4a^3+5a^2+6a+1}$	$\frac{-2a^3+a^2+4a+1}{-4a^3+5a^2+6a+1}$	$\frac{4a^4-9a^3+2a^2+a-2}{2(4a^3-5a^2-6a-1)}$
$\alpha_4 \leq a \leq \alpha_5$	$\frac{a^3+a+1}{-4a^3+3a^2+6a+1}$	$\frac{2a^2+4a+1}{-4a^3+3a^2+6a+1}$	$\frac{4a^4-4a^3+a-2}{2(4a^3-3a^2-6a-1)}$
$\alpha_5 \leq a \leq \alpha_6$	$-\frac{3a^2+a-1}{4a^3-4a^2-4a+1}$	$\frac{-4a^2-2a+1}{4a^3-4a^2-4a+1}$	$\frac{8a^3-4a^2-5a+2}{2(4a^3-4a^2-4a+1)}$
$\alpha_6 \leq a \leq \alpha_7$	$\frac{2a+1}{7a+1}$	$\frac{8a^2+6a+1}{7a^2+8a+1}$	$\frac{-2a^2+3a+2}{2(a+1)(7a+1)}$
$\alpha_7 \leq a \leq 1$	$\frac{(a+1)^2}{a(7a+4)}$	$\frac{(a+1)(4a+1)}{a(7a+4)}$	$\frac{-7a^4+6a^3+6a^2-2a-1}{2a^2(7a+4)}$
$1 \leq a \leq \alpha_8$	$\frac{(a+1)^2}{a^4+2a^3+3a^2+2a+3}$	$\frac{(a+1)(a^2+2a+2)}{a^4+2a^3+3a^2+2a+3}$	$\frac{a^4-a^2-2a+4}{2a(a^4+2a^3+3a^2+2a+3)}$
$\alpha_8 \leq a$	$\frac{a+1}{a^2+2a+3}$	$\frac{a^2+2a+2}{a^2+2a+3}$	$\frac{1}{2a(a^2+2a+3)}$

Here, the quantities $\alpha_1,\ldots,\alpha_8$ stand for the following algebraic numbers, where $\operatorname{Root}(p,I)$ *denotes the unique real root of the polynomial p in the interval I:*

