Tweaking the Beukers integrals in search of more miraculous irrationality proofs à la Apéry

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- More precisely: $d_n^3 v_n \in \mathbb{Z}$ where $d_n := \operatorname{lcm}(1, 2, \dots, n)$

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he studied the following sequence of double integrals:

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$$I(n) = v_n - u_n \frac{\pi^2}{6} \quad \text{and} \quad \lim_{n \to \infty} \frac{v_n}{u_n} = \frac{\pi^2}{6}$$

Hence, the family of integrals I(n) yields a sequence of rational approximations to $\zeta(2)$:

$$\lim_{n \to \infty} \frac{v_n}{u_n} = \frac{\pi^2}{6},$$

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By estimating the size of the integral I(n), one can show, by denoting $I'(n) = u'_n \frac{\pi^2}{6} - v'_n$:

$$\lim_{n \to \infty} |I'(n)| = 0 \quad \text{and} \quad I'(n) \neq 0.$$

Starting from the integral

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and more generally: $I(n)=u_n\zeta(3)-v_n.~$ In fact, I(n) satisfies

$$(n+2)^3 I(n+2) = (2n+3)(17n^2 + 51n + 39)I(n+1) - (n+1)^3 I(n).$$

Zeilberger

An e-mail from Doron Zeilberger:

For the Beukers integral for Zeta(3)

B(n):=int(int(int((x*(1-x)*y*(1-y)*z*(1-z))^n/(1-z+x*y*z)^(n+1),x=0..1),y=0..1),z=0..1) even without any extra parameters it takes a VERY long time. In an optimized version, that targets these kind of integrals it still takes about 2000 seconds.

Our questions are:

1. Can your package find these recurrence in one "key-stroke" or does it need some pre-processing?

2. How fast can your package find the recurrence for B(n), and similar integrals where you stick in the integrand $x^{(a1)*(1-x)a2*...}$ (for numeric a1,a2, ..)

Definition: A function f(x) is called **holonomic** if it satisfies a linear ordinary differential equation with polynomial coefficients:

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 \longrightarrow In both cases, one needs only finitely many initial conditions.

Differential Equations and Recurrences

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Many special functions can be characterized as solutions to systems of linear differential equations and recurrences, and in fact are holonomic.

Multivariate Holonomic Functions

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$$\frac{\mathrm{d}^{i_1}}{\mathrm{d}x_1^{i_1}} \dots \frac{\mathrm{d}^{i_s}}{\mathrm{d}x_s^{i_s}} f(x_1, \dots, x_s, n_1 + j_1, \dots, n_r + j_r)$$

with $i_1, \ldots, i_s, j_1, \ldots, j_r \in \mathbb{N}$ such that any shifted partial derivative of f (of the above form) can be expressed as a $\mathbb{K}(x_1, \ldots, x_s, n_1, \ldots, n_r)$ -linear combination of the basis functions (plus some further, technical assumptions), then f is **holonomic**.

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 \longrightarrow Finitely many initial conditions suffice.

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The **holonomic systems approach** (Zeilberger 1990) is a versatile toolbox for solving many different kinds of mathematical problems:

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$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$
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$$\sum_{k=1}^{\infty} \frac{1}{k(k+n)} = \frac{\gamma + \psi(n)}{n}$$

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$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \qquad \text{Bad: no parameter!}$$

$$\sum_{\substack{k=1\\ =: f_n}}^{\infty} \frac{1}{k(k+n)} \rightsquigarrow (n+2)^2 f_{n+2} = (n+1)(2n+3)f_{n+1} - n(n+1)f_n$$

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Method for doing integrals and sums (already mentioned in van der Poorten's report of Apéry's proof!)

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 $c_r(n)f(n+r,k) + \dots + c_0(n)f(n,k) = g(n,k+1) - g(n,k).$

Summing from a to b yields a recurrence for F(n):

$$c_r(n)F(n+r) + \dots + c_0(n)F(n) = g(n,b+1) - g(n,a).$$

Creative Telescoping

Method for doing integrals and sums (already mentioned in van der Poorten's report of Apéry's proof!)

Consider the following integration problem: $F(x) := \int_a^b f(x, y) \, dy$

Telescoping: write
$$f(x, y) = \frac{d}{dy}g(x, y)$$
.
Then $F(n) = \int_{a}^{b} \left(\frac{d}{dy}g(x, y)\right) dy = g(x, b) - g(x, a)$.

Creative Telescoping: write

$$c_r(x)\frac{\mathrm{d}^r}{\mathrm{d}x^r}f(x,y)+\cdots+c_0(x)f(x,y) = \frac{\mathrm{d}}{\mathrm{d}y}g(x,y).$$

Integrating from a to b yields a differential equation for F(x):

$$c_r(x)\frac{\mathrm{d}^r}{\mathrm{d}x^r}F(x) + \dots + c_0(x)F(x) = g(x,b) - g(x,a)$$

Zeilberger

An e-mail from Doron Zeilberger:

For the Beukers integral for Zeta(3)

B(n):=int(int(int((x*(1-x)*y*(1-y)*z*(1-z))^n/(1-z+x*y*z)^(n+1),x=0..1),y=0..1),z=0..1) even without any extra parameters it takes a VERY long time. In an optimized version, that targets these kind of integrals it still takes about 2000 seconds.

Our questions are:

1. Can your package find these recurrence in one "key-stroke" or does it need some pre-processing?

2. How fast can your package find the recurrence for B(n), and similar integrals where you stick in the integrand $x^{(a1)*(1-x)a2*...}$ (for numeric a1,a2, ..)

$$\int_0^1 \int_0^1 \frac{(x(1-x)y(1-y))^n}{(1-xy)^{n+1}} \,\mathrm{d}x \,\mathrm{d}y.$$

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$$\int_0^1 \int_0^1 \int_0^1 \frac{\left(x(1-x)y(1-y)z(1-z)\right)^n}{(1-z+xyz)^{n+1}} \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z$$

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$$\sum_{k=0}^{n} \binom{n+k}{k}^2 \binom{n}{k}^2$$

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\sum_{n}^{n}	(n	+	$k \rangle^2$	$(n)^2$
$\sum_{k=0}$		k)	$\binom{k}{k}$

$$\sum_{k=0}^{n} \binom{n+k}{k}^{2} \binom{n}{k}^{2} \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^{3} \binom{n+m}{m} \binom{n}{m}}$$

$$\int_0^1 \int_0^1 \frac{(x(1-x)y(1-y))^n}{(1-xy)^{n+1}} \, \mathrm{d}x \, \mathrm{d}y.$$

$$\int_0^1 \int_0^1 \int_0^1 \frac{\left(x(1-x)y(1-y)z(1-z)\right)^n}{(1-z+xyz)^{n+1}} \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z$$

