Holonomic Function Identities

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Motivation

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{k+n}{k}^2 = \sum_{k=0}^{n} \binom{n}{k} \binom{k+n}{k} \sum_{j=0}^{k} \binom{k}{j}^3 \tag{1}$$

$$\int_0^\infty \frac{1}{(x^4+2ax^2+1)^{m+1}}\, \mathrm{d}x = \frac{\pi P_m^{(m+\frac{1}{2}, -m-\frac{1}{2})}(a)}{2^{m+\frac{3}{2}}(a+1)^{m+\frac{1}{2}}} \tag{2}$$

$$e^{-x} x^{a/2} n! L_n^a(x) = \int_0^\infty e^{-t} t^{\frac{a}{2}+n} J_a\left(2\sqrt{tx}\right)\, \mathrm{d}t \tag{3}$$

$$\int_{-\infty}^\infty \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{H_m(x) H_n(x) r^m s^n e^{-x^2}}{m!n!}\, \mathrm{d}x = \sqrt{\pi} e^{2rs} \tag{4}$$

$$\int_{-1}^1 \left(1-x^2\right)^{\nu-\frac{1}{2}} e^{iax} C_n^{(\nu)}(x)\, \mathrm{d}x = \frac{\pi 2^{1-\nu} i^n \Gamma(n+2\nu) a^{-\nu} J_{n+\nu}(a)}{n!\Gamma(\nu)} \tag{5}$$

$$\frac{\sin\left(\sqrt{z^2+2tz}\right)}{z} = \sum_{n=0}^\infty \frac{(-t)^n y_{n-1}(z)}{n!} \tag{6}$$

Historical Context

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2008: Further improvements and Mathematica implementation by CK

Notation

- $\mathbb{K}$: field of characteristic 0
- $A_n = A_n(\mathbb{K})$: the n-th Weyl algebra
- D_x: differential operator w.r.t. x, i.e., $D_x \bullet f(x) = f'(x)$
- S_n: shift operator w.r.t. n, i.e., $S_n \bullet f(n) = f(n+1)$
- $\mathbb{O}$: an Ore algebra
- $\operatorname{Ann}_{\mathbb{O}} f$: the ideal of annihilating operators of f in $\mathbb{O}$, i.e., $\operatorname{Ann}_{\mathbb{O}} f = \{P \in \mathbb{O} \mid P \bullet f = 0\}$
  Note: We often denote also subideals of the full annihilator by $\operatorname{Ann}_{\mathbb{O}} f$.

Definition: Ore Algebra (1)

Let $\mathcal{F}$ be a $\mathbb{K}$-algebra (of "functions"), and let $\sigma, \delta \in \mathrm{End}_{\mathbb{K}} \mathcal{F}$ with

$$\delta(fg) = \sigma(f)\delta(g) + \delta(f)g \quad \text{for all } f, g \in \mathcal{F} \quad \text{(skew Leibniz law).}$$

The endomorphism δ is called a σ-*derivation*.

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Let $\mathbb{A}$ be a $\mathbb{K}$-subalgebra of $\mathcal{F}$ (e.g., $\mathbb{A} = \mathbb{K}[x]$ or $\mathbb{A} = \mathbb{K}(x)$) and assume that σ, δ restrict to a σ-derivation on $\mathbb{A}$.

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Define the skew polynomial ring $\mathbb{O} := \mathbb{A}[\partial; \sigma, \delta]$:

- polynomials in ∂ with coefficients in $\mathbb{A}$
- usual addition
- product that makes use of the commutation rule

$$\partial a = \sigma(a)\partial + \delta(a) \text{ for all } a \in \mathbb{A}$$

Definition: Ore Algebra (2)

We turn $\mathcal{F}$ into an $\mathbb{O}$-module by defining an action of elements in $\mathbb{O}$ on a function $f \in \mathcal{F}$, e.g., by

$$\begin{aligned} a \bullet f &:= a \cdot f \quad (a \in \mathbb{A}), \\ \partial \bullet f &:= \delta(f). \end{aligned}$$

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Example 1: $\mathbb{A} = \mathbb{K}[x]$, $\sigma = 1$, $\delta = \frac{\mathrm{d}}{\mathrm{d}x}$.

