Tweaking the Beukers integrals in search of more miraculous irrationality proofs à la Apéry

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March 30, 2023
Workshop on Effective Aspects in Diophantine Approximation
ENS Lyon

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- The numbers u_n are all integers.
- The denominators of v_n are growing moderately.
- More precisely: $d_n^3 v_n \in \mathbb{Z}$ where $d_n := \operatorname{lcm}(1, 2, \ldots, n)$

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$$I(0) = \frac{\pi^2}{6}, \quad I(1) = 5 - \frac{\pi^2}{2}, \quad I(2) = -\frac{125}{4} + \frac{19\pi^2}{6}, \quad \dots$$

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One sees that

$$I(n) = v_n - u_n \frac{\pi^2}{6}$$

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One sees that

$$I(n) = v_n - u_n\frac{\pi^2}{6} \quad \text{and} \quad \lim_{n\to\infty}\frac{v_n}{u_n} = \frac{\pi^2}{6}$$

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Hence, the family of integrals $I(n)$ yields a sequence of rational approximations to $\zeta(2)$:

$$\lim_{n\to\infty} \frac{v_n}{u_n} = \frac{\pi^2}{6},$$

and u_n, v_n satisfy $(n+1)^2 a_{n+1} + (11n^2 + 11n + 3)a_n = n^2 a_{n-1}$.

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Since $v_n \in \mathbb{Q}$, we clear denominators and write

$$\frac{v_n}{u_n} = \frac{v_n'}{u_n'}, \quad u_n', v_n' \in \mathbb{Z}.$$

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Since $v_n \in \mathbb{Q}$, we clear denominators and write

$$\frac{v_n}{u_n} = \frac{v'_n}{u'_n}, \quad u'_n, v'_n \in \mathbb{Z}.$$

By estimating the size of the integral $I(n)$, one can show, by denoting $I'(n) = u'_n \frac{\pi^2}{6} - v'_n$:

$$\lim_{n\to\infty} |I'(n)| = 0 \quad \text{and} \quad I'(n) \neq 0.$$

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Starting from the integral

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and more generally: $I(n) = u_n\zeta(3) - v_n$. In fact, $I(n)$ satisfies

$$(n+2)^3 I(n+2) = (2n+3)(17n^2+51n+39)I(n+1) - (n+1)^3 I(n).$$

Zeilberger

An e-mail from Doron Zeilberger:

For the Beukers integral for Zeta(3)

```
B(n):=int(int(int( (x*(1-x)*y*(1-y)*z*(1-z))^n/(1-z+x*y*z)^(n+1),x=0..1),y=0..1),z=0..1)
```

even without any extra parameters it takes a VERY long time.
In an optimized version, that targets these kind of integrals it still takes about 2000 seconds.

Our questions are:

1. Can your package find these recurrence in one "key-stroke" or does it need some pre-processing?

2. How fast can your package find the recurrence for B(n), and similar integrals where you stick in the integrand x^(a1)*(1-x)^a2*...
(for numeric a1,a2, ..)

Holonomic Functions

Definition: A function $f(x)$ is called **holonomic** if it satisfies a linear ordinary differential equation with polynomial coefficients:

$$p_r(x)f^{(r)}(x) + \cdots + p_1(x)f'(x) + p_0(x)f(x) = 0,$$

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$\longrightarrow$ In both cases, one needs only **finitely many** initial conditions.

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Many special functions can be characterized as solutions to systems of linear differential equations and recurrences, and in fact are holonomic.

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$$\frac{\mathrm{d}^{i_1}}{\mathrm{d}x_1^{i_1}} \cdots \frac{\mathrm{d}^{i_s}}{\mathrm{d}x_s^{i_s}} f(x_1, \ldots, x_s, n_1 + j_1, \ldots, n_r + j_r)$$

with $i_1, \ldots, i_s, j_1, \ldots, j_r \in \mathbb{N}$ such that any shifted partial derivative of f (of the above form) can be expressed as a $\mathbb{K}(x_1, \ldots, x_s, n_1, \ldots, n_r)$-linear combination of the basis functions (plus some further, technical assumptions), then f is **holonomic**.