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$$\frac{1}{n!} \left(\frac{\mathrm{d}}{\mathrm{d}z} + \frac{\gamma z + \delta}{z(z-a)}\right)^n z^n (z-a)^n$$

15 / 36

Beukers Integral

Task: Show that the Beukers integral for $\zeta(3)$ satisfies Apéry's second-order recurrence:

$$(n+2)^3 I(n+2) = (2n+3)(17n^2 + 51n + 39)I(n+1) - (n+1)^3 I(n).$$

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In[97]:= << RISC`HolonomicFunctions`</pre>

HolonomicFunctions Package version 1.7.3 (21-Mar-2017) written by Christoph Koutschan Copyright Research Institute for Symbolic Computation (RISC), Johannes Kepler University, Linz, Austria

```
--> Type ?HolonomicFunctions for help.
```

In[98]:= CreativeTelescoping[CreativeTelescoping[CreativeTelescoping[

 $(x * (1 - x) * y * (1 - y) * z * (1 - z)) ^n / (1 - z + x * y * z) ^ (n + 1),$

Der[x], {S[n], Der[y], Der[z]}][[1]], Der[y]][[1]], Der[z]][[1]] // Timing

 $\text{Out[98]=} \left\{ \texttt{2.07527,} \left\{ \left(\texttt{8} + \texttt{12} \ \texttt{n} + \texttt{6} \ \texttt{n}^2 + \texttt{n}^3 \right) \ \texttt{S}_n^2 + \left(-\texttt{117} - \texttt{231} \ \texttt{n} - \texttt{153} \ \texttt{n}^2 - \texttt{34} \ \texttt{n}^3 \right) \ \texttt{S}_n + \left(\texttt{1} + \texttt{3} \ \texttt{n} + \texttt{3} \ \texttt{n}^2 + \texttt{n}^3 \right) \right\} \right\}$

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→ Wow, we are really impressed! We will rave about your package in our forthcoming paper...

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for another function S(x) (and their multidimensional analogs).

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for another function S(x) (and their multidimensional analogs). Of course I(0) = C.

Generalization of the Beukers Integral

$$\int_0^1 \int_0^1 \int_0^1 x^{a_1} (1-x)^{a_2} y^{b_1} (1-y)^{b_2} z^{c_1} (1-z)^{c_2} \\ \times \frac{\left(x(1-x)y(1-y)z(1-z)\right)^n}{(1-z+xyz)^{n+d+1}} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

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► Look at many different choices for the parameters $a_1, a_2, b_1, b_2, c_1, c_2, d$.

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- ► Look at many different choices for the parameters *a*₁, *a*₂, *b*₁, *b*₂, *c*₁, *c*₂, *d*.
- Hope that this gives irrationality proofs of some interesting constants...

$$\int_0^1 \int_0^1 \int_0^1 x^{1/3} (1-x)^{1/5} y^{2/3} (1-y)^{4/5} z^{2/5} (1-z)^{3/5} \\ \times \frac{\left(x(1-x)y(1-y)z(1-z)\right)^n}{(1-z+xyz)^{n+1}} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

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In[108]:= CreativeTelescoping[CreativeTelescoping[CreativeTelescoping[

 $(x \land (1 / 3) * (1 - x) \land (1 / 5) * y \land (2 / 3) * (1 - y) \land (4 / 5) * z \land (2 / 5) * (1 - z) \land (3 / 5)) *$

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Der[x], {S[n], Der[y], Der[z]}][[1]], Der[y]][[1]], Der[z]][[1]] // Timing

$$\label{eq:second} \begin{split} & \{4.1699, \{\{809156\,506\,601963\,520\,+\,5067\,425\,510\,376\,800\,160\,\,n\,,14\,542\,081\,347\,310\,357\,120\,\,n^2\,+\\ & 25\,319\,953\,606\,388\,665\,760\,\,n^3\,,2\,98\,42\,83\,920\,776\,537\,400\,\,n^4\,,25\,142\,793\,811\,471\,3199\,500\,\,n^5\,,\\ & 15\,577\,799\,653\,225\,653\,760\,\,n^6\,+\,71\,36\,224\,321\,91\,159\,375\,\,n^7\,+\,2\,46\,22\,883\,94\,34\,21\,875\,\,n^6\,+\,6\,223\,381\,733\,800\,156\,250\,\,n^9\,+\\ & 112\,528\,920\,664\,37\,500\,\,n^{10}\,+\,13\,748\,203\,880\,859\,375\,\,n^{11}\,+\,10\,18\,941\,240\,234\,375\,\,n^{12}\,+\,34\,599\,023\,437\,500\,\,n^{13}\,\}\,\,s^2_n\,+\\ & (-17\,125\,635\,748\,64\,5552\,128\,-\,109\,729\,476\,620\,207\,403\,520\,n\,-\,322\,769\,689\,989\,785\,724\,288\,n^2\,-\,577\,188\,476\,311\,327\,527\,680\,\,n^3\,-\\ & 700\,15\,1928\,007\,331\,611\,200\,\,n^4\,-\,608\,446\,931\,731\,545\,645\,900\,\,n^6\,-\\ & 186\,337\,566\,996\,167\,643\,750\,n^7\,-\,64\,986\,692\,529\,n^6\,-\,17\,496\,721\,516\,131\,56\,2500\,n^9\,-\\ & 32\,99\,344\,288\,917\,187\,500\,\,n^{10}\,-\,422\,270\,445\,565\,5593\,750\,\,n^{11}\,-\,32\,879\,45\,197\,265\,62\,50\,\,n^{12}\,-\,1176\,366\,796\,875\,900\,\,n^{13}\,)\,5_n\,+\\ & (208\,791\,484\,354\,252\,800\,+\,1448\,758\,522\,297\,558\,880\,n\,+\,4\,606\,818\,936\,47\,867\,520\,\,n^2\,-\,8\,888\,945\,87\,84\,83\,621\,920\,\,n^3\,+\\ & 116\,11\,92\,10\,70002\,41\,90\,00\,n^4\,+\,14\,84\,52\,92\,255\,551\,809\,500\,n^6\,-\,7\,450\,932\,284\,137\,37\,50\,n^6\,+\\ & 38\,12\,727\,944\,06\,7609\,375\,n^7\,+\,1453\,21\,851\,93\,357\,5\,n^6\,+\,467\,551\,51\,823\,906\,250\,\,n^{13}\,,\\ & 8\,17\,19\,325\,81\,56\,25\,000\,\,n^{13}\,,\\ & 8\,17\,19\,325\,81\,65\,25\,000\,\,n^{13}\,,\\ & 8\,17\,19\,325\,81\,145\,14\,150\,921\,87\,5\,n^{13}\,,\\ & 8\,17\,19\,32\,8\,31\,32\,37\,5\,n^{13}\,,\\\\ & 8\,17\,19\,32\,$$

$$\int_0^1 \int_0^1 \int_0^1 x^3 (1-x) y^2 (1-y)^4 z^5 (1-z)^3 \\ \times \frac{\left(x(1-x)y(1-y)z(1-z)\right)^n}{(1-z+xyz)^{n+1}} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