Then $\mathbb{K}[x][D_x; 1, \frac{\mathrm{d}}{\mathrm{d}x}] = \mathbb{K}[x][D_x; 1, D_x]$ is the Weyl algebra A_1.

Example 2: $\mathbb{A} = \mathbb{K}[n]$, $\sigma(n) = n + 1$, $\sigma(c) = c$ for $c \in \mathbb{K}$, $\delta = 0$.

Then $\mathbb{K}[n][S_n; S_n, 0]$ is a shift algebra.

Example 3: $\mathbb{K}(n, x, y)[S_n; S_n, 0][D_x; 1, D_x][D_y; 1, D_y]$

Holonomic functions

Definition:
A function $f(x_1, \ldots, x_n) \in \mathcal{F}$ is said to be holonomic if $A_n / \operatorname{Ann}_{A_n} f$ is a holonomic A_n-module, i.e., the Bernstein dimension is minimal (according to Bernstein's inequality).

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Definition:
A sequence $f(k_1, \dots, k_n) \in \mathbb{C}^{\mathbb{N}^n}$ is holonomic if its multivariate generating function

$$F(x_1, \dots, x_n) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} f(k_1, \dots, k_n) x_1^{k_1} \cdots x_n^{k_n}$$

is a holonomic function.

Properties of holonomic functions

Closure properties:

- sum
- product
- definite integration
- ...

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Elimination property:
Given an ideal I in A_n s.t. A_n/I is holonomic; then for any choice of $n+1$ among the $2n$ generators of A_n there exists a nonzero operator in I that depends only on these. In other words, we can eliminate $n-1$ variables.

Definite integration via elimination

Given: Integration bounds $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$ and $\operatorname{Ann}_{\mathbb{O}} f$, the annihilator of a holonomic function $f(x, y)$ in the Ore algebra $\mathbb{O} = \mathbb{K}[x, y][D_x; 1, D_x][D_y; 1, D_y] = A_2$

Find: A differential equation for $F(y) = \int_a^b f(x, y)\,\mathrm{d}x$

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Find: A differential equation for $F(y) = \int_a^b f(x, y)\,\mathrm{d}x$

Since $\mathbb{O}/\operatorname{Ann}_{\mathbb{O}} f$ is holonomic, there exists $P \in \operatorname{Ann}_{\mathbb{O}} f$ that does not contain x (by elimination property). Write

$$P(y, D_x, D_y) = Q(y, D_y) + D_x \cdot R(y, D_x, D_y)$$

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Apply $\int_a^b \ldots \mathrm{d}x$ to $P \bullet f = 0$:

$$\int_a^b Q(y, D_y) \bullet f \,\mathrm{d}x + \int_a^b D_x R(y, D_x, D_y) \bullet f \,\mathrm{d}x = 0$$

$$Q(y, D_y) \bullet F(y) + \Big[R(y, D_x, D_y) \bullet f \Big]_{x=a}^{x=b} = 0$$

Definite summation via elimination

Given: Summation bounds $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$ and $\operatorname{Ann}_{\mathbb{O}} f$, the annihilator of a holonomic sequence $f(k,n)$ in the Ore algebra $\mathbb{O} = \mathbb{K}[k,n][S_k; S_k, 0][S_n; S_n, 0]$
Find: A recurrence for $F(n) = \sum_{k=a}^{b} f(k,n)$

By the elimination property there exists $P \in \operatorname{Ann}_{\mathbb{O}} f$ that does not contain k. Write

$$P(n, S_k, S_n) = Q(n, S_n) + (S_k - 1) \cdot R(n, S_k, S_n)$$

Sum over the equation $P \bullet f = 0$:

$$\sum_{k=a}^{b} Q(n, S_n) \bullet f + \sum_{k=a}^{b} (S_k - 1) R(n, S_k, S_n) \bullet f = 0$$

$$Q(n, S_n) \bullet F(n) + \Big[R(n, S_k, S_n) \bullet f(k,n) \Big]_{k=a}^{k=b+1} = 0$$

Example: Orthogonality of Hermite polynomials (1)

The Hermite polynomials $H_n(x)$ are a family of orthogonal polynomials w.r.t. to the weight funtion e^{-x^2}.
Let's prove this, i.e.,