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Let $f(x_1, \dots, x_s, n_1, \dots, n_r)$ be a function in the continuous variables $x_1, \dots, x_s$ and in the discrete variables $n_1, \dots, n_r$.
If there is a finite set of basis functions of the form

$$\frac{\mathrm{d}^{i_1}}{\mathrm{d}x_1^{i_1}} \cdots \frac{\mathrm{d}^{i_s}}{\mathrm{d}x_s^{i_s}} f(x_1, \dots, x_s, n_1 + j_1, \dots, n_r + j_r)$$

with $i_1, \dots, i_s, j_1, \dots, j_r \in \mathbb{N}$ such that any shifted partial derivative of f (of the above form) can be expressed as a $\mathbb{K}(x_1, \dots, x_s, n_1, \dots, n_r)$-linear combination of the basis functions (plus some further, technical assumptions), then f is **holonomic**.

$\longrightarrow$ Finitely many initial conditions suffice.

Holonomic Functions

A function $f_{\ell,m,\dots,n}(x,y,\dots,z)$ is called holonomic, if it is the solution of a system

- of **linear** differential equations or recurrences,

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- $f_n(h(x))$, where $h(x)$ is an algebraic function.

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A holonomic function a priori is an infinite object (e.g., $\mathbb{R}^2 \to \mathbb{R}^2$).

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- fast numerical evaluation of mathematical functions
- evaluate symbolic determinants (e.g., in combinatorics)
- number theory (e.g., irrationality proofs)

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$$\underbrace{\sum_{k=1}^{\infty} \frac{1}{k(k+n)}}_{=:\, f_n} \rightsquigarrow (n+2)^2 f_{n+2} = (n+1)(2n+3) f_{n+1} - n(n+1) f_n$$

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Method for doing integrals and sums
(already mentioned in van der Poorten's report of Apéry's proof!)

Consider the following summation problem: $F(n) := \sum_{k=a}^{b} f(n,k)$

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Then $F(n) = \sum_{k=a}^{b} \big(g(n,k+1) - g(n,k)\big) = g(n,b+1) - g(n,a)$.

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Creative Telescoping: write

$$c_r(n)f(n+r,k) + \cdots + c_0(n)f(n,k) = g(n,k+1) - g(n,k).$$

Summing from a to b yields a recurrence for $F(n)$:

$$c_r(n)F(n+r) + \cdots + c_0(n)F(n) = g(n,b+1) - g(n,a).$$

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Method for doing integrals and sums
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Consider the following integration problem: $F(x) := \int_a^b f(x,y)\,\mathrm{d}y$

Telescoping: write $f(x,y) = \frac{\mathrm{d}}{\mathrm{d}y} g(x,y)$.

Then $F(n) = \int_a^b \left(\frac{\mathrm{d}}{\mathrm{d}y} g(x,y)\right)\mathrm{d}y \qquad = g(x,b) - g(x,a)$.

Creative Telescoping: write

$$c_r(x)\frac{\mathrm{d}^r}{\mathrm{d}x^r} f(x,y) + \cdots + c_0(x) f(x,y) \;= \frac{\mathrm{d}}{\mathrm{d}y} g(x,y).$$

Integrating from a to b yields a differential equation for $F(x)$:

$$c_r(x)\frac{\mathrm{d}^r}{\mathrm{d}x^r} F(x) + \cdots + c_0(x) F(x) \;= g(x,b) - g(x,a)$$

Zeilberger

An e-mail from Doron Zeilberger:

For the Beukers integral for Zeta(3)

```
B(n):=int(int(int( (x*(1-x)*y*(1-y)*z*(1-z))^n/(1-z+x*y*z)^(n+1),x=0..1),y=0..1),z=0..1)
```

even without any extra parameters it takes a VERY long time.
In an optimized version, that targets these kind of integrals it still takes about 2000 seconds.