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$$\begin{split} & \text{In}[182] \coloneqq \text{CreativeTelescoping}[\text{CreativeTelescoping}[\text{CreativeTelescoping}[\\ & x^3 * (1-x) * y^2 * (1-y) ^4 * z^5 * (1-z) ^3 * \\ & (x * (1-x) * y^2 * (1-y)) * z * (1-z) ^n / (1-z + x * y * z) ^(n+1) , \\ & \text{Der}[x], (S[n], \text{Der}[y], \text{Der}[z]) [[1]], \text{Der}[y]] [[1]], \text{Der}[z]] [[1]] // \text{Timing} \\ \\ & \text{Out}[182] = \left\{ 3.44204, \left\{ \left(-142\,334\,280 - 343\,227\,108\,n - 357\,150\,418\,n^2 - 211221\,795\,n^3 - \right. \\ & 78\,696\,369\,n^4 - 19\,325\,330\,n^5 - 3\,172\,216\,n^6 - 344\,195\,n^7 - 23\,661\,n^8 - 932\,n^9 - 16\,n^{10} \right) S_n^3 + \\ & (8\,634\,592\,800 + 18\,280\,850\,800\,n + 16\,901\,127\,872\,n^2 + 9\,023\,153\,352\,n^3 + 3\,089\,809\,298\,n^4 + \\ & 710\,664\,515\,n^5 + 111\,371\,203\,n^6 + 117\,57\,433\,n^7 + 800\,987\,n^8 + 31\,820\,n^9 + 560\,n^{10} \right) S_n^2 + \\ & \left(-17\,235\,247\,680 - 31\,662\,217\,276\,n - 25\,995\,705\,428\,n^2 - 12\,561\,638\,841\,n^3 - 3\,956\,545\,763\,n^4 - \\ & 848\,851\,634\,n^5 - 125\,646\,202\,n^6 - 126\,72\,109\,n^7 - 833\,567\,n^8 - 32\,300\,n^9 - 560\,n^{10} \right) S_n + \\ & \left(285\,956\,160 + 586\,168\,912\,n + 525\,286\,576\,n^2 + 272\,628\,648\,n^3 + 91\,123\,028\,n^4 + \\ & 20\,554\,053\,n^5 + 3\,175\,443\,n^6 + 332\,327\,n^7 + 22\,577\,n^8 + 900\,n^9 + 16\,n^{10} \right) \right\} \end{split}$$

Generalized Integral with Symbolic Parameters Question: When do we get a second-, when a third-order rec.?

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► Compute symbolically the third-order recurrence and check under which conditions it can be reduced? ~> failed.

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- (Infinite?) family of six-parameter families

$$a_1 = b, \quad a_2 = c - f, \quad b_1 = e, \quad b_2 = a + f + i,$$

 $c_1 = a, \quad c_2 = c, \qquad d = d,$

where a, b, c, d, e, f are arbitrary (i.e., symbolic) parameters, while *i* must be a nonnegative integer.

Question: When do we get a second-, when a third-order rec.?

- ► Compute symbolically the third-order recurrence and check under which conditions it can be reduced? ~> failed.
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$$a_1 = b, \quad a_2 = c - f, \quad b_1 = e, \quad b_2 = a + f + i,$$

 $c_1 = a, \quad c_2 = c, \qquad d = d,$

where a, b, c, d, e, f are arbitrary (i.e., symbolic) parameters, while i must be a nonnegative integer.

Computational data:

	(a, b, c, d, e, f, n)-deg	points	time/pt	total time	size
i = 0	(6, 6, 10, 6, 6, 8, 13)	960	170 s	45 h + 0.5 h	18 M
i = 1	(7, 7, 12, 7, 7, 10, 15)	1512	300 s	126 h + 3 h	47 M
i = 2	(8, 8, 14, 8, 8, 12, 17)	2240	700 s	18d $+8h$	106 M

Generalized Integral with Six Symbolic Parameters $\int_0^1 \int_0^1 \int_0^1 x^b (1-x)^{c-f} y^e (1-y)^{a+f} z^a (1-z)^c$ $\times \frac{\left(x(1-x)y(1-y)z(1-z)\right)^n}{(1-z+xyz)^{n+d+1}} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$

(.2.a.b.c.f.m (2.a.c.d.m (2.d.m (2.b.c.d.f.m (2.b.c.e.f.m (2.e.f.m (2.a.b.e.f.m (2.