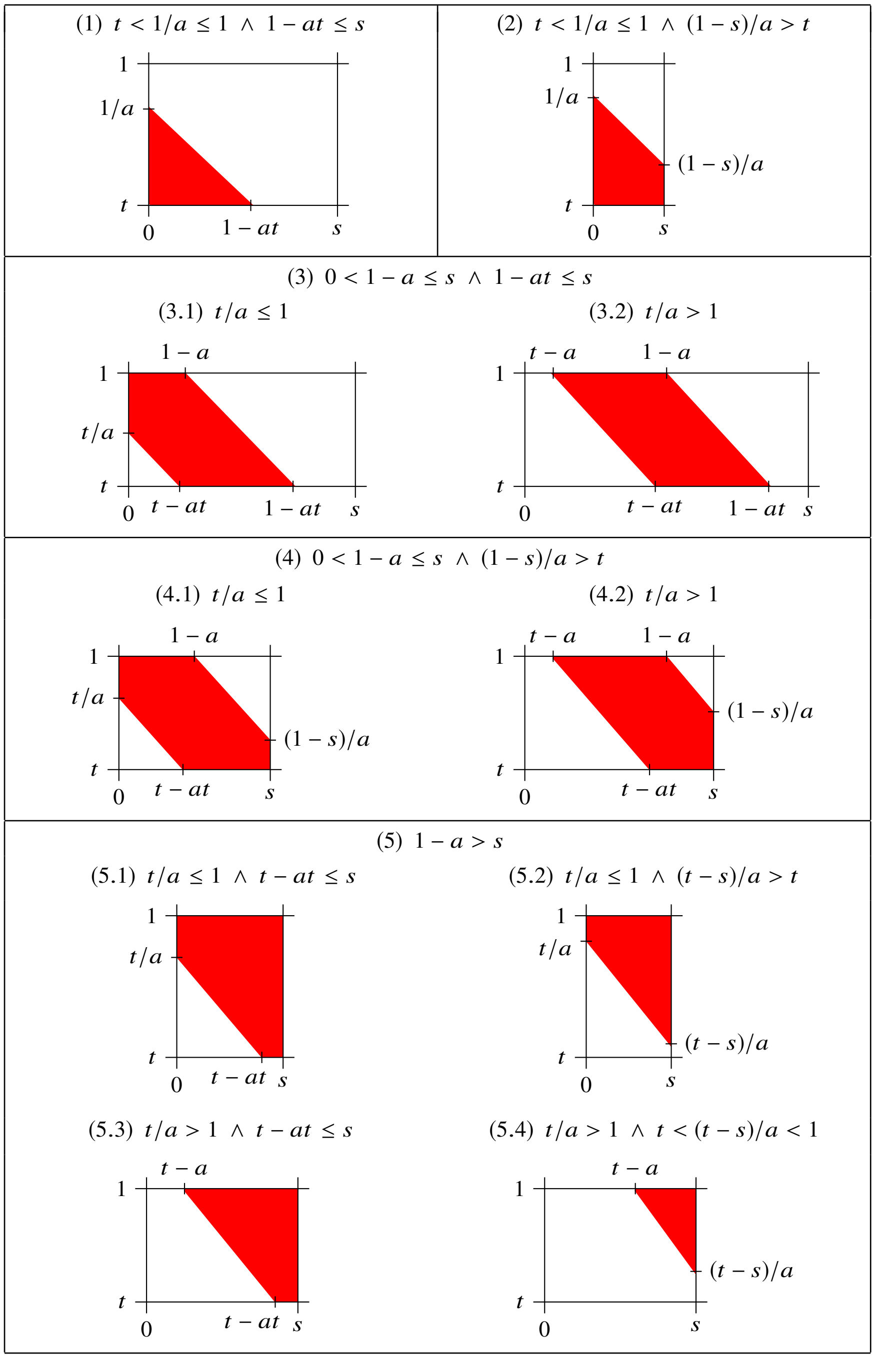

Fig. 4 Case distinctions for polygon 133, showing all possibilities of admissible regions in the top left corner (depending on conditions for a, s, t). The empty cases (not shown) correspond to the conditions $1/a \le t$ or $t - a \ge s$.

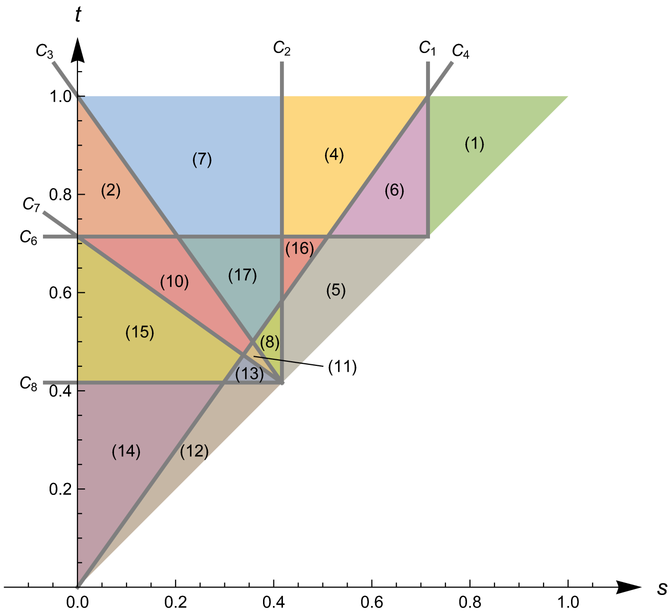

Fig. 5 Domains of definition of $A(s, t, a)$ for $a = 1.4$, according to Lemma 3. Note that not all 17 cases listed in the lemma are present for this particular choice of a.

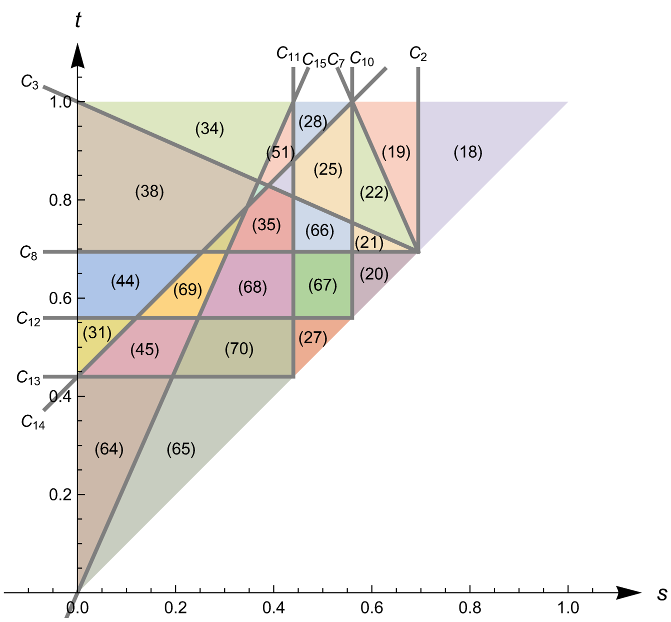

Fig. 6 Domains of definition of the area function $A(s, t, a)$ for $a = 0.44$.

$$\begin{aligned}
\alpha_1 &= 0.295597\ldots = \operatorname{Root}\bigl(a^3 + a^2 + 3a - 1, [0,1]\bigr),\\
\alpha_2 &= 0.395065\ldots = \operatorname{Root}\bigl(a^5 - 9a^2 + a + 1, [0,1]\bigr),\\
\alpha_3 &= 0.405669\ldots = \operatorname{Root}\bigl(2a^4 - a^3 - 6a^2 + 1, [0,1]\bigr),\\
\alpha_4 &= 0.553409\ldots = \operatorname{Root}\bigl(12a^4 - 15a^3 - 24a^2 + 5a + 6, [0,1]\bigr),\\
\alpha_5 &= 0.622179\ldots = \operatorname{Root}\bigl(4a^3 - 8a^2 - 3a + 4, [0,1]\bigr),\\
\alpha_6 &= 0.647363\ldots = \operatorname{Root}\bigl(8a^2 + a - 4, [0,1]\bigr) = \tfrac{1}{16}\bigl(\sqrt{129} - 1\bigr),\\
\alpha_7 &= 0.931478\ldots = \operatorname{Root}\bigl(7a^3 - 5a - 1, [0,1]\bigr),\\
\alpha_8 &= 1.174559\ldots = \operatorname{Root}\bigl(a^3 + a^2 - 3, [1,2]\bigr).
\end{aligned}$$

Proof We locate the minimum in a similar fashion as in Section 2, by identifying points (s,t) where the gradient of the area function A vanishes. What complicates our task is the additional parameter a. Since A is defined in pieces, it may not be differentiable at the boundaries between different regions, and therefore, we should be aware that such locations could contain the minimum. For each region (R_i), $1 \leq i \leq 70$, on which $A(s,t,a)$ is defined, we perform the following steps:

- compute the gradient $\left(\frac{\partial A}{\partial s}, \frac{\partial A}{\partial t}\right)$,
- find all points (s,t) where the gradient is zero, and
- for each such point determine for which values of a it actually lies in (R_i).