Our questions are:

1. Can your package find these recurrence in one "key-stroke" or does it need some pre-processing?

2. How fast can your package find the recurrence for B(n), and similar integrals where you stick in the integrand x^(a1)*(1-x)^a2*...
(for numeric a1,a2, ..)

Demo

$$\int_0^1 \int_0^1 \frac{(x(1-x)y(1-y))^n}{(1-xy)^{n+1}} \, \mathrm{d}x \, \mathrm{d}y.$$

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$$\sum_{k=0}^{n} \binom{n+k}{k}^2 \binom{n}{k}^2$$

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$$\sum_{k=0}^{n} \binom{n+k}{k}^2 \binom{n}{k}^2$$

$$\sum_{k=0}^{n} \binom{n+k}{k}^2 \binom{n}{k}^2 \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n+m}{m} \binom{n}{m}}$$

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$$\frac{1}{n!} \left(\frac{\mathrm{d}}{\mathrm{d}z} + \frac{\gamma z + \delta}{z(z-a)} \right)^n z^n (z-a)^n$$

Beukers Integral

Task: Show that the Beukers integral for $\zeta(3)$ satisfies Apéry's second-order recurrence:

$$(n+2)^3 I(n+2) = (2n+3)(17n^2+51n+39)I(n+1) - (n+1)^3 I(n).$$

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```
In[97]:= << RISC`HolonomicFunctions`
```

HolonomicFunctions Package version 1.7.3 (21-Mar-2017)
written by Christoph Koutschan
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

```
--> Type  ?HolonomicFunctions  for help.

In[98]:= CreativeTelescoping[CreativeTelescoping[CreativeTelescoping[
           (x * (1 - x) * y * (1 - y) * z * (1 - z)) ^ n / (1 - z + x * y * z) ^ (n + 1),
           Der[x], {S[n], Der[y], Der[z]}][[1]], Der[y]][[1]], Der[z]][[1]] // Timing
```

Out[98]= $\{2.07527, \{(8+12n+6n^2+n^3)S_n^2 + (-117-231n-153n^2-34n^3)S_n + (1+3n+3n^2+n^3)\}\}$

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$\longrightarrow$ *Wow, we are really impressed!*
We will rave about your package in our forthcoming paper...

General Strategy

Start with a constant C given by an explicit integral

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Then introduce a sequence of integrals

$$I(n) = \int_0^1 K(x)\,(x(1-x)K(x))^n\,\mathrm{d}x$$

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for another function $S(x)$ (and their multidimensional analogs).
Of course $I(0) = C$.

Generalization of the Beukers Integral

$$\int_0^1\int_0^1\int_0^1 x^{a_1}(1-x)^{a_2}y^{b_1}(1-y)^{b_2}z^{c_1}(1-z)^{c_2} \times \frac{\big(x(1-x)y(1-y)z(1-z)\big)^n}{(1-z+xyz)^{n+d+1}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

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- Look at many different choices for the parameters $a_1, a_2, b_1, b_2, c_1, c_2, d$.
- Hope that this gives irrationality proofs of some interesting constants...

Generalized Integral with Numeric Parameters

$$\int_0^1\int_0^1\int_0^1 x^{1/3}(1-x)^{1/5}y^{2/3}(1-y)^{4/5}z^{2/5}(1-z)^{3/5} \times \frac{\big(x(1-x)y(1-y)z(1-z)\big)^n}{(1-z+xyz)^{n+1}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

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```
In[108]:= CreativeTelescoping[CreativeTelescoping[CreativeTelescoping[
            (x^(1/3)*(1-x)^(1/5)*y^(2/3)*(1-y)^(4/5)*z^(2/5)*(1-z)^(3/5))*
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            Der[x], {S[n], Der[y], Der[z]}][[1]], Der[y]][[1]], Der[z]][[1]] // Timing
```