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x)\, \mathrm{d}x = \delta_{m,n} \sqrt{\pi} 2^n n!$$

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First we compute an annihilator of the integrand:

```
ann = Annihilator[Exp[-x^2]*HermiteH[m,x]*HermiteH[n,x],
                  {Der[x], S[m], S[n]}]
```

$$\begin{gathered}\{-2x + D_x + S_m + S_n, \\ S_n^2 - 2xS_n + 2n + 2, \\ S_m^2 - 2xS_m + 2m + 2\}\end{gathered}$$

Example: Orthogonality of Hermite polynomials (2)

Next step is to compute a Gröbner basis w.r.t. lexicographical order in order to eliminate x:

```
gb = OreGroebnerBasis[
        ann, OreAlgebra[x, m, n, Der[x], S[m], S[n]],
        MonomialOrder -> Lexicographic]
```

$$\begin{aligned}\{&-D_x S_n - S_m S_n + 2n + 2,\\ &-D_x S_m - S_m S_n + 2m + 2,\\ &D_x + S_m + S_n - 2x \quad\}\end{aligned}$$

Example: Orthogonality of Hermite polynomials (3)

In the first operator, the part $R = -S_n$, in the second $R = -S_m$. We have to check that $[R \bullet f]_{x=-\infty}^{x=\infty}$ indeed vanishes. Clearly

$$\lim_{x \to \pm\infty} -e^{-x^2} H_m(x) H_{n+1}(x) = 0$$

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Hence we take the first two operators (which do not involve the integration variable x) and set D_x to 0:

```
OrePolynomialSubstitute[Take[gb, 2], {Der[x] -> 0}]
```

$$\{-S_m S_n + 2n + 2, -S_m S_n + 2m + 2\}$$

Example: Orthogonality of Hermite polynomials (4)