(12 - 15 a - 5 a² - 15 b - 18 a b - 6 a² b - 5 b² - 6 a b² - 2 a² b² - 30 a² c - 32 a b c - 32 a b c - 32 a b c - 32 a² b c - 6 b² c - 20 a² c - 5 a² c² - 5 a² c² - 5 a² c² - 3 b² c² - 3 b² c² - 3 b² c² - 4 b² c² - 2 a² b² 14bd - 12abd - 2a²bd - 3b²d - 2bc²d - 2bcd - 21acd - 3a²cd - 18bcd - 12abcd - a²bcd - 2b²cd - a²b²cd - 18c²d - a²c²d - a²c²d - a²b²cd - 3bc²d - 4c²d - 2ac²d - 3ad² - 3bd² - 3bd² - 2ab²d - 3bc²d - 3bc 14 a e - 3 a' e - 18 b e - 15 a b e - 1 a' b e - 6 b' e - 5 a b' e - a' b' e - 32 c e - 34 a c e - 4 a' c e - 32 b c e - 2 a b' c e - 3 b c' e - 3 a c' e - 13 b c' e - 4 a b c' e - 5 a c' e - 2 a c' e - 2 b c' e - 6 a c' e - 2 a c' e - 2 b c' e - 13 b c e - 12 b d 8abde-afbde-2bfde-abfde-21cde-8acde-12bcde-4abcde-bfcde-11cfde-2acfde-3bcfde-2cfde-3bfde-adfe-2bdfe-adfe-2cdfe-bcdfe-cfdfe-5cdfe-5aef-6bef-3aef-6bef-3aef-2bfe-abfe-2bfe-4accf-7bcef-2abce¹ +b²ce² +5c²e¹ + ac²e² + 2bc²e¹ - c²e¹ - 2bde² - abde² - 2cde² - bcde² - c²de² - 4at - 3a²t + 4bt - 2a²bt + 3b²t + 3act - 3a²ct - 3b²ct - 3b²ct - 3b²ct - 3b²ct - ac²t - ac²t - ac²t - 7b²c²t - 7b²t abc'f+b'c'f+4c'f+2bc'f+c'f+3adf+abdf-b'df-3cdf+4acdf+a'cdf-5bcdf+5c'df+3c'df+3bc'df+2c'df+ad'f+bd'f+cd'f+acd'f+bcd'f+c'd'f+4ef+5aef+2a'ef+2abef+a'bef+2b'ef+ab'ef+ 3cef-4acef-a²cef-2bcef+b²cef-c²ef-ac²ef-bc²ef-3def+2adef+2adef+2adef+c²def-d²ef-cd²ef-2ae²f-2be²f-3c²f-bce²f-bc²f-d²f-de²f-de²f-de²f-de²f-de²f-2abf²-2abf²-2abf²-6cf²-3acf-3bcf-abcf-4c'f-ac'f-bcf-c'f-3df-adf-bdf-5cdf-acdf-bcdf-2c'df-d'f-cf'f-3af-3aef-2bef-2bef-2bef-2cef-bcef-c'ef-cef-bcef-c'ef-adef-bcef-c'ef-12 a' bn - 15 b' n - 12 a b' n - 2 a' b' n - 126 cn - 56 a cn - 16 a' cn - 64 b cn - 7 a' b cn - 12 b' cn - 5 a b' cn - 64 c' n - 64 c' n - 64 a c' n 2x2 bdn 6b2 dn 2xb2 dn Mcdn 42xcdn 3x2 cdn 3b2 cdn 22b cdn 22b cdn 3x2 dn 14x2 dn 6b2 dn 4c2 dn 15d2 n 6x2 n 2bd2 n 2bd2 n 2ac2 n 2xc2 3 a' ben - 12 b' en - 5 a b' en - 96 cen - 44 a cen - 44 cen - 54 b cen - 15 a b cen - 56 ' cen - 42 c' en - 12 b c' en - 62 den - 18 a den - a' den - 24 b den - 26 b den - 42 cden - 8 a c den - 12 b c den - 12 b c den - 12 c' den - 6d' en - a d' en -2 b dr en - 3 c dr en - 12 b er n - 6 a er n - 12 b er n - 3 a b er n - 2 b r er n - 12 c f n - 14 c er n - 3 c r er n - 6 d er n - 2 a b f n - 6 d er n - 2 a d er n - 12 a f n - 6 a' f n - 12 a f n - 6 a' f n - 12 c f n - 6 a c f n - 3 a' c f n - 18 b c f n - 2 a b c f n -3b' cfn - 12c' fn - 12c' fn - 12c' fn - 6 adfn - 12 dfn - 6 bdfn - b' dfn - 6 cdfn - 4 acdfn - 6 bcdfn - 5 c' dfn - 10 d' fn - 12 efn - 12 aefn - 2 a' efn - 2 ab efn - 2 b' efn - 6 cefn - 4 ac efn - 2 b cefn - c' efn - 6 defn -2adefn-4cdefn-6e'fn-2ae'fn-2be'fn-2be'fn-2be'fn-2ef'n-6af'n-6af'n-6af'n-2abf'n-12cf'n-3acf'n-4c'f'n-6df'n-4df'n-bdf'n-bdf'n-bdf'n-5cdf'n-d'f'n-6ef'n-2aef'n-2bef'n-3cef'n-4ef'n+120f' 99 an² - 15 a² n² - 99 bn² - 54 a bn² - 54 a bn² - 5 a b² n² - 5 a b² n² - 94 a² n² - 97 a ban² - 5 a² cn² - 94 a² n² - 10 b² cn² cn² cn² - 10 b² cn² cn² cn² cn² cn² cn² 18 c[°]dn' + 15 d'n' + 5 a d'n' + 5 b d'n' + 6 c d'n' + 42 a c n' + 5 a c n' + 5 a b c n' + 5 b c c n' + 24 a c c n' + 24 c c c n' + 24 c c c n' - 42 d c n' - 9 a d c n' - 12 c d c n' - 32 c d c n' + 5 c c' n' + 3 a c' n' + 5 c c' n' + 3 c c' n' + 3 c c c' + 3 d c c n' + 2 c c n' + 2 c c c n' 12afn'- 3a'fn'- 12bfn'- 3b'fn'- 12cfn'- 3acfn'- 9bcfn'- 6c'fn'- 3acfn'- 3acfn'- 3cefn'- 3cefn'- 3cefn'- 3cefn'- 3defn'- 3e'fn'- 12f'n'- 3af'n'- 3bf'n'- 6cf'n'- 3df'n'- 3ef'n'- 3bf'n'- 6cf'n'- 3df'n'- 3ef'n'- 9bn'-18 abn', 5 b'n', 120 cn', 32 a cn', 28 b cn', 28 c'n', 60 dn', 14 adn', 14 b dn', 28 c dn', 5 d'n', 60 en', 18 b en', 32 c en', 14 d en', 5 e'n', 4 a f n', 4 f n', 4 f n', 60 n', 15 an', 15 b n', 30 c n', 15 an', 15 e n', 12 n', 15 e n', 10 e n', 12 n', 10 e n', 12 n', 10 e n', 10 144 923 + 166 416 a + 160 632 a² + 47 144 a³ + 169 28 a⁴ + 160 6 a³ + 166 416 b + 248 688 a b + 233 648 a³ b + 199 224 a³ b + 2328 a⁴ b + 129 233 a⁴ b + 2328 a³ b + 199 632 b² + 233 648 a b² + 219 252 a³ b³ + 199 224 a³ b³ + 231 2 a³ b³ + 199 224 a³ b + 199 224 a³ b + 199 224 a³ b³ + 