Out[108]= $\{4.1699, \{(809\,156\,506\,601\,963\,520 + 5\,067\,425\,510\,376\,860\,160\,n + 14\,542\,081\,347\,310\,357\,120\,n^2 + 25\,319\,953\,606\,388\,665\,760\,n^3 + 29\,842\,834\,920\,776\,537\,400\,n^4 + 25\,142\,793\,811\,471\,399\,500\,n^5 + 15\,577\,799\,653\,225\,653\,750\,n^6 + 7\,186\,224\,321\,391\,359\,375\,n^7 + 2\,468\,228\,839\,434\,421\,875\,n^8 + 623\,381\,733\,800\,156\,250\,n^9 + 112\,528\,920\,684\,375\,000\,n^{10} + 13\,748\,203\,880\,859\,375\,n^{11} + 1\,018\,941\,240\,234\,375\,n^{12} + 34\,599\,023\,437\,500\,n^{13})\,S_n^2 + (-17\,125\,635\,748\,645\,552\,128 - 109\,729\,476\,620\,207\,403\,520\,n - 322\,769\,689\,989\,785\,724\,288\,n^2 - 577\,188\,476\,311\,327\,527\,680\,n^3 - 700\,151\,928\,007\,931\,611\,200\,n^4 - 608\,446\,931\,731\,545\,645\,000\,n^5 - 389\,745\,966\,708\,905\,310\,000\,n^6 - 186\,337\,566\,996\,167\,643\,750\,n^7 - 66\,498\,692\,729\,896\,406\,250\,n^8 - 17\,496\,721\,516\,131\,562\,500\,n^9 - 3\,299\,344\,288\,917\,187\,500\,n^{10} - 422\,270\,445\,058\,593\,750\,n^{11} - 32\,879\,451\,972\,656\,250\,n^{12} - 1\,176\,366\,796\,875\,000\,n^{13})\,S_n + (208\,791\,484\,354\,252\,800 + 1\,448\,758\,522\,297\,658\,880\,n + 4\,606\,818\,936\,047\,867\,520\,n^2 + 8\,888\,945\,878\,483\,621\,920\,n^3 + 11\,611\,921\,070\,002\,419\,000\,n^4 + 10\,845\,296\,255\,561\,809\,500\,n^5 + 7\,450\,983\,284\,163\,738\,750\,n^6 + 3\,812\,727\,944\,067\,609\,375\,n^7 + 1\,453\,218\,514\,321\,359\,375\,n^8 + 407\,501\,515\,823\,906\,250\,n^9 + 81\,719\,325\,815\,625\,000\,n^{10} + 11\,098\,995\,099\,609\,375\,n^{11} + 915\,144\,169\,921\,875\,n^{12} + 34\,599\,023\,437\,500\,n^{13})\}\}$

Generalized Integral with Numeric Parameters

$$\int_0^1\int_0^1\int_0^1 x^3(1-x)y^2(1-y)^4z^5(1-z)^3 \times \frac{\big(x(1-x)y(1-y)z(1-z)\big)^n}{(1-z+xyz)^{n+1}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

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In[182]:=
```
CreativeTelescoping[CreativeTelescoping[CreativeTelescoping[
    x^3 * (1 - x) * y^2 * (1 - y) ^4 * z^5 * (1 - z) ^3 *
     (x * (1 - x) * y * (1 - y) * z * (1 - z)) ^n / (1 - z + x * y * z) ^ (n + 1),
    Der[x], {S[n], Der[y], Der[z]}][[1]], Der[y]][[1]], Der[z]][[1]] // Timing
```