By computing again a Gröbner basis of the previous, we get a nicer result:

```
OreGroebnerBasis[%, OreAlgebra[m, n, S[m], S[n]]]
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$$\{m - n, -S_m S_n + 2n + 2\}$$

This proves that the right hand side can only be nonzero if $m = n$.

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This proves that the right hand side can only be nonzero if $m = n$.
By similar computations we obtain the recurrence

$$\left(4n^2 + 8n + 4\right) f(n) + (-4n - 6) f(n + 1) + f(n + 2) = 0$$

for the right hand side when we set $m = n$.
Together with the initial values $f(0) = \sqrt{\pi}$ and $f(1) = 2\sqrt{\pi}$ we have a full and simple description of the desired result.

Definite integration with Takayama's algorithm

Given: $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$ and $\operatorname{Ann}_{\mathbb{O}} f$, the annihilator of a holonomic function $f(x, y)$ (which must have natural boundaries at a and b) in the Ore algebra $\mathbb{O} = \mathbb{K}[x, y][D_x; 1; D_x][D_y; 1, D_y]$.
Find: The annihilator of $F(y) = \int_a^b f(x, y)\,\mathrm{d}x$ in the Ore algebra $\mathbb{O}' = \mathbb{K}[y][D_y; 1, D_y]$

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Find: The annihilator of $F(y) = \int_a^b f(x, y)\,\mathrm{d}x$ in the Ore algebra $\mathbb{O}' = \mathbb{K}[y][D_y; 1, D_y]$

Compute a $P \in \operatorname{Ann}_{\mathbb{O}} f$ that can be written in the form

$$P(x, y, D_x, D_y) = Q(y, D_y) + D_x \cdot R(x, y, D_x, D_y)$$

By the same reasoning as before, we get

$$Q(y, D_y) \bullet F(y) = 0$$

The operator Q can be computed with Takayama's algorithm.

Comparison

Zeilberger:

1. eliminate x
2. reduce modulo $D_x\mathbb{O}$

Takayama (variant due to Chyzak/Salvy):

1. reduce modulo $D_x\mathbb{O}$
2. eliminate x

How to eliminate x?

Problem: After reducing modulo $D_x\mathbb{O}$, no multiplication by x is allowed any more!

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$$P + D_x Q$$

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$$\big\downarrow \cdot x$$

$$xP + (D_x x - 1)Q$$

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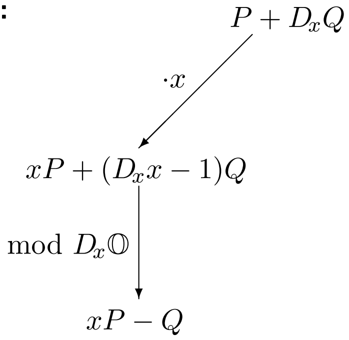

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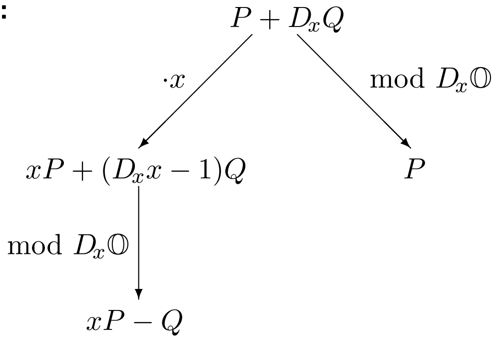

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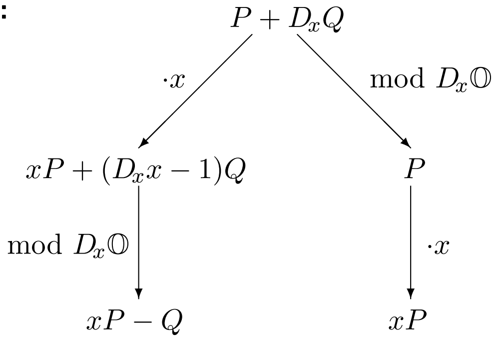

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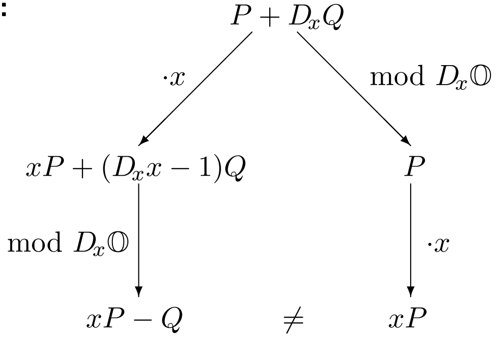

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Eliminate x by computing a Gröbner basis in the $\mathbb{O}'$-module w.r.t. the basis $x^\alpha, \alpha \in \mathbb{N}$:

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$\longrightarrow$ gives $(-1, 0, 1+y, 0, \dots)$

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$$x + D_x D_y + y \equiv x + y \mod D_x\mathbb{O}$$

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As an input for this Gröbner basis computation, we take the generators of $\operatorname{Ann}_{\mathbb{O}} f$ plus some multiples by x^α, e.g.,

$$x^2 + xD_xD_y + xy = x^2 + D_x x D_y - xD_y + xy \equiv x^2 - x(D_y + y)$$

$\longrightarrow$ gives $(0, D_y + y, 1, 0, \dots)$

Task: We have to find an element of the form $(Q, 0, 0, \dots)$.

Takayama's algorithm

Input: a set of generators $\{G_1, \ldots, G_m\}$ for $\mathrm{Ann}_{\mathbb{O}}\, f$
Output: $\mathrm{Ann}_{\mathbb{O}'}\, F$

1. set $d = \max_{1 \leq i \leq m} \deg_x G_i$
2. set $A = \{G_1, \ldots, G_m\} \cup \bigcup_{i=1}^{m} \{x^\alpha G_i \,|\, 1 \leq \alpha \leq \deg_x G_i\}$
3. reduce all elements in A modulo $D_x\mathbb{O}$
4. compute a Gröbner basis in the corresponding (truncated) module in order to eliminate x
5. if no $(Q, 0, \ldots, 0)$ is found, increase d

Since f is holonomic the algorithm is guaranteed to terminate.

Example: Proof of Gessel's conjecture (1)

(joint work with M. Kauers and D. Zeilberger)

Definition: A Gessel walk is a walk in the integer lattice $\mathbb{N}^2$ which uses only steps from the set $\{\leftarrow, \rightarrow, \swarrow, \nearrow\}$. Let $f(n; i, j)$ denote the number of Gessel walks with exactly n steps starting at the origin $(0, 0)$ and ending at the point (i, j).

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Ira Gessel in 2001 conjectured that

$$f(n; 0, 0) = \begin{cases} 16^k \frac{(5/6)_k (1/2)_k}{(2)_k (5/3)_k} & \text{if } n = 2k \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

The function $f(n; 0, 0)$ counts the number of closed Gessel walks of length n.

Example: Proof of Gessel's conjecture (2)

Let $\mathbb{O} = \mathbb{Q}(i,j,n)[S_i;S_i,0][S_j;S_j,0][S_n;S_n,0]$.

The quasi-holonomic ansatz: Find an operator $R \in \operatorname{Ann}_{\mathbb{O}} f(n;i,j)$ of the form

$$\begin{aligned} R(n,i,j,S_n,S_i,S_j) = P(n,S_n) \quad &+ \quad iQ_1(n,i,j,S_n,S_i,S_j) \\ &+ \quad jQ_2(n,i,j,S_n,S_i,S_j) \end{aligned}$$

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Let $\mathbb{O} = \mathbb{Q}(i, j, n)[S_i; S_i, 0][S_j; S_j, 0][S_n; S_n, 0]$.

The quasi-holonomic ansatz: Find an operator $R \in \operatorname{Ann}_{\mathbb{O}} f(n; i, j)$ of the form

$$\begin{aligned} R(n, i, j, S_n, S_i, S_j) = P(n, S_n) &+ iQ_1(n, i, j, S_n, S_i, S_j) \\ &+ jQ_2(n, i, j, S_n, S_i, S_j) \end{aligned}$$

- $R(n, i, j, S_n, S_i, S_j)$ annihilates $f(n; i, j)$
- set $i = j = 0$
- $P(n, S_n)$ annihilates $f(n; 0, 0)$

Problem: $R(n, i, j, S_n, S_i, S_j)$ is too big to be computed.

Example: Proof of Gessel's conjecture (3)