199 224 a³ b³ + 199 224 a³ b + 199 224 a³ b + 199 224 a³ b + 199 232 b³ b³ + 199 24 a b³ + 199 24 a b³ + 199 224 a³ b³ + 199 224 a³ b + 199 232 b³ b³ + 199 232 b³ b³ + 199 24 a b³ + 199 224 a³ + 199 224 48.312 m² b² + 11.324 m² b² + 10.920 b⁴ + 25.328 m b⁴ + 25.026 m² b⁴ + 11.324 m² b⁴ + 2676 m² b⁴ + 2676 m² b⁴ + 2000 b² + 2221 m b² + 2221 m² b² + 2022 m² b² + 262 m 980 700 a b c - 899 542 a' b c - 374 942 a' b c - 81138 a' b c - 6958 a' b c - 375 600 b' c - 606 126 ab' c - 606 595 a' b' c - 506 595 a' b' c - 506 595 a' b' c - 5505 a' b' c - 5505 a' b' c - 152 904 b' c - 322 906 a b' c - 216 37 a' b' c - 2116 a' b' c - 25107 a' b' c - 2016 30 20 20 a' b' c - 500 595 a' b' c - 5505 a' b' c - 5505 a' b' c - 152 904 b' c - 322 906 a b' c - 216 377 a' b' c - 5107 a' b' c - 2116 a' b' c - 2016 30 20 a' b' c - 500 595 a' b' c - 2116 37 a' b' c - 2116 37 a' b' c - 500 595 a' b' c 61042 a b¹ c , 53175 a³ b¹ c , 32365 a³ b¹ c , 4661 a⁴ b¹ c , 307 a⁴ b¹ c , 2228 b¹ c , 4766 a b¹ c , 3091 a² b¹ c , 307 a⁴ b¹ 1122 108 al bei - MARTE al bei - 110 138 al bei - 100 138 al bei - 110 138 al bei - 110 138 al bei - 120 170 ab el el - 100 170 ab el el - 304 100 al bi el - 304 100 35 556 b¹ c² · 45 845 a b¹ c² · 45 845 a b¹ c² · 17 945 a³ b¹ c² · 218 a³ b¹ c² · 2212 b¹ c¹ · 3931 a b¹ c² · 2764 a³ b¹ c² · 158 a¹ b¹ c² · 158 a¹ b¹ c² · 561 584 c³ · 1083 168 a c¹ · 829 881 a² c¹ · 318 371 a¹ c¹ · 5911 a⁴ c¹ · 4221 a³ c¹ · 919 888 b c¹ · 1712510 8 b c² + 171277 8³ b c² + 465309 8³ b c² + 50468³ b c² + 50468³ b c² + 577780 b² c² + 1029128 a b² c² + 777707 a³ b² c² + 246168³ b² c² + 246168³ b² c² + 2671803 a² b² c² + 663168³ b 15510 d f¹ n² + 2474 a d f¹ n³ + 2474 b d f¹ n³ + 3592 c d f¹ n³ + 15520 e f¹ n³ + 15520 e f¹ n³ + 2558 a n¹⁰ + 2474 b d f¹ n³ + 208 b f¹ n³ 28 941 a¹ n¹¹ , 1177 a¹ n¹¹ , 125 556 b n¹¹ , 70 290 a b n¹¹ , 5599 a¹ b n¹¹ , 5599 a¹ b n¹¹ , 1277 b¹ n¹¹ , 1217 13 552 c^{*}n¹¹ - 129 558 d^{*}n¹¹ - 24568 d^{*}n¹¹ - 4568 b^{*}d^{*}n¹¹ - 4568 b^{*}d^{*}n¹¹ - 11138 d^{*}b^{*}d^{*}n¹¹ - 128 172 c^{*}d^{*}n¹¹ - 28 328 c^{*}d^{**} 64 666 a en¹⁰ - 4665 a¹ en¹¹¹ - 70 298 b e n¹⁰ - 12 232 a b e n¹⁰ - 5559 b² e n¹⁰ - 134 376 c e n¹⁰ - 22 396 a c e n¹⁰ - 22 396 c² e n¹¹¹ - 64 886 d e n¹⁰ - 11 198 b d e n¹⁰ - 11 198 b d e n¹⁰ - 11 362 c d e n¹¹¹ - 4665 a¹ e n¹¹¹ - 4665 a¹ e n¹¹¹ - 4565 a¹¹¹ e n¹¹¹ 10 104 c e¹ n¹⁰ - 4565 d e¹ n¹⁰ - 1177 e¹ n¹¹ - 6204 a f n¹¹ - 1034 a d f n¹¹ - 6204 a f n¹¹ - 1034 a d f n¹¹ - 1 1894 e² f n¹⁰ - 1894 e¹ n¹⁰ - 13888 e¹ - 1754 e¹ n¹¹ - 12888 e¹ - 1754 e¹ n¹¹ - 12888 e¹ - 1754 e¹ n¹¹ - 12888 e¹ - 128888 e¹ - 12888 e¹ - 128888 e¹ - 12888 e¹ - 12888 3864 b d n¹¹ - 7768 c d n¹¹ - 1754 d² n¹¹ - 23654 e n¹¹ - 3864 b e n¹¹ - 3864 d e n¹¹ - 376 b f n¹¹ - 376 b f n¹¹ - 376 c f 12 - a - m (1 - b - m (1 - c - m (1 - b - c - d - m) 1 - c - d - e - m (1 - c - d - e - m) 1 - c - f - m (334 - 240 a - 40a² + 240 b - 144 a b - 24a³ b - 40b² - 24a³ b - 40b² - 256 a c - 35a² c - 224 b c - 308 a b c - 34a³ b c - 24b² c - 10 a b² c - a³ b³ c - a³ b³ c - a³ b³ c - 24b³ c - 10 a b³ c - a³ b³ c - a³ b³ c - 24b³ c - 10 a b³ c - 256 a c - 35a³ c - 224 b c - 308 a b c - 34a³ b c - 24b³ c - 10 a b³ c - a³ b³ c - a³ b³ c - a³ b³ c - 24b³ c - 10 a b³ c - a³ b³ c - a³ b³ c - 24b³ c - 256 a c - 35a³ c - 224 b c - 308 a b c - 34a³ b c - 24b³ c - 10 a b³ c - a³ b³ c - a³ b³ c - 24b³ c - 256 a c - 35a³ c - 224 b c - 308 a b c - 34a³ b c - 24b³ c - 10 a b³ c - a³ b³ c - a³ b³ c - 24b³ c - 10 a b³ c - a³ b³ c - 24b³ c - 256 a c - 35a³ c - 224 b³ c - 256 a c - 35a³ c - 224 b³ c - 256 a c - 35a³ c - 224 b³ c - 256 a c - 35a³ c - 224 b³ c - 256 a c - 35a³ c - 224 b³ c - 256 a c - 35a³ c - 224 b³ c - 256 a c - 35a³ c - 224 b³ c - 256 a c - 35a³ c - 224 b³ c - 256 a c - 35a³ c - 224 b³ c - 256 a c - 35a³ c - 224 b³ c - 256 a c - 35a³ c - 224 b³ c - 256 a c - 35a³ c - 224 b³ c - 256 a c - 35a³ c - 224 b³ c - 256 a c - 35a³ c - 224 b³ c - 256 a c - 256 224 c² + 96 a c² + 10 a² c² + 72 b c² + 25 a b c² + 25 a b c² + 26 a c² + 16 c² + 16 c² + 16 c² + 26 c² + 26 c² + 26 c² + 