Out[182]= $\{3.44204, \{(-142\,334\,280 - 343\,227\,108\,n - 357\,150\,418\,n^2 - 211\,221\,795\,n^3 - 78\,696\,369\,n^4 - 19\,325\,330\,n^5 - 3\,172\,216\,n^6 - 344\,195\,n^7 - 23\,661\,n^8 - 932\,n^9 - 16\,n^{10})\,S_n^3 + (8\,634\,592\,800 + 18\,280\,850\,800\,n + 16\,901\,127\,872\,n^2 + 9\,023\,153\,352\,n^3 + 3\,089\,809\,298\,n^4 + 710\,664\,515\,n^5 + 111\,371\,203\,n^6 + 11\,757\,433\,n^7 + 800\,987\,n^8 + 31\,820\,n^9 + 560\,n^{10})\,S_n^2 + (-17\,235\,247\,680 - 31\,662\,217\,276\,n - 25\,995\,705\,428\,n^2 - 12\,561\,638\,841\,n^3 - 3\,956\,545\,763\,n^4 - 848\,851\,634\,n^5 - 125\,646\,202\,n^6 - 12\,672\,109\,n^7 - 833\,567\,n^8 - 32\,300\,n^9 - 560\,n^{10})\,S_n + (285\,956\,160 + 586\,168\,912\,n + 525\,286\,576\,n^2 + 272\,628\,648\,n^3 + 91\,123\,028\,n^4 + 20\,554\,053\,n^5 + 3\,175\,443\,n^6 + 332\,327\,n^7 + 22\,577\,n^8 + 900\,n^9 + 16\,n^{10})\}\}$

Generalized Integral with Symbolic Parameters

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- (Infinite?) family of six-parameter families

$$
\begin{aligned}
a_1 &= b, \quad a_2 = c - f, \quad b_1 = e, \quad b_2 = a + f + i, \\
c_1 &= a, \quad c_2 = c, \qquad\quad d = d,
\end{aligned}
$$

where a, b, c, d, e, f are arbitrary (i.e., symbolic) parameters, while i must be a nonnegative integer.

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  where a, b, c, d, e, f are arbitrary (i.e., symbolic) parameters, while i must be a nonnegative integer.
- Computational data:

	(a,b,c,d,e,f,n)-deg	points	time/pt	total time		size
$i=0$	$(6,6,10,6,6,8,13)$	960	170 s	45 h	+ 0.5 h	18 M
$i=1$	$(7,7,12,7,7,10,15)$	1512	300 s	126 h	+ 3 h	47 M
$i=2$	$(8,8,14,8,8,12,17)$	2240	700 s	18 d	+ 8 h	106 M

Generalized Integral with Six Symbolic Parameters

$$\int_0^1\int_0^1\int_0^1 x^b(1-x)^{c-f}y^e(1-y)^{a+f}z^a(1-z)^c \times \frac{\big(x(1-x)y(1-y)z(1-z)\big)^n}{(1-z+xyz)^{n+d+1}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

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Fix a family of integrals $I(n)$ with $C = I(0)$ to be proven irrational.

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- Then frequently it happens that $I(0)$ and $I(1)$ are rationally-related:

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- Hence one can write $I(n) = u_n C - v_n$ for two sequences of rational numbers (u_n) and (v_n) that both satisfy the same recurrence as $I(n)$.
- Let $E(n)$ be an **integer-ating factor** so that $u'_n := u_n\, E(n)$ and $v'_n := v_n\, E(n)$ are always integers and $\gcd(u'_n, v'_n) = 1$.

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- Check whether $\beta > e^{\nu}$, or equivalently, whether

$$\delta = \frac{\log \beta - \nu}{\log \alpha + \nu} > 0.$$

Some Results

Generalizing the Alladi-Robinson family of integrals

$$I(n) := \int_0^1 \frac{1}{1+cx}\left(\frac{x(1-x)}{1+cx}\right)^n \mathrm{d}x,$$

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- Many of these constants are expressible terms of algebraic numbers and/or logarithms of rational numbers.
- Hence proving them irrational is not that exciting...
- However, there are also some unidentified cases.