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Remark: The result will be $P(n, S_n)$ as above, but Q_1 and Q_2 are not computed at all.

$\longrightarrow$ Computation becomes feasible!

Example: Proof of Gessel's conjecture (4)

We found an operator $P(n, S_n)$ annihilating $f(n; 0, 0)$ with

- order 32
- polynomial coefficients of degree 172
- and integer coefficients up to 385 digits.

The computation took 7 hours.

Example: Proof of Gessel's conjecture (5)

Doron Zeilberger's bet:
"I offer a prize of one hundred (100) US-dollars for a short, self-contained, human-generated (and computer-free) proof of Gessel's conjecture, not to exceed five standard pages typed in standard font. The longer that prize would remain unclaimed, the more (empirical) evidence we would have that a proof of Gessel's conjecture is indeed beyond the scope of humankind."

∂-finite functions

Definition: A "function" $f(x_1, \ldots, x_n)$ is called ∂-finite w.r.t. $\mathbb{O} = \mathbb{K}(x_1, \ldots, x_n)[\partial_1; \sigma_1, \delta_1] \cdots [\partial_n; \sigma_n, \delta_n]$ if it is 0 or if $0 < \dim_{\mathbb{K}(x_1,\ldots,x_n)} \mathbb{O} / \operatorname{Ann}_{\mathbb{O}} f < \infty$.

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In other words, f is ∂-finite if all its "derivatives" span a finite-dimensional $\mathbb{K}(x_1, \ldots, x_n)$-vector space.

Example: All derivatives (w.r.t. x and y) of $\sin\left(\frac{x+y}{x-y}\right)$ are of the form

$$r_1(x,y)\sin\left(\frac{x+y}{x-y}\right) + r_2(x,y)\cos\left(\frac{x+y}{x-y}\right), \quad r_1, r_2 \in \mathbb{Q}(x,y)$$

e.g.,

$$D_x^3 D_y^2 \bullet \sin\left(\frac{x+y}{x-y}\right) = \frac{32\left(3x^4+12yx^3-30y^2x^2-4y^3x+9y^4\right)}{(x-y)^9}\sin\left(\frac{x+y}{x-y}\right) - \frac{16\left(6x^5-33yx^4+80y^3x^2-54y^4x+3y^5\right)}{(x-y)^{10}}\cos\left(\frac{x+y}{x-y}\right)$$

Closure properties of ∂-finite functions

Closure properties:

- sum
- product
- application of an Ore operator
- algebraic substitution (of a continuous variable)
- subsequence / $\mathbb{Q}$-linear substitution (of a discrete variable)
- definite summation and integration

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In contrast to holonomic closure properties, the closure properties for ∂-finite functions can be computed quite easily (using linear algebra and an FGLM-like algorithm).

Holonomic vs. ∂-finite

holonomic description:	$\mathbb{K}[x_1, \ldots, x_n][\partial_1; \sigma_1, \delta_1] \cdots [\partial_n; \sigma_n, \delta_n]$
∂-finite description:	$\mathbb{K}(x_1, \ldots, x_n)[\partial_1; \sigma_1, \delta_1] \cdots [\partial_n; \sigma_n, \delta_n]$

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In the shift case there are only some subtle exceptions:

- $\frac{1}{k^2+n^2}$ is ∂-finite but not holonomic.
- $\delta_{i,j}$ is holonomic but not ∂-finite.

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In the shift case there are only some subtle exceptions:

- $\frac{1}{k^2+n^2}$ is ∂-finite but not holonomic.
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In practice, consider functions that are both holonomic and ∂-finite:

- ∂-finite: better algorithmic treatment
- holonomic: guarantees termination of algorithms

Implementation of ∂-finite closure properties

The function `Annihilator` automatically executes ∂-finite closure properties.

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The function `Annihilator` automatically executes ∂-finite closure properties.

The base cases are stored in a "database". Up to now it contains: Exp, Log, Sqrt, Sin, Cos, ArcSin, ArcCos, ArcTan, ArcCot, ArcSec, ArcCsc, Sinh, Cosh, ArcSinh, ArcCosh, ArcTanh, ArcCoth, ArcSech, ArcCsch, BesselJ, BesselY, BesselI, BesselK, HankelH1, HankelH2, AiryAi, AiryAiPrime, AiryBi, AiryBiPrime, StruveH, StruveL, KelvinBei, KelvinBer, KelvinKei, KelvinKer, SphericalBesselJ, SphericalBesselY, SphericalHankelH1, SphericalHankelH2, Fibonacci, LucasL, HermiteH, LaguerreL, LegendreP, ChebyshevT, ChebyshevU, GegenbauerC, JacobiP, Fibonacci, Factorial, Factorial2, Pochhammer, Binomial, Multinomial, CatalanNumber, Gamma, GammaRegularized, Subfactorial, PolyGamma, HarmonicNumber, Beta, BetaRegularized, Erf, Erfc, Erfi, FresnelS, FresnelC, ExpIntegralE, ExpIntegralEi, SinIntegral, CosIntegral, SinhIntegral, CoshIntegral, HypergeometricPFQ, EllipticE, EllipticK, EllipticPi

Creative telescoping: Chyzak's extension of Zeilberger's fast algorithm

Given: $\operatorname{Ann}_{\mathbb{O}} f$, the annihilator of a ∂-finite function $f(x, y)$ in the Ore algebra $\mathbb{O} = \mathbb{K}(x, y)[D_x; 1; D_x][D_y; 1, D_y]$.
Find: $Q(y, D_y)$ and $R(x, y, D_x, D_y)$ such that $Q + D_x \cdot R \in \operatorname{Ann}_{\mathbb{O}} f$.

1. compute a Gröbner basis G of $\operatorname{Ann}_{\mathbb{O}} f$ in order to know the set $U = \{u_1, \ldots, u_k\}$ of monomials that can not be reduced by $\operatorname{Ann}_{\mathbb{O}} f$, i.e., the elements under the stairs of G
2. make an ansatz for $Q(y, D_y) = \sum_{i=0}^{d} \eta_i(y) D_y^i$ and $R(x, y, D_x, D_y) = \sum_{j=1}^{k} \phi_j(x, y) u_j$
3. reduce $Q + D_x \cdot R$ with G and set all coefficients to zero
4. solve the corresponding coupled system of differential equations (for rational solutions)
5. if there is no solution, increase d

Example: One of Olver's problems (1)

Prove the following identity:

$$\frac{1}{z}\sinh\sqrt{z^2-2izt}=\sum_{n=0}^{\infty}\frac{(-it)^n}{n!}\sqrt{\frac{\pi}{2z}}I_{\frac{1}{2}-n}(z)$$

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For the left hand side, we can immediately compute annihilating operators:

```
F[t_,z_] := Sinh[Sqrt[z^2 - 2*I*z*t]]/z
lhs = Annihilator[F[t,z], {Der[t], Der[z]}]
```

$$\begin{aligned}&\{(-t-iz)D_t+zD_z+1,\\&(-z^4+3itz^3+2t^2z^2)D_z^2+(-2z^3+6itz^2+5t^2z)D_z\\&+(z^4-3itz^3-3t^2z^2+it^3z+t^2)\}\end{aligned}$$

Example: One of Olver's problems (2)

On the right hand side $\sum_{n=0}^{\infty} \frac{(-it)^n}{n!} \sqrt{\frac{\pi}{2z}} I_{\frac{1}{2}-n}(z)$, we do creative telescoping:

```
f[n_,t_,z_] := (-I*t)^n/n!*Sqrt[Pi/2/z]*BesselI[-n+1/2,z];
{opQ, opR} =
  CreativeTelescoping[f[n,t,z], S[n]-1, {Der[t],Der[z]}]
```

We obtain two operators $Q_1 + (S_n - 1) \cdot R_1 \in \mathrm{Ann}\, f$ and $Q_2 + (S_n - 1) \cdot R_2 \in \mathrm{Ann}\, f$ where

$$\begin{aligned}
Q_1 &= -t(t+iz)D_t + tzD_z + t, \\
Q_2 &= (t+iz)(2t+iz)z^2D_z^2 - z(-5t^2 - 6izt + 2z^2)D_z \\
&\quad +i(-iz^4 - 3tz^3 + 3it^2z^2 + t^3z - it^2), \\
R_1 &= -inz, \\
R_2 &= i(n^2+n)(t+iz)zS_n + 2t^2n^2 - z^2n^2 + 3itzn^2 - t^2n \\
&\quad +z^2n - 3itzn
\end{aligned}$$

Example: One of Olver's problems (3)

Next we have to verify that $[R_1 \bullet f]_{n=0} = 0$ and that $R_1 \bullet f$ tends to 0 when n goes to infinity (the same for R_2):