On the region (R_1) from Lemma 3, the gradient of A is $\frac{1}{a}(s-t+1, t-s-1)$, which vanishes on all points $(s, s+1)$; however, since the region (R_1) is characterized by $\overline{C_1} \equiv s > \frac{1}{a}$ (and the general condition $s \leq t \leq 1$), one sees that none of these points lie in it. Continuing in this manner, we find that in each of the regions $(R_2) - (R_{70})$ there is exactly one point (s,t) for which the gradient of A vanishes, but in most cases this point lies outside the region for all a. For example, on (R_2) the gradient is $\frac{1}{a}(2as - 2at + 2s - t + a, t - 2as - s)$, which equals zero for

$$(s,t) = \left(\frac{a}{4a^2 + 2a - 1}, \frac{a(2a+1)}{4a^2 + 2a - 1}\right). \tag{7}$$

In order to find the values of a that give us that $(s,t) \in (R_2)$, the conditions defining (R_2) (plus the global assumptions) need to be satisfied, namely:

$$as + t \leq 1 \ \wedge\ t \geq as \ \wedge\ at > 1 \ \wedge\ 0 < s < t < 1.$$

After substituting s and t with the right hand side of (7) and clearing denominators, one gets a system of polynomial inequalities, involving only the variable a. Cylindrical algebraic decomposition simplifies it to

$$a \geq \operatorname{Root}\bigl(2a^3 - 3a^2 - 2a + 1, [1,2]\bigr) = 1.889228559\ldots$$

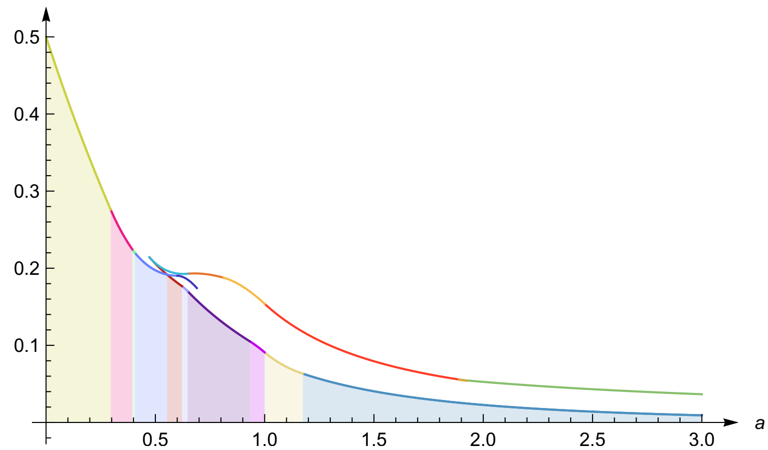

Fig. 7 Plot of $A(s, t, a)$ on the 17 different intervals of a identified from the 17 local minima in the proof of Lemma 4 for $0 \leq a \leq 3$; the shading under the graph indicates the main 10 intervals that are needed to describe the global minimum function $m(a)$.

Hence, for each a satisfying this condition we have a local minimum at the point given in (7).

We proceed in similar fashion and identify 17 local minima, each occurring only for a in a certain interval. Some of these intervals partly overlap, which means that we have to study a subdivision of the positive real line that is a refinement of all 17 intervals. When two functions intersect in the interior of an interval, it is split into two subintervals. CAD is once again employed to find the smallest among the local minima; this is done individually for each of the refined intervals. As a result, we obtain the piecewise description of the function $m(a)$ given above; see Figure 7 and the accompanying electronic material [7] for details.

It is clear from construction that $A(s, t, a)$ must be a continuous function, since the admissible polygons (shaded regions in Figure 3) cannot jump or disappear if the parameters a, s, t are changed infinitesimally, i.e., if the lines in Figure 3 are shifted or slanted by a little bit. In contrast, it is not obvious why it should be differentiable. Therefore, there is a possibility that the minimum can occur where the derivative does not exist. Hence, it is necessary to study the values of $A(s, t, a)$ along the boundaries of the different domains of definition. To accomplish this task, we view A as a bivariate function in s and t, with a parameter a. For each inequality in the list of conditions (5), the corresponding equation defines a line in $\mathbb{R}^2$. For each such line, we proceed to determine the range of a for which the line intersects the triangle $0 \leq s \leq t \leq 1$. On the resulting line segment, the pieces of $A(s, t, a)$ are given by univariate polynomials, still involving the parameter a. Equating their derivatives to zero, we find all of the local minima on this line segment, which could give rise to local minima of $A(s, t, a)$. After looking at all 16 lines, each of which splits into at most 70 segments, we find 225 candidates for minima. CAD confirms that none of them are actually smaller than the one given by $m(a)$. This fact also becomes apparent by plotting these candidates against the function $m(a)$, as shown in Figure 8 (top part).