112 a d + 112 b d + 4a b d + 4a² b d + 12b² d + 4ab² d + 4a² d + 224 c d + 84 a c d + 6a² c d + 72 b c d + 24 a b c d + a² b c d + 26 a b c d 4 b' c d - a b' c d - 72 c' d - 22 a c' d - 3 c' d - 12 b c' d - 3 a b c' d - 3 a b c' d - 3 a b' d - 40 d' + 12 a d' + 12 b d' + 40 d' + 12 a d' + 40 d' + 24 c d' + 6 a c d' + 6 b c d' + 6 b c d' + 6 c d' + 6 b c d' + 24 b c d' + 6 b c d' + 356 ce - 66 a ce - 84° ce - 188° ce - 32° b ce - 181° ce - 3 b° ce - 3 b° ce - 66 c° e - 36 b c° e - 46 b c° e - 66 c° e - 66 c° e - 36 b c 16acde - 24bcde - 4abcde - b¹cde - 22c¹de - 2ac¹de - 3bc¹de - 2c¹de - 12d¹e - 2ad¹e - 4bd²e - abd¹e - 6bd¹e - 6cd¹e - 6cd¹e - 40e² - 12ae¹ - 14be² - 4b¹e¹ - 3bc² - 3bce¹ - 3bce¹ - 14bce¹ - 14bce¹ - 10ce² - b¹ce² - 10c²e¹ ac'e' - 2 bc'e' - c'e' - 126e' - 2 ade' - 4 bde' - abde' - 6 cde' - bcde' - c'de' - 32 af - 12 a' f - 32 b f - 4 a' b f - 12 b' f - 4 ab' f - 52 c f - 12 ac f - 6 a' c f - 3 b c f - 4 ab c f - a' bc f - 6 b' c f - ab' c' f - 2 a c' f - 3' c' f - 14 bc' f abc² f - b² c² f - 2 bc² f - 2 bc² f - c⁴ f - 12 adf - 2 a² df - 12 bdf - 2 b² df - 12 cdf - 12 bcdf - b² cdf - 12 bcdf - b² cdf - 12 bcdf - b² cdf - 2 bc² df - 2 c³ df - 2 bd² f - 2 cd² f - 2 cd² f - 2 cd² f - 2 cd² f - 2 dd² f - 2 bd² f - 2 cd² f - 2 bd² f - 2 cd² f - 2 cd² f - 2 bd² f - 2 cd² f - 2 cd² f - 2 bd² f - 2 cd² f - 2 a'bef-4b'ef-ab'ef-12cef-3acef-a'cef-4bcef-b'cef-2c'ef-ac'ef-bce'f-12def-4adef-3cdef-2acdef-c'def-2d'ef-12d'f-4ae'f-4be'f-abe'f-6ce'f-ace'f-bce'f-c'e'f-2de'fcde'f - 32 f' - 12 a f' - 12 b f' - 4 a b f' - 34 c f' - 6 b c f' - 8 b c f' - 8 c' f' - 8 c' f' - 9 c' f' - 12 d f' - 2 b d f' - 2 b d f' - 8 c d f' - 8 c d f' - 5 c' d f' - 2 d f f' - 2 bceff - c'eff - 2 deff - 6 deff - 980 n - 400 n - 400 n - 216 abn - 24 a' bn - 60 b'n - 24 ab' n - 2 a' b'n - 960 cn - 384 a cn - 350 a' cn - 100 ab cn - 7 a' b cn - 24 b' cn - 5 ab' cn - 350 a' cn - 950 ac' n - 5 a' c' n - 12 ab c' n - 10 ab cn - 100 ab cn - 7 a' b cn - 24 b' cn - 5 ab' cn - 350 a' cn - 950 ac' n - 5 a' c' n - 12 ab c' n - 2 b² c² n. 48 c³ n. 8 x c³ n. 4 b c³ n. 2 c⁴ n. 48 b d n. 168 b d n. 18 b d n. 48 b d n. 2 a³ b d n. 12 b² d n. 38 c d n. 8a c d n. 8a² c d n. 72 b c d n. 72 c³ d n. 10 c³ d n. 6 b c³ d n. 12 a b³ n c d n. 2 a b³ d n. 8 b c³ n c d n. 8 b c³ n c d n. 8 b c³ n c d n. 8 b c d n. 8 b c d n. 72 c d n. 72 c d n. 72 b c d n. 72 12bd²n - 2abd²n - 3acd²n - 3acd²n - 2bcd²n - 2c²d²n - 460 en - 100 aen - 12a² en - 216 ben - 3a² ben - 34² en - 34b² en - 58b² en - 364 cen - 4a² cen - 108 b cen - 16 ab cen - 56² cen - 96 c² en - 12 ac² en - 13 b c² en - 10 b 8c²en - 168den - 36aden - a²den - 48bden - 8abden - 2b²den - 84cden - 8acden - 12bcden - 11c²den - 12d²en - 2b²en - 5cd²en - 50d²en - 12ae²n - 12ae²n - 2b²d²en - 36be²n - 2b²d²en - 7bce²n 12 de² n. ade² n. 2 bde² n. 3 cde² n. 48 af n. 12 a² f n. 48 bf n. 2 a² bf n. 12 b² f n. 48 cf n. 12 a cf n. 3a² cf n. 36 bcf n. 3b cf n. 3b² cf 4acdfn - 6bcdfn - 5c'dfn - ad'fn - bd'fn - cd'fn - 48efn - 24aefn - 2a'efn - 2abefn - 2b'efn - 12cefn - 4acefn - 2bcefn - 2bcefn - 12defn - 12defn - 4cdefn - 4cdefn - 12defn 48 f² n - 12 a f² n - 12 b f² n - 2 a b f² n - 3 a c f⁴ n - 3 a c f⁴ n - 3 b c f² n - 4 c² f¹ n - 12 d f² n - 3 d d f² n - 6 d f² n - 6 d f² n - 6 d f² n - 2 b d f² n - 2 b d f² n - 3 c d f⁴ n - 5 d d f⁴ n - 6 d d n - 3 c d f⁴ n - 6 d d n - 3 c d f⁴ n - 6 d d n - 3 c d f⁴ n - 6 d d n - 3 c d f⁴ n - 6 d d n - 3 c d f⁴ n - 6 d d n - 3 c d f⁴ n - 6 d d n - 3 c d f⁴ n - 6 d d n - 3 c d f⁴ n - 6 d d n - 3 c d f⁴ n - 6 d d n - 3 c d f⁴ n - 6 d d n - 3 c d f⁴ n - 6 d d n - 3 c d f⁴ n - 6 d d n - 3 c d f⁴ n - 6 d d n - 3 c d f⁴ n - 6 d d n - 3 c d f⁴ n - 6 d d n - 3 c 6 abini - 728 cni - 192 a cni - 9 ai cni - 106 b cni - 17 a b cni - 16 b cni - 16 b cni - 16 b cni - 18 b cni - 18 b cni - 12 ch cni - 3 ai dni - 3 b d ni - 12 a b dni - 12 a b dni - 11 a c dni - 18 b c dni - 18 b c dni - 18 b c dni - 3 adini - 3 b d ni - 3 b d ni - 10 b c dni - 12 a b dni - 10 b c dni - 11 a b c dni - 3 b d ni - 3 b d ni - 10 b c dni - 10 b c d