Some Results

Generalizing the Beukers Integral for $\zeta(2)$, we define

$$I_2(a_1, a_2, b_1, b_2)(n) := \frac{1}{B(1-a_1, 1-a_2)B(1-b_1, 1-b_2)}$$
$$\times \int_0^1 \int_0^1 \frac{x^{-a_1}(1-x)^{-a_2} y^{-b_1}(1-y)^{-b_2}}{1-xy} \cdot \left(\frac{x(1-x)y(1-y)}{1-xy} \right)^n \, \mathrm{d}x \, \mathrm{d}y$$

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It allows us to realize the following constants as weak Apéry limits:

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- Several hundred cases with positive δ, but many of them are equivalent via transformations $C \mapsto \frac{a+bC}{c+dC}$ with integer coeffs.
- Again, there are some cases that could not be identified.

Examples

$$C_2(0,0,\tfrac{1}{2},0) = {}_3F_2(1,1,\tfrac{1}{2};2,\tfrac{3}{2};1) = 2\log 2$$

$$C_2(0,0,\tfrac{1}{3},-\tfrac{2}{3}) = {}_3F_2(1,1,\tfrac{2}{3};2,\tfrac{7}{3};1) = -6+4\pi\sqrt{3}/3$$

$$C_2(-\tfrac{3}{4},-\tfrac{3}{4},-\tfrac{1}{4},-\tfrac{3}{4}) = {}_3F_2(1,\tfrac{7}{4},\tfrac{5}{4};\tfrac{7}{2},3;1) = -240 + \frac{512}{3}\sqrt{2}$$

$$C_2(-\tfrac{4}{5},-\tfrac{4}{5},-\tfrac{2}{5},-\tfrac{3}{5}) = {}_3F_2(1,\tfrac{9}{5},\tfrac{7}{5};\tfrac{18}{5},3;1) = -\frac{845}{2} + \frac{2275}{12}\sqrt{5}$$

$$C_2(-\tfrac{5}{6},-\tfrac{5}{6},-\tfrac{1}{2},-\tfrac{1}{2}) = {}_3F_2(1,\tfrac{11}{6},\tfrac{3}{2};\tfrac{11}{3},3;1) = -\frac{1344}{5} + \frac{16384\sqrt{3}}{105}$$

$$C_2(-\tfrac{5}{6},-\tfrac{5}{6},-\tfrac{1}{3},-\tfrac{2}{3}) = {}_3F_2(1,\tfrac{11}{6},\tfrac{4}{3};\tfrac{11}{3},3;1) = \frac{972\,2^{2/3}}{5} - \frac{1536}{5}$$

Some Results

Using the generalized Beukers integral for $\zeta(3)$,

$$J_3(a_1, a_2, b_1, b_2, c_1, c_2; e)(n) := \int_0^1\int_0^1\int_0^1 \left(\frac{x(1-x)y(1-y)z(1-z)}{1-z+xyz}\right)^n \times \frac{x^{a_1}(1-x)^{a_2}y^{b_1}(1-y)^{b_2}z^{c_1}(1-z)^{c_2}}{(1-z+xyz)^e}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

we define

$$I_3(a_1, a_2, b_1, b_2, c_1, c_2; e)(n) := \frac{J_3(a_1, a_2, b_1, b_2, c_1, c_2; e+1)(n)}{J_3(a_1, a_2, b_1, b_2, c_1, c_2; e)(0)}.$$

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$$I_3(a_1,a_2,b_1,b_2,c_1,c_2;e)(n) := \frac{J_3(a_1,a_2,b_1,b_2,c_1,c_2;e+1)(n)}{J_3(a_1,a_2,b_1,b_2,c_1,c_2;e)(0)}.$$

Using the previously derived symbolic recurrence, allows us to study the constants

$$K(a,b,c,d,e)(n) := I_3(b,c,e,a,a,c,d)(n)$$

Some Results

Using the generalized Beukers integral for $\zeta(3)$,

$$J_3(a_1,a_2,b_1,b_2,c_1,c_2;e)(n) := \int_0^1\int_0^1\int_0^1 \left(\frac{x(1-x)y(1-y)z(1-z)}{1-z+xyz}\right)^n \times \frac{x^{a_1}(1-x)^{a_2}y^{b_1}(1-y)^{b_2}z^{c_1}(1-z)^{c_2}}{(1-z+xyz)^e}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

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The output file contains many such conjectured evaluations and we challenge the birthday boy [WZ], or anyone else, to prove them.