Finally, we should also check all points where any two lines defined by (5) intersect. We find 54 points that lie inside the triangle $0 \leq s \leq t \leq 1$, at least for certain choices of a. The value of $A(s, t, a)$ at a particular point is given by a piecewise function depending on a. Assembling all pieces for all points, we obtain 348 cases. For each of them, CAD confirms (rigorously!) that the value of $A(s, t, a)$ does not go below $m(a)$. A "non-rigorous proof" of this fact is shown in Figure 8 (bottom part).

Summarizing, we have shown that, for each particular choice of $a > 0$, the minimum of the function $A(s, t, a)$ on the triangle $0 \leq s \leq t \leq 1$ is given by $m(a)$, and we have determined the location (s_0, t_0) where this minimum is attained. This immediately establishes an asymptotic lower bound for MGSTs on$[n]$, as n goes to infinity. □

We wrap up this section with some remarks on the consequences of Lemma 4 and on what appears to be erratic (jumpy) behavior for some values of a in Figure 8. We assure the reader that it is not due to the amount of alcohol that was consumed throughout this meal, but rather an indication of the appearance and disappearance of certain admissible regions for the MGSTs as a changes.

First, we would like to note that Lemma 4 explains why the asymptotic formula for MGSTs for integral $a \geq 2$ given in [1, 13, 14] does not specialize to the previously known case $a = 1$: this phenomenon is due to the piecewise definition of $m(a)$, with a transition at $1 < \alpha_8 < 2$. Geometrically speaking, α_8 marks the point where the polygon 133 (see Figure 3) disappears, when a increases from 1 to 2, and $s = s_0(a)$ and $t = t_0(a)$ are updated constantly.

A second interesting finding that follows from Lemma 4 is that there is a jump of $\big(s_0(a), t_0(a)\big)$ at $a = \alpha_4 = 0.5534\ldots$; the function $m(a)$ however is continuous. In Figure 7 one sees that at $a = \alpha_4$ the functions of two local minima intersect, and therefore this point marks the jump from one branch to another one. In Figure 9 the situation is shown for two different values of a close to α_4: while the shaded area in both parts of the figure is almost the same, the values of s and t change quite dramatically. We invite the reader to play with such transitions in the accompanying electronic material [7].

In the next section, we bring up the fact that the coloring pattern of three blocks that we generously assumed for $a > 0$ does not actually give the global minimum on $0 < a < 1$ over any 2-coloring of $[n]$ and we take care to emphasize this in the statement of the theorems. This will therefore explain the erratic behavior at $a = 1$ in both graphs of Figure 8.

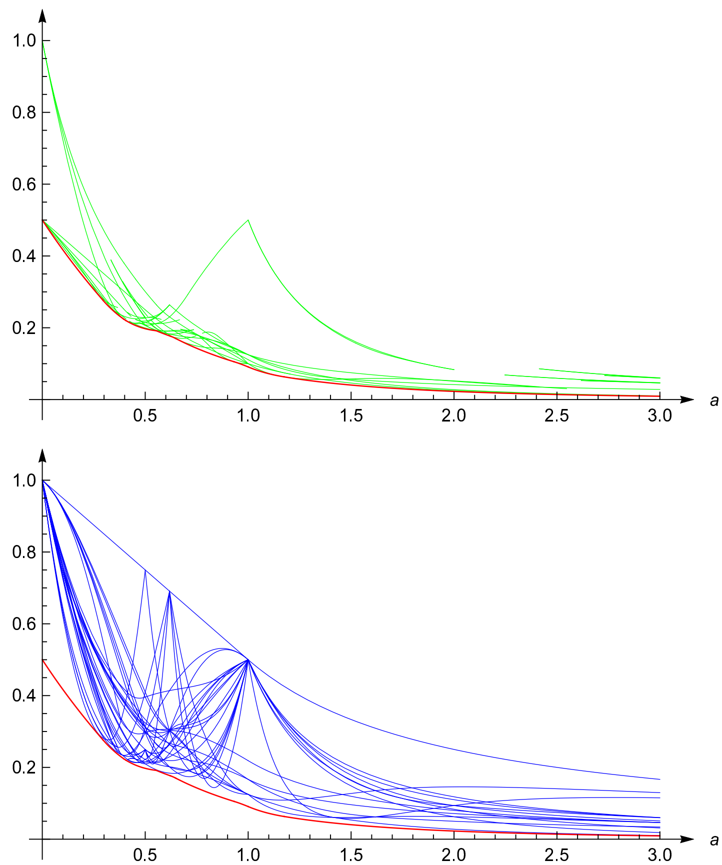

Fig. 8 Global minimum of $A(s, t, a)$ (red curve) compared to potential minima along lines (green curves, top part) and potential minima on intersection points (blue curves, bottom part).

4 Exact bounds for generalized Schur triples

In this section we apply the results from the last section, i.e., from the continuous setting, to the discrete enumeration problem of monochromatic generalized Schur triples (MGSTs). Hence, s and t are now integers with $1 \leq s \leq t \leq n$ that describe the coloring $R^s B^{t-s} R^{n-t}$ of $[n]$. Throughout this section we use the convention that a sum whose lower bound is greater than its upper bound is zero, i.e.,

$$\sum_{x=i}^{j} f(x) = \begin{cases} f(i) + \cdots + f(j), & \text{if } i \leq j \\ 0, & \text{if } i > j. \end{cases}$$

Analogous to Section 2 we use the notation $\mathcal{M}^{(a)}$ to count MGSTs. More precisely, we define $\mathcal{M}^{(a)}(n, s, t)$ and $\mathcal{M}^{(a)}(n)$, as follows:

$$\mathcal{M}^{(a)}(n,s,t) := \Bigl|\bigl\{T = (x,y,x+\lfloor ay\rfloor) \in [n]^3 : \\ T \in ([s] \cup \{t+1,\dots,n\})^3 \vee T \in \{s+1,\dots,t\}^3\bigr\}\Bigr|,$$

$$\mathcal{M}^{(a)}(n) := \min_{1\le s\le t\le n} \mathcal{M}^{(a)}(n,s,t).$$

In contrast to the previous section, we will now mostly look at special cases for a, since we cannot hope to get an exact formula for the minimal number of MGSTs for general $a \in \mathbb{R}^+$.