Fix a family of integrals I(n) with C = I(0) to be proven irrational.

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 \blacktriangleright Check whether $\beta > e^{\nu},$ or equivalently, whether

$$\delta = \frac{\log \beta - \nu}{\log \alpha + \nu} > 0.$$

Generalizing the Alladi-Robinson family of integrals

$$I(n) := \int_0^1 \frac{1}{1 + cx} \left(\frac{x(1-x)}{1 + cx}\right)^n \, \mathrm{d}x,$$

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led us to quite a few irrationality proofs of constants of the form $I_1(0) = {}_2F_1(1, a + 1; a + b + 2; -c).$

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- However, there are also some unidentified cases.

Generalizing the Beukers Integral for $\zeta(2)$, we define

$$I_2(a_1, a_2, b_1, b_2)(n) := \frac{1}{B(1 - a_1, 1 - a_2)B(1 - b_1, 1 - b_2)}$$
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It allows us to realize the following constants as weak Apéry limits:

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- Several hundred cases with positive δ, but many of them are equivalent via transformations C → ^{a+bC}/_{c+dC} with integer coeffs.
- ► Again, there are some cases that could not be identified.

Examples

$$\begin{aligned} C_2(0,0,\frac{1}{2},0) &= {}_3F_2(1,1,\frac{1}{2};2,\frac{3}{2};1) &= 2\log 2\\ C_2(0,0,\frac{1}{3},-\frac{2}{3}) &= {}_3F_2(1,1,\frac{2}{3};2,\frac{7}{3};1) &= -6 + 4\pi\sqrt{3}/3\\ C_2(-\frac{3}{4},-\frac{3}{4},-\frac{1}{4},-\frac{3}{4}) &= {}_3F_2(1,\frac{7}{4},\frac{5}{4};\frac{7}{2},3;1) &= -240 + \frac{512}{3}\sqrt{2}\\ C_2(-\frac{4}{5},-\frac{4}{5},-\frac{2}{5},-\frac{3}{5}) &= {}_3F_2(1,\frac{9}{5},\frac{7}{5};\frac{18}{5},3;1) &= -\frac{845}{2} + \frac{2275}{12}\sqrt{5}\\ C_2(-\frac{5}{6},-\frac{5}{6},-\frac{1}{2},-\frac{1}{2}) &= {}_3F_2(1,\frac{11}{6},\frac{3}{2};\frac{11}{3},3;1) &= -\frac{1344}{5} + \frac{16384\sqrt{3}}{105}\\ C_2(-\frac{5}{6},-\frac{5}{6},-\frac{1}{3},-\frac{2}{3}) &= {}_3F_2(1,\frac{11}{6},\frac{4}{3};\frac{11}{3},3;1) &= \frac{9722^{2/3}}{5} - \frac{1536}{5} \end{aligned}$$

Using the generalized Beukers integral for $\zeta(3)$,

$$J_{3}(a_{1}, a_{2}, b_{1}, b_{2}, c_{1}, c_{2}; e)(n) := \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \left(\frac{x(1-x)y(1-y)z(1-z)}{1-z+xyz} \right)^{n} \\ \times \frac{x^{a_{1}}(1-x)^{a_{2}}y^{b_{1}}(1-y)^{b_{2}}z^{c_{1}}(1-z)^{c_{2}}}{(1-z+xyz)^{e}} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z,$$

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$$K(a, b, c, d, e)(n) := I_3(b, c, e, a, a, c, d)(n)$$

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The output file contains many such conjectured evaluations and we challenge the birthday boy [WZ], or anyone else, to prove them.

The Birthday Boy Problem

Wadim writes:

In their recent preprint arXiv:2101.08308, Robert Dougherty-Bliss, Christoph Koutschan and Doron Zeilberger come up with a powerful strategy to prove the irrationality, in a quantitative form, of some numbers that are given as multiple integrals or quotients of such. What is really missing there, for many examples given, is an explicit identification of those irrational numbers. Without an identification, the numbers are hardly appealing to human (number theorists). The goal of this note is to outline a strategy to do the job and illustrate it on several promising entries discussed in the preprint above.

Zudilin

$$\begin{split} &K(0,0,0,\frac{2}{3},\frac{1}{3}) = -\frac{K_1 - 2}{2(K_1 - 3)}, & \text{where } K_1 = \log 3 + \frac{\pi}{\sqrt{3}} \\ &K(0,0,0,\frac{1}{3},\frac{2}{3}) = -\frac{2(K_2 + 1)}{K_2 + 1/2}, & \text{where } K_2 = \log 3 + \frac{\pi}{\sqrt{3}} \\ &K(0,\frac{1}{3},\frac{2}{3},\frac{1}{3},\frac{2}{3}) = -\frac{20(7 - 54K_3)}{52 - 405K_3}, \text{where } K_3 = \frac{\Gamma(2/3)^3}{\Gamma(1/3)^3} \\ &K(0,\frac{1}{5},0,\frac{3}{5},\frac{2}{5}) = -\frac{4(1 - 4K_4)}{5 - 24K_4}, & \text{where } K_4 = \frac{1}{\sqrt{5}}\log\frac{\sqrt{5} + 1}{2} \\ &K(\frac{1}{7},0,\frac{2}{7},\frac{3}{7},\frac{4}{7}) = -\frac{189(8 - 5K_5)}{832 - 525K_5}, \text{where } K_5 = \frac{2^{2/7}\sqrt{\pi}\,\Gamma(9/14)}{\cos(3\pi/14)\,\Gamma(4/7)^2} \end{split}$$

Perhaps, a real pearl in this collection of "quantitatively" irrational numbers is the number K_3 .