The Birthday Boy Problem

Wadim writes:

In their recent preprint arXiv:2101.08308, Robert Dougherty-Bliss, Christoph Koutschan and Doron Zeilberger come up with a powerful strategy to prove the irrationality, in a quantitative form, of some numbers that are given as multiple integrals or quotients of such. What is really missing there, for many examples given, is an explicit identification of those irrational numbers. Without an identification, the numbers are hardly appealing to human (number theorists). The goal of this note is to outline a strategy to do the job and illustrate it on several promising entries discussed in the preprint above.

Zudilin

$$K(0,0,0,\tfrac{2}{3},\tfrac{1}{3}) = -\frac{K_1-2}{2(K_1-3)}, \qquad \text{where } K_1 = \log 3 + \frac{\pi}{\sqrt{3}}$$

$$K(0,0,0,\tfrac{1}{3},\tfrac{2}{3}) = -\frac{2(K_2+1)}{K_2+1/2}, \qquad \text{where } K_2 = \log 3 + \frac{\pi}{\sqrt{3}}$$

$$K(0,\tfrac{1}{3},\tfrac{2}{3},\tfrac{1}{3},\tfrac{2}{3}) = -\frac{20(7-54K_3)}{52-405K_3}, \text{ where } K_3 = \frac{\Gamma(2/3)^3}{\Gamma(1/3)^3}$$

$$K(0,\tfrac{1}{5},0,\tfrac{3}{5},\tfrac{2}{5}) = -\frac{4(1-4K_4)}{5-24K_4}, \qquad \text{where } K_4 = \frac{1}{\sqrt{5}} \log \frac{\sqrt{5}+1}{2}$$

$$K(\tfrac{1}{7},0,\tfrac{2}{7},\tfrac{3}{7},\tfrac{4}{7}) = -\frac{189(8-5K_5)}{832-525K_5}, \text{ where } K_5 = \frac{2^{2/7}\sqrt{\pi}\,\Gamma(9/14)}{\cos(3\pi/14)\,\Gamma(4/7)^2}$$

Perhaps, a real pearl in this collection of "quantitatively" irrational numbers is the number K_3.

Another Integral

Wadim Zudilin suggested to study the double integral

$$J_n(z) = \int_0^1 \int_0^1 \frac{x^{n-1/2}(1-x)^{n-1/2}y^{n-1/2}(1-y)^n}{(1-zxy)^{n+1/2}} \, \mathrm{d}x \, \mathrm{d}y$$

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A recurrence equation can be obtained by

- continuous creative telescoping on the double integral
- Zeilberger's fast algorithm on the hypergeometric representation

Third-Order Recurrence

$$
\begin{aligned}
&4z^4(2n+1)^2(n+1)^2\big(16(27z-32)n^4-16(69z-86)n^3\\
&\quad+8(108z-143)n^2-4(55z-76)n+3(7z-10)\big)J_{n+1}\\
&\;+z^2\big(256(3z+8)(27z-32)n^8-256(3z+8)(15z-22)n^7\\
&\quad-64(651z^2+661z-1744)n^6+192(59z^2-186)n^5\\
&\quad+16(1503z^2+697z-3610)n^4-16(79z^2-290z+116)n^3\\
&\quad-4(569z^2-381z-580)n^2+4(11z^2-44z+18)n+3(4z+3)(7z-10)\big)J_n\\
&\;+4n\big(64(3z^2-20z+16)(27z-32)n^7-384(3z^2-20z+16)(7z-9)n^6\\
&\quad-16(411z^3-2698z^2+3988z-1696)n^5+64(183z^3-1372z^2+2339z-1134)n^4\\
&\quad+4(531z^3-1400z^2-424z+1240)n^3-8(571z^3-4001z^2+6532z-3060)n^2\\
&\quad+(151z^3-4742z^2+11596z-6888)n+12(14z^2-29z-30)(z-1)\big)J_{n-1}\\
&\;+4n(n-1)(2n-3)^2(z-1)\big(16(27z-32)n^4+48(13z-14)n^3\\
&\quad+8(18z-11)n^2-4(19z-24)n-(7z+6)\big)J_{n-2}=0.
\end{aligned}
$$