Lemma 5 *Let $a \in \mathbb{R}$ with $a \ge 1$ and let $n, s, t \in \mathbb{N}$ with $1 \le s \le t \le n$. Furthermore, assume that the inequalities $as + t \ge n$, $t \ge as$, and $s + as \le t$ hold. Then the number $\mathcal{M}^{(a)}(n,s,t)$ of monochromatic generalized Schur triples of $[n]$ under the coloring $R^s B^{t-s} R^{n-t}$ is given by*

$$\sum_{y=1}^{\lfloor s/a\rfloor} \sum_{x=1}^{s-\lfloor ay\rfloor} 1 \; + \sum_{y=s+1}^{\lfloor (t-s)/a\rfloor} \sum_{x=s+1}^{t-\lfloor ay\rfloor} 1 \; + \sum_{y=1}^{\lfloor (n-t)/a\rfloor} \sum_{x=t+1}^{n-\lfloor ay\rfloor} 1 \; + \sum_{y=t+1}^{\lfloor n/a\rfloor} \sum_{x=1}^{n-\lfloor ay\rfloor} 1.$$

Moreover, the explicit list of these MGSTs $(x, y, x + \lfloor ay\rfloor)$ can be directly read off from the above formula.

Proof Under the given assumptions, we have to consider monochromatic triples of types 111, 222, 313, and 133, see, e.g., Figure 3. Obviously, the four sums correspond exactly to these four cases. Note that if $at > n$, then the case 133 is not present, which is reflected by the fact that the corresponding sum is zero in this case. □

The assumed inequalities in Lemma 5 tell us that we are either in (R_7) (when $at > n$) or in (R_{17}) (when $at \le n$); these regions were introduced in Lemma 3. Recall $\alpha_8 = 1.174559\ldots$ from Lemma 4, and also that the global minimum of the area function $A(s,t,a)$ is located in (R_7) (when $a \ge \alpha_8$) or in (R_{17}) (when $1 \le a \le \alpha_8$).

Theorem 2 *The minimal number of monochromatic generalized Schur triples of the form $(x, y, x + 2y)$ that can be attained under any 2-coloring of $[n]$ of the form $R^s B^{t-s} R^{n-t}$ is*

$$\mathcal{M}^{(2)}(n) = \left\lfloor \frac{n^2 - 10n + 33}{44} \right\rfloor.$$

Proof For $a = 2$ we clearly have $\alpha_8 \le a$, and by Lemma 4 it follows that the optimal choice for s and t is expected around the point

$$n \cdot \left(\frac{a+1}{a^2+2a+3}, \frac{a^2+2a+2}{a^2+2a+3}\right) = \left(\frac{3n}{11}, \frac{10n}{11}\right).$$

The three conditions $2s + t \ge n$, $t \ge 2s$, $3s \le t$ are satisfied (at least for large n), and therefore we can use Lemma 5 to compute the exact number of MGSTs:

$$\mathcal{M}^{(2)}(n,s,t) = \sum_{y=1}^{\lfloor s/2 \rfloor} \sum_{x=1}^{s-2y} 1 + \sum_{y=s+1}^{\lfloor (t-s-1)/2 \rfloor} \sum_{x=s+1}^{t-2y} 1 + \sum_{y=1}^{\lfloor (n-t)/2 \rfloor} \sum_{x=t+1}^{n-2y} 1 =$$
$$= \left\lfloor \frac{s}{2} \right\rfloor \left\lfloor \frac{s-1}{2} \right\rfloor + \left\lfloor \frac{n-t}{2} \right\rfloor \left\lfloor \frac{n-t-1}{2} \right\rfloor + \left\lfloor \frac{t-s}{2} \right\rfloor \left\lfloor \frac{t-s-1}{2} \right\rfloor + 2s^2 - st + s.$$

From now on, we proceed in an analogous fashion as in the proofs of Lemma 2 and Theorem 1. Empirically, we find that for each $n \in \mathbb{N}$, the minimum of $\mathcal{M}^{(2)}(n,s,t)$ is attained at

$$s_0 = \left\lfloor \frac{3n+1}{11} \right\rfloor, \qquad t_0 = \left\lfloor \frac{10n}{11} \right\rfloor + \begin{cases} -1, & \text{if } n = 22k+10, \\ 0, & \text{otherwise.} \end{cases}$$