Wadim Zudilin suggested to study the double integral

$$J_n(z) = \int_0^1 \int_0^1 \frac{x^{n-1/2}(1-x)^{n-1/2}y^{n-1/2}(1-y)^n}{(1-zxy)^{n+1/2}} \,\mathrm{d}x \,\mathrm{d}y$$

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= $\frac{\Gamma(n+\frac{1}{2})^3\Gamma(n+1)}{\Gamma(2n+1)\Gamma(2n+\frac{3}{2})} \cdot {}_3F_2 \left(\begin{array}{c} n+\frac{1}{2}, n+\frac{1}{2}, n+\frac{1}{2}\\ 2n+1, 2n+\frac{3}{2} \end{array} \right) z \, \mathrm{d}z$

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A recurrence equation can be obtained by

- continuous creative telescoping on the double integral
- Zeilberger's fast algorithm on the hypergeometric representation

Third-Order Recurrence

$$\begin{split} &4z^4(2n+1)^2(n+1)^2 \left(16(27z-32)n^4-16(69z-86)n^3\right.\\ &+8(108z-143)n^2-4(55z-76)n+3(7z-10)\right)J_{n+1}\\ &+z^2 \left(256(3z+8)(27z-32)n^8-256(3z+8)(15z-22)n^7\right.\\ &-64(651z^2+661z-1744)n^6+192(59z^2-186)n^5\\ &+16(1503z^2+697z-3610)n^4-16(79z^2-290z+116)n^3\\ &-4(569z^2-381z-580)n^2+4(11z^2-44z+18)n+3(4z+3)(7z-10)\right)J_n\\ &+4n \left(64(3z^2-20z+16)(27z-32)n^7-384(3z^2-20z+16)(7z-9)n^6\right.\\ &-16(411z^3-2698z^2+3988z-1696)n^5+64(183z^3-1372z^2+2339z-1134)n^4\\ &+4(531z^3-1400z^2-424z+1240)n^3-8(571z^3-4001z^2+6532z-3060)n^2\\ &+(151z^3-4742z^2+11596z-6888)n+12(14z^2-29z-30)(z-1))J_{n-1}\\ &+4n(n-1)(2n-3)^2(z-1)\left(16(27z-32)n^4+48(13z-14)n^3\right.\\ &+8(18z-11)n^2-4(19z-24)n-(7z+6)\right)J_{n-2}=0. \end{split}$$

Initial Values

We have

$$\begin{aligned} J_0(z) &= \lambda(z), \\ J_1(z) &= -\frac{3+4z}{4z^2} \lambda(z) - \frac{5(1-z)}{z^2} \rho_1(z) + \frac{13}{2z^2} \rho_2(z), \\ J_2(z) &= \frac{105+480z+64z^2}{64z^4} \lambda(z) + \frac{3151-2167z-984z^2}{144z^4} \rho_1(z) - \frac{7247+3452z}{288z^4} \rho_2(z), \end{aligned}$$

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where

$$\lambda(z) = \int_0^1 \int_0^1 \frac{\mathrm{d}x \,\mathrm{d}y}{\sqrt{x(1-x)y(1-zxy)}}$$
$$\rho_1(z) = \int_0^1 \frac{\mathrm{d}x}{\sqrt{x(1-x)(1-zx)}}$$
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where

$$\begin{split} \lambda(z) &= \int_0^1 \int_0^1 \frac{\mathrm{d}x \,\mathrm{d}y}{\sqrt{x(1-x)y(1-zxy)}} = 2\pi \,_3F_2 \begin{pmatrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, \frac{3}{2} \\ \end{pmatrix},\\ \rho_1(z) &= \int_0^1 \frac{\mathrm{d}x}{\sqrt{x(1-x)(1-zx)}} = \pi \,_2F_1 \begin{pmatrix} \frac{1}{2}, \frac{1}{2} \\ 1 \\ \end{pmatrix},\\ \rho_2(z) &= \int_0^1 \frac{\sqrt{1-zx}}{\sqrt{x(1-x)}} \,\mathrm{d}x = \pi \,_2F_1 \begin{pmatrix} -\frac{1}{2}, \frac{1}{2} \\ 1 \\ \end{pmatrix}. \end{split}$$

Hence, each integral can be written as a linear combination of $\lambda,\rho_1,\rho_2:$

$$J_n(z) = a_n(z)\lambda(z) + b_n(z)\rho_1(z) + c_n(z)\rho_2(z)$$

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For $z^{-1} \in \mathbb{Z} \setminus \{\pm 1\}$, the coefficients a_n, b_n, c_n seem to satisfy

$$z^n 2^{4n} a_n, \ z^n 2^{4n} d_{2n}^2 b_n, \ z^n 2^{4n} d_{2n}^2 c_n \in \mathbb{Z}$$
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Eliminating $\rho_2(z)$ we get

$$\det\begin{pmatrix} J_n & J_{n+1} \\ c_n & c_{n+1} \end{pmatrix} = \underbrace{\det\begin{pmatrix} a_n & a_{n+1} \\ c_n & c_{n+1} \end{pmatrix}}_{\bullet} \cdot \lambda(z) + \underbrace{\det\begin{pmatrix} b_n & b_{n+1} \\ c_n & c_{n+1} \end{pmatrix}}_{\bullet} \cdot \rho_1(z)$$

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The sequences A_n and B_n satisfy again a third-order recurrence, which is the exterior square of the recurrence for J_n .
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$$\begin{split} &z^{2n+2}2^{2n}d_{2n}(n+1)(2n+1)^2A_n\in\mathbb{Z},\\ &z^{2n+2}2^{2n}d_{2n}^2(n+1)(2n+1)^2B_n\in\mathbb{Z},\quad\text{for }n=0,1,2,\ldots\,. \end{split}$$

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In other words, the number λ/ρ_1 (but also the quotients λ/ρ_2 and ρ_1/ρ_2) are Apéry limits for the considered values of z.

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In other words, the number λ/ρ_1 (but also the quotients λ/ρ_2 and ρ_1/ρ_2) are Apéry limits for the considered values of z. Note that

$$\lambda\left(\frac{1}{2}\right) = 2\sqrt{2}\,\pi L'(E,0) = 16\sqrt{2}\,\frac{L(E,2)}{\pi}, \quad \rho_1\left(\frac{1}{2}\right) = 4\sqrt{2}\,L(E,1).$$

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