Initial Values

We have

$$J_0(z) = \lambda(z),$$

$$J_1(z) = -\tfrac{3+4z}{4z^2}\,\lambda(z) - \tfrac{5(1-z)}{z^2}\,\rho_1(z) + \tfrac{13}{2z^2}\,\rho_2(z),$$

$$J_2(z) = \tfrac{105+480z+64z^2}{64z^4}\,\lambda(z) + \tfrac{3151-2167z-984z^2}{144z^4}\,\rho_1(z) - \tfrac{7247+3452z}{288z^4}\,\rho_2(z),$$

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where

$$\lambda(z) = \int_0^1\int_0^1 \frac{\mathrm{d}x\,\mathrm{d}y}{\sqrt{x(1-x)y(1-zxy)}}$$

$$\rho_1(z) = \int_0^1 \frac{\mathrm{d}x}{\sqrt{x(1-x)(1-zx)}}$$

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$$\lambda(z) = \int_0^1\int_0^1 \frac{\mathrm{d}x\,\mathrm{d}y}{\sqrt{x(1-x)y(1-zxy)}} = 2\pi\,{}_3F_2\left(\begin{matrix}\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\\ 1,\ \frac{3}{2}\end{matrix}\,\middle|\, z\right),$$

$$\rho_1(z) = \int_0^1 \frac{\mathrm{d}x}{\sqrt{x(1-x)(1-zx)}} = \pi\,{}_2F_1\left(\begin{matrix}\frac{1}{2},\ \frac{1}{2}\\ 1\end{matrix}\,\middle|\, z\right),$$

$$\rho_2(z) = \int_0^1 \frac{\sqrt{1-zx}}{\sqrt{x(1-x)}}\,\mathrm{d}x = \pi\,{}_2F_1\left(\begin{matrix}-\frac{1}{2},\ \frac{1}{2}\\ 1\end{matrix}\,\middle|\, z\right).$$

Eliminate

Hence, each integral can be written as a linear combination of λ, ρ_1, ρ_2:

$$J_n(z) = a_n(z)\lambda(z) + b_n(z)\rho_1(z) + c_n(z)\rho_2(z)$$

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For $z^{-1} \in \mathbb{Z} \setminus \{\pm 1\}$, the coefficients a_n, b_n, c_n seem to satisfy

$$z^n 2^{4n} a_n,\ z^n 2^{4n} d_{2n}^2 b_n,\ z^n 2^{4n} d_{2n}^2 c_n \in \mathbb{Z} \qquad \text{for} \quad n = 0, 1, 2, \ldots$$

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Eliminating $\rho_2(z)$ we get

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The sequences A_n and B_n satisfy again a third-order recurrence, which is the exterior square of the recurrence for J_n.

Quotients of L-values as Apéry limits

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In other words, the number λ/ρ_1 (but also the quotients λ/ρ_2 and ρ_1/ρ_2) are Apéry limits for the considered values of z.

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In other words, the number λ/ρ_1 (but also the quotients λ/ρ_2 and ρ_1/ρ_2) are Apéry limits for the considered values of z. Note that

$$\lambda\Big(\frac{1}{2}\Big) = 2\sqrt{2}\,\pi L'(E,0) = 16\sqrt{2}\,\frac{L(E,2)}{\pi}, \quad \rho_1\Big(\frac{1}{2}\Big) = 4\sqrt{2}\,L(E,1).$$

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