When we plug in $s_0 + i$ and $t_0 + j$ into the above formula for $\mathcal{M}^{(2)}(n,s,t)$, we need to make a case distinction $n = 22k + \ell$ for $0 \le \ell \le 21$ in order to get rid of the floors. Moreover, we need to distinguish even and odd i (resp. j). Evaluating and simplifying

$$\mathcal{M}^{(2)}(22k+\ell, s_0 + 2i_1 + i_2, t_0 + 2j_1 + j_2), \quad 0 \le \ell \le 21,\ i_2, j_2 \in \{0,1\},$$

we obtain 88 polynomials in i_1, j_1, k. Applying CAD individually to each of these polynomials and by checking a few values explicitly (not unlike what we did in the proof of Lemma 5), one proves that the minimum is indeed attained at (s_0, t_0). Finally, one evaluates $\mathcal{M}^{(2)}(22k+\ell, s_0, t_0)$ for all $\ell = 0, \ldots, 21$ and finds that it is always of the form $\frac{1}{44}\left(n^2 - 10n\right) + \delta_\ell$, where the values $\delta_0, \ldots, \delta_{21}$ are

$$0, \tfrac{9}{44}, \tfrac{4}{11}, \tfrac{21}{44}, \tfrac{6}{11}, \tfrac{25}{44}, \tfrac{6}{11}, \tfrac{21}{44}, \tfrac{4}{11}, \tfrac{9}{44}, 0, \tfrac{3}{4}, \tfrac{5}{11}, \tfrac{5}{44}, \tfrac{8}{11}, \tfrac{13}{44}, -\tfrac{2}{11}, \tfrac{13}{44}, \tfrac{8}{11}, \tfrac{5}{44}, \tfrac{5}{11}, \tfrac{3}{4}.$$

Since the largest value is $\frac{3}{4}$ and since the smallest value is greater than $-\frac{1}{4}$ (i.e., all values δ_ℓ lie inside an interval of length 1), the claimed formula follows.

One last detail: we still have to examine for which n the conditions $2s + t \ge n$, $t \ge 2s$, $3s \le t$ are satisfied, as it could happen that for small n the point (s_0, t_0) lies not inside the correct region (R_{17}), due to the rounding errors. With the (somewhat generous) assumptions $\frac{3n+1}{11} - 1 \le s \le \frac{3n+1}{11}$ and $\frac{10n}{11} - 2 \le t \le \frac{10n}{11}$ we find that the above conditions are satisfied for all $n \ge 25$. For the remaining values $n < 25$, the claimed formula can be verified by an explicit computation. □

Theorem 3 *The minimal number of monochromatic generalized Schur triples of the form* $(x, y, x+3y)$ *that can be attained under any* 2*-coloring of* $[n]$ *of the form* $R^s B^{t-s} R^{n-t}$ *is*

$$\mathcal{M}^{(3)}(n) = \left\lfloor \frac{n^2 - 18n + 101}{108} \right\rfloor + \begin{cases} 1, & \text{if } n = 54k+36, \\ -1, & \text{if } n = 54k+30 \text{ or } n = 54k+42 \\ 0, & \text{otherwise.} \end{cases}$$

Proof For $a = 3$, it follows by Lemma 4 that the optimal choice for s and t is expected around the point

$$n \cdot \left(\frac{a+1}{a^2+2a+3}, \frac{a^2+2a+2}{a^2+2a+3}\right) = \left(\frac{4n}{18}, \frac{17n}{18}\right).$$

This means that the proof will require $18 \cdot a = 54$ case distinctions $n = 54k + \ell$ for $0 \le \ell \le 53$. Empirically, we find that for each $n \in \mathbb{N}$, the minimum of $\mathcal{M}^{(3)}(n, s, t)$ is attained at

$$s_0 = \left\lfloor \frac{4n}{18} \right\rfloor - \begin{cases} 1, & \text{if } n = 54k + 18, \\ 0, & \text{otherwise}, \end{cases}$$

$$t_0 = \left\lfloor \frac{17n}{18} \right\rfloor - \begin{cases} 1, & \text{if } n = 9k + i \text{ for } i \in \{3,4,7,8\}, \\ 2, & \text{if } n = 54k + 18, \\ 0, & \text{otherwise}. \end{cases}$$

Applying CAD to the 486 polynomials

$$\mathcal{M}^{(3)}(54k+\ell, s_0 + 3i_1 + i_2, t_0 + 3j_1 + j_2), \quad 0 \le \ell \le 53,\ i_2, j_2 \in \{0,1,2\},$$

proves that our choice of (s_0, t_0) locates the minimum. Evaluating $\mathcal{M}^{(3)}(n, s_0, t_0)$ for $n = 54k + \ell$, one obtains $\frac{1}{108}\left(n^2 - 18n\right) + \delta_\ell$, where $\delta_{36} = 1$, $\delta_{30} = \delta_{42} = -\frac{1}{3}$, and all remaining δ_ℓ range from $-\frac{1}{27}$ to $\frac{101}{108}$. Hence, the claimed formula follows. □

Theorem 4 *The minimal number of monochromatic generalized Schur triples of the form $(x, y, x + 4y)$ that can be attained under any 2-coloring of $[n]$ of the form $R^s B^{t-s} R^{n-t}$ is*

$$\mathcal{M}^{(4)}(n) = \left\lfloor \frac{n^2 - 28n + 245}{216} \right\rfloor - \begin{cases} 1, & \text{if } n = 108k + i \text{ for } i \in I, \\ 0, & \text{otherwise}, \end{cases}$$

where $I = \{0, 1, 27, 28, 43, 47, 48, 53, 58, 63, 67, 68, 69, 73, 78, 83, 88, 89, 93\}$.