```
ApplyOreOperator[opR, f[n,t,z]] /. n->0
```

$$\{0, 0\}$$

$\longrightarrow$ The delta part vanishes.
Hence Q_1 and Q_2 are annihilating operators for the sum. In fact, we find that they agree with the annihilating operators that we computed for the left hand side.

Example: One of Olver's problems (4)

In order to establish equality, we have to compare initial values. Look at the vector space under the stairs of the Gröbner basis:

```
u = UnderTheStaircase[lhs]
```

$$\{1, D_z\}$$

This means we have to compute two initial values:

```
ApplyOreOperator[u, F[t,z]] /. {t->0,z->1} //FullSimplify
```

$$\{\sinh(1), \tfrac{1}{e}\}$$

```
ApplyOreOperator[u, f[n,t,z]] /. {t->0,z->1}
```

$$\left\{ \frac{0^n \sqrt{\frac{\pi}{2}} I_{\frac{1}{2}-n}(1)}{n!}, \frac{0^n \sqrt{\frac{\pi}{2}} \left(I_{-n-\frac{1}{2}}(1) + I_{\frac{3}{2}-n}(1) \right)}{2n!} - \frac{0^n \sqrt{\frac{\pi}{2}} I_{\frac{1}{2}-n}(1)}{2n!} \right\}$$

```
% /. (0^n)->1 /. n->0 // FullSimplify
```

$$\{\sinh(1), \tfrac{1}{e}\}$$

Application in Finite Element Methods (1)

(joint work with Joachim Schöberl, RWTH Aachen)

Task: Compute electromagnetic waves using the Maxwell equations:

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \operatorname{curl} E, \quad \frac{\mathrm{d}E}{\mathrm{d}t} = -\operatorname{curl} H$$

where H and E are the magnetic and the electric field respectively.

Method: Divide the domain into small triangles (finite elements). Approximate the solution by certain basis functions that are defined on each finite element.

In this application we define the basis functions as follows:

$$\varphi_{i,j}(x,y) := (1-x)^i P_j^{(2i+1,0)}(2x-1) P_i\left(\frac{2y}{1-x} - 1\right)$$

In order to solve the above equations, one needs to represent the partial derivatives of $\varphi_{i,j}(x,y)$ in the basis (i.e., as linear combinations of the $\varphi_{i,j}(x,y)$ itself).

Application in Finite Element Methods (2)

```
phi[i_,j_,x_,y_] :=
 LegendreP[i,2*y/(1-x)-1]*(1-x)^i*JacobiP[j,2*i+1,0,2*x-1]
ann = Annihilator[phi[i,j,x,y], {Der[x], S[i], S[j]}]
```

⟨*quite big output*⟩

In order to see better the structure of the output, we look only at the support of each operator:

```
Support /@ ann
```

$$\{\{S_j^2, S_j, 1\}, \{S_i S_j, D_x, S_i, S_j, 1\}, \{S_i^2, D_x, S_i, S_j, 1\}, \{D_x S_j, D_x, S_i, S_j, 1\}, \{D_x S_i, D_x, S_i, S_j, 1\}, \{D_x^2, D_x, S_i, S_j, 1\}\}$$

$\longrightarrow$ We see that the second and the third operator match exactly our needs!

Application in Finite Element Methods (3)

BUT: The numerists need a relation that is free of x and y! In change, they allow also shifts in the derivative, i.e., we are now looking for a relation of the following form:

$$\sum_{(k,l)\in A} a_{k,l}(i,j)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+k,j+l}(x,y) = \sum_{(m,n)\in B} b_{m,n}(i,j)\varphi_{i+m,j+n}(x,y),$$

where $A, B \subset \mathbb{N}^2$ are finite index sets.

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where $A, B \subset \mathbb{N}^2$ are finite index sets.

- Use Gröbner basis computation in order to eliminate x and y.
- After some time we get an operator of the desired form, that is even not too big (about 2 pages).
- Because of extension/contraction problem we can not be sure that we obtain the smallest operator.

Application in Finite Element Methods (4)

$$\sum_{(k,l)\in A} a_{k,l}(i,j)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+k,j+l}(x,y) = \sum_{(m,n)\in B} b_{m,n}(i,j)\varphi_{i+m,j+n}(x,y),$$

New idea: Similar approach as in creative telescoping.

1. we work in $\mathbb{O} = \mathbb{K}(i,j,x,y)[D_x;1,D_x][S_i;S_i,0][S_j;S_j,0]$
2. choose index sets A and B
3. reduce the corresponding ansatz with the Gröbner basis of $\operatorname{Ann}_{\mathbb{O}}\varphi$
4. do coefficient comparison w.r.t. x and y
5. solve the resulting linear system for $a_{k,l}$ and $b_{m,n}$ in $\mathbb{K}(i,j)$

Application in Finite Element Methods (5)

With this method, we find (in less than 1 minute) the following relation:

$$
\begin{aligned}
&(2i+j+3)(2i+2j+7)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i,j+1}(x,y)\\
&+2(2i+1)(i+j+3)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i,j+2}(x,y)\\
&-(j+3)(2i+2j+5)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i,j+3}(x,y)\\
&+(j+1)(2i+2j+7)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+1,j}(x,y)\\
&-2(2i+3)(i+j+3)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+1,j+1}(x,y)\\
&-(2i+j+5)(2i+2j+5)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+1,j+2}(x,y) =\\
&2(i+j+3)(2i+2j+5)(2i+2j+7)\varphi_{i,j+2}(x,y)\\
&+2(i+j+3)(2i+2j+5)(2i+2j+7)\varphi_{i+1,j+1}(x,y)
\end{aligned}
$$

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&+2(i+j+3)(2i+2j+5)(2i+2j+7)\varphi_{i+1,j+1}(x,y)
\end{aligned}$$

Schöberl: “Now I am really impressed: this is exactly what I need!”

Thank you for your attention!