Proof For $a = 4$, it follows by Lemma 4 that the optimal choice for s and t is expected around the point

$$n \cdot \left(\frac{a+1}{a^2+2a+3}, \frac{a^2+2a+2}{a^2+2a+3}\right) = \left(\frac{5n}{27}, \frac{26n}{27}\right).$$

This means that the proof will require $27 \cdot a = 108$ case distinctions of the form $n = 108k + \ell$ for $0 \le \ell \le 107$. Empirically, we find that for each $n \in \mathbb{N}$, the minimum of $\mathcal{M}^{(4)}(n, s, t)$ is attained at

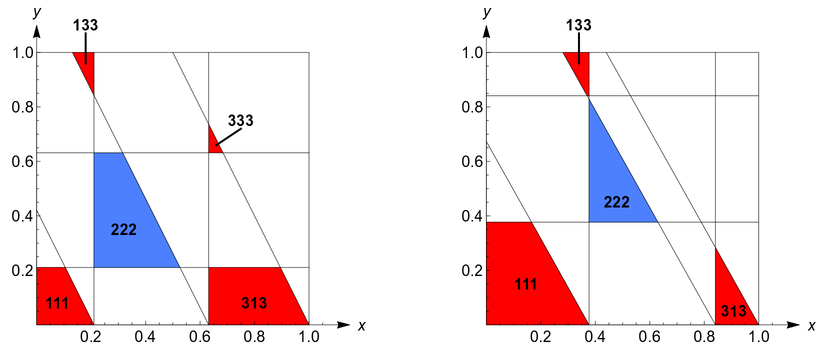

Fig. 9 The red and blue polygons correspond to monochromatic generalized Schur triples for $a = \frac{1}{2}$, $s = \frac{4}{19}$, $t = \frac{12}{19}$ (left) and $a = 0.56$, $s = 0.377$, $t = 0.841$ (right).

$$s_0 = \left\lfloor \frac{5n-4}{27} \right\rfloor + \begin{cases} -1, & \text{if } n = 108k + 28, \\ 1, & \text{if } n = 108k + i \text{ for } i \in \{0, 87, 103\}, \\ 0, & \text{otherwise.} \end{cases}$$

$$t_0 = \left\lfloor \frac{26n-34}{27} \right\rfloor + \begin{cases} -1, & \text{if } n = 108k + i \text{ for } i \in \{28, 33, 38, 43\}, \\ 1, & \text{if } n = 108k + i \\ & \quad \text{for } i \in \{1, 77, 78, 82, 83, 88, 93, 98, 104\}, \\ 2, & \text{if } n = 108k + i \text{ for } i \in \{0, 87, 103\}, \\ 0, & \text{otherwise.} \end{cases}$$

Applying CAD to the 1728 polynomials

$$\mathcal{M}^{(4)}(108k + \ell, s_0 + 4i_1 + i_2, t_0 + 4j_1 + j_2), \quad 0 \le \ell \le 107, \ i_2, j_2 \in \{0, 1, 2, 3\},$$

proves that our choice of (s_0, t_0) locates the minimum. Evaluating $\mathcal{M}^{(4)}(n, s_0, t_0)$ for $n = 108k + \ell, 0 \le \ell \le 107$, one obtains 108 polynomials of the form $\frac{1}{216}\left(n^2 - 28n\right) + \delta_\ell$. At this point, the analysis deviates a bit from the previous two theorems, because we observe that the range of the computed δ_ℓ's is much larger than 1. Therefore, we would like to choose an appropriate interval to contain the largest number of δ_ℓ such that we minimize the number of exceptional cases (i.e., the necessary corrections resulting from applying the floor function to numbers that are out of range).

To accomplish this, we find that shifting all of the values down by $\frac{29}{216}$ gives the minimum number (19, to be precise) of δ_ℓ that are not within range (i.e., not in $[0, 1)$). We now realize that these are the values that give us our desired count, so we add 1 to make sure it is recognized by the floor function. Hence, the optimal delta is $\frac{29}{216} + 1 = \frac{245}{216}$. Finally, for each of the nineteen δ_ℓ's that are out of bounds (in this case, less than 0), we remove 1 and this gives us our claimed formula. □

Theorem 5 *The minimal number of monochromatic generalized Schur triples of the form* $\big(x, y, x + \lfloor \frac{1}{2} y \rfloor\big)$ *that can be attained under any* 2*-coloring of* $[n]$ *of the form* $R^s B^{t-s} R^{n-t}$ *is given by*

$$\mathcal{M}^{(1/2)}(n) = \left\lfloor \frac{15n^2 + 72}{76} \right\rfloor + \begin{cases} 1, & \textit{if } n = 38k + 18 \textit{ or } n = 38k + 20 \\ -1, & \textit{if } n = 38k + 19, \\ 0, & \textit{otherwise}. \end{cases}$$

Proof For $a = \frac{1}{2}$, it follows by Lemma 4 that the optimal choice for s and t is expected around the point

$$n \cdot \left(\frac{-2a^2 + a + 1}{-4a^3 + 5a^2 + 6a + 1}, \frac{-2a^3 + a^2 + 4a + 1}{-4a^3 + 5a^2 + 6a + 1} \right) = \left(\frac{4n}{19}, \frac{12n}{19} \right).$$

For this choice of parameters we end up in region (R_{69}) (see Figure 6). Under the conditions that characterize this region, more precisely

$$\frac{n}{2} \le t \le \frac{2n}{3} \ \wedge \ t - s \le \frac{n}{2} \ \wedge \ 2s \le t,$$

the number of MGSTs is given by

$$\mathcal{M}^{(1/2)}(n, s, t) = \sum_{y=1}^{s} \sum_{x=1}^{s-\lfloor y/2 \rfloor} 1 + \sum_{y=s+1}^{t} \sum_{x=s+1}^{t-\lfloor y/2 \rfloor} 1 + \sum_{y=1}^{s} \sum_{x=t+1}^{n-\lfloor y/2 \rfloor} 1 + \\ + \sum_{y=2t-2s+1}^{n} \sum_{x=t+1-\lfloor y/2 \rfloor}^{s} 1 + \sum_{y=t+1}^{2n-2t-1} \sum_{x=t+1}^{n-\lfloor y/2 \rfloor} 1.$$

The five double sums correspond to the cases 111, 222, 313, 133, 333, respectively, and the summation ranges are chosen such that they actually agree with the first two coordinates of the monochromatic triples in question, see Figure 9.

In order to eliminate all floor functions, a case distinction $n = 38k + \ell$ is made. It is conjectured that the minimum is attained at $(s, t) = (s_0, t_0)$ with

$$s_0 = \left\lfloor \frac{4n + 7}{19} \right\rfloor + \begin{cases} 1, & \text{if } n = 19k + 17, \\ 0, & \text{otherwise}, \end{cases}$$

$$t_0 = \left\lfloor \frac{12n + 6}{19} \right\rfloor + \begin{cases} 1, & \text{if } n = 19k + 4, \\ 0, & \text{otherwise}. \end{cases}$$

This conjecture is proven by case distinction and CAD, as in Theorem 2. As a final result, one obtains the claimed formula, see the accompanying electronic material for details [7]. □

It has to be noted that all results presented so far in this section (Theorems 2–5) are based on the assumption of the optimal coloring being of the form $R^s B^{t-s} R^{n-t}$.

While we have strong evidence that this assumption is valid for $a > 1$ (and in fact we know it to be true [10] for $a = 1$), it seems to be inappropriate for $0 < a < 1$. More concretely, we can construct explicit examples where we get fewer MGSTs for $a = \frac{1}{2}$ than predicted in Theorem 5: the first instance is $n = 4$, where Theorem 5 yields four MGSTs for the coloring $RBBR$, namely $(1, 1, 1), (4, 1, 4), (2, 2, 3), (2, 3, 3)$, but where the better coloring $RBRB$ exists, that allows only three MGSTs, namely $(1, 1, 1), (3, 1, 3)$, and $(2, 4, 4)$. Note, however, that this is not a counter-example to the theorem because the coloring $RBRB$ is not of the form $R^s B^{t-s} R^{n-t}$.

We close this section by stating a conjecture about what we believe is the true minimum for $a = \frac{1}{2}$.

Conjecture 1 For $n \geq 12$, the minimal number of monochromatic generalized Schur triples of the form $\left(x, y, x + \lfloor \frac{1}{2} y \rfloor\right)$ that can be attained under any 2-coloring of $[n]$ is given by

$$\left\lfloor \frac{n^2 + 5}{6} \right\rfloor,$$

and it occurs at the coloring $R^s B^{t-s} R^{u-t} B^{n-u}$ for

$$s = \left\lfloor \frac{n+3}{6} \right\rfloor, \qquad t = \left\lfloor \frac{n+1}{2} \right\rfloor, \qquad u = \left\lfloor \frac{5n+3}{6} \right\rfloor.$$

Curiously, the conjectured formula is not valid for $n = 11$, where it would give a minimum number of 21 MGSTs with a four-block coloring. The true minimum is 20 and it is attained at the coloring $RBRBBRRBRBB$.

5 Conclusions and outlook

In this paper we have presented, for the first time, exact formulas for the minimum number of monochromatic (generalized) Schur triples. We give such formulas explicitly only for the few cases $a = 1, 2, 3, 4$, but we want to point out that we could do many more special cases, say $a = 5, 6, 7, \ldots$ or $a = \frac{3}{2}, \frac{5}{4}, \ldots$, based on the general analysis carried out in Section 3. In fact, the proofs would be done in completely analogous fashion, requiring only little human interaction, but an increasing amount of computation time. In this sense, our paper contains a hidden treasure, which is an infinite set of theorems that just have to be unveiled.

For future research, we propose to look more closely at the cases of generalized Schur triples $(x, y, x + \lfloor ay \rfloor)$ with $0 < a < 1$. Our analysis is based on the assumption that the optimal coloring that produces the least number of monochromatic triples consists of three blocks. Computational experiments suggest that this assumption is not valid for $0 < a < 1$. For example, we believe that four blocks are necessary to capture the minimum in the case $a = \frac{1}{2}$, as conjectured in the previous section. For some less nice rational numbers $a < 1$ we were even not able to detect a block pattern in the optimal coloring, but that may be an artifact due to the limited size of n for which we can do exhaustive searches (note that there are 2^n possible colorings).

Our results are heavily based on symbolic computation techniques, such as cylindrical algebraic decomposition and symbolic summation. Often our proofs require case distinctions into several dozens or even several hundred cases, and it would be too tedious to check all of them by hand. The reader should be convinced by now that symbolic computation can be extremely useful and that it could be adapted to solve problems in many different areas of mathematics. We provide all details of our calculations in the supplementary electronic material [7], which we hope is instructive for readers who would like to become more acquainted with the techniques that we used here.

Acknowledgment. We would like to thank our colleagues Thotsaporn Thanatipanonda, Thibaut Verron, Herwig Hauser, and Carsten Schneider for inspiring discussions, comments, and encouragement. The first author was supported by the Austrian Science Fund (FWF): P29467-N32. The second author was supported by the Austrian Science Fund (FWF): F5011-N15.

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