The combinatorics of motion polynomials

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Linkages

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Restriction: We consider only **planar linkages**.

There are two different types of joints:

1. rotational joints
2. translational joints

$\longrightarrow$ Show animation!

Motivation and Goal

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Theorem (Kempe's Universality Theorem).
Let $f \in \mathbb{R}[x, y]$ be a polynomial, and let $B \subseteq \mathbb{R}^2$ be a closed disk.
Then there exists a planar linkage which draws the curve

$$B \cap \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 0\}.$$

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Goal 2: Construct a linkage that realizes a certain planar motion.

Model (Denavit-Hartenberg)

1. Not a single frame of reference for the configuration of a linkage, but each link has its own frame of reference. Every frame of reference is modeled by a Euclidean affine plane.
2. Self-collisions of the links are not taken into account.
3. Thus the actual shape of the links doesn't matter, just the position of the joints.

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$$(x_0, \ldots, x_3) \sim (y_0, \ldots, y_3) :\Longleftrightarrow \exists c \in \mathbb{C}^* : (x_0, \ldots, x_3) = (cy_0, \ldots, cy_3).$$

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Embedding: We embed SE_2 in $\mathbb{P}^3_{\mathbb{C}}$ as the set of real points of the open subset

$$\mathcal{U} = \mathbb{P}^3_{\mathbb{C}} \setminus \left\{(x_1 : x_2 : y_1 : y_2) \in \mathbb{P}^3_{\mathbb{C}} \mid x_1^2 + x_2^2 = 0\right\}.$$

Geometric interpretation: Hence $\mathcal{U}$ is the complement of the two conjugate complex planes $x_1 + i\,x_2 = 0$ and $x_1 - i\,x_2 = 0$.

Action

Let $\sigma \in \mathrm{SE}_2$ be a direct isometry, given by the projective point $(x_1 : x_2 : y_1 : y_2) \in \mathbb{P}^3_{\mathbb{C}}$.

The action of σ on a point (x, y) in the plane is given by:

$$\frac{1}{x_1^2 + x_2^2} \left[\begin{pmatrix} x_1^2 - x_2^2 & -2x_1x_2 \\ 2x_1x_2 & x_1^2 - x_2^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_1y_1 - x_2y_2 \\ x_1y_2 + x_2y_1 \end{pmatrix} \right].$$

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- Note that the points for which $x_1^2 + x_2^2 = 0$ were excluded.
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- The translational part vanishes if $y_1 = y_2 = 0$.
- Action is compatible with $\sim$ in $\mathbb{P}^3_{\mathbb{C}}$.

Product

The product in SE_2 becomes a bilinear map:

$$(x_1 : x_2 : y_1 : y_2) \cdot (x_1' : x_2' : y_1' : y_2') =$$

$$\big(x_1x_1' - x_2x_2' : x_1x_2' + x_2x_1' : x_1y_1' + x_2y_2' + y_1x_1' - y_2x_2' \\ : x_1y_2' - x_2y_1' + y_1x_2' + y_2x_1'\big)$$

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Notation 1: Write a point $(x_1 : x_2 : y_1 : y_2)$ as a pair of complex numbers (z, w) where $z = x_1 + i\,x_2$ and $w = y_1 + i\,y_2$.

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Using this notation, the product in SE_2 can be rewritten as

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Notation 2: This operation can be further simplified by using "dual numbers": write (z, w) as $z + \eta\,w$ where η satisfies $\eta\,z = \overline{z}\,\eta$ and $\eta^2 = 0$. Denote $\mathbb{K} = \mathbb{C}[\eta]/\langle i\,\eta + \eta\,i, \eta^2\rangle$.

Rational motions and motion polynomials

Definition. A **rational motion** is a map $\mathbb{R} \longrightarrow \mathbb{P}^3_{\mathbb{C}}$ given by four real polynomials $X_1, X_2, Y_1, Y_2 \in \mathbb{R}[t]$ such that $X_1^2 + X_2^2$ is not identically zero. Hence for almost every t this map yields a direct isometry in SE_2.

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Definition. Using the notation introduced before, a rational motion can be written as a single polynomial $P(t) \in \mathbb{K}[t]$, where $P(t) = Z(t) + \eta W(t)$ with $Z, W \in \mathbb{C}[t]$. A polynomial $P \in \mathbb{K}[t]$ is called a **motion polynomial**.

Set of rational motions

Definition. The set of rational motions is defined as $\mathbb{K}[t]/\sim$ where

$$P_1(t) \sim P_2(t) :\Longleftrightarrow \exists\, R_1, R_2 \in \mathbb{R}[t] \setminus \{0\} \colon P_1 R_1 = P_2 R_2.$$

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In other words, the (commutative) multiplication by a real polynomial changes a motion polynomial, but not the rational motion it describes.

Normedness and boundedness

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Definition. We say that P is **bounded** if Z does not have real roots.

$\longrightarrow$ Recall the prefactor $\frac{1}{x_1^2 + x_2^2}$ in the definition of the action.
If Z has a real root, this causes division by zero.

Characterization of degree 1 motions

Lemma. Let $L \subseteq \mathbb{P}^3_{\mathbb{C}}$ be a real line, namely a line defined by real equations, and define $L_{\mathcal{U}} = L \cap \mathcal{U}$. Let X be the set-theoretical intersection of L and the complement of $\mathcal{U}$ in $\mathbb{P}^3_{\mathbb{C}}$. Then:

1. if X has cardinality 1, then $L_{\mathcal{U}}$ corresponds to the set of isometries $\sigma \in \mathrm{SE}_2$ such that $\sigma(L_1) = L_2$ for some lines $L_1, L_2 \subseteq \mathbb{R}^2$ (translational motion).
2. if X has cardinality 2, then $L_{\mathcal{U}}$ corresponds to the set of isometries $\sigma \in \mathrm{SE}_2$ such that $\sigma(p_1) = p_2$ for some fixed points $p_1, p_2 \in \mathbb{R}^2$ (rotational motion = "revolution").

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Lemma. Let $P \in \mathbb{K}[t]$ be a normed motion polynomial of degree 1, i.e., $P(t) = t + i\, x_2 + \eta\, (y_1 + i\, y_2)$, $x_2, y_1, y_2 \in \mathbb{R}$. Then:

1. if $x_2 = 0$ then P describes a translational motion in direction $\binom{y_1}{y_2}$.
2. if $x_2 \neq 0$ then P describes a revolution around the point $\frac{1}{2x_2}\binom{-y_2}{y_1}$.

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3. Multiplication of motion polynomials corresponds to composition of motions, e.g.,

   $$(t + i) \cdot (t - i + \eta) = (t^2 + 1) + \eta\,(t - i)$$

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   $\longrightarrow$ This parametrizes the circle with radius $\frac{1}{2}$ around the point $(0, -\frac{1}{2})$. Hence we get a circular translation.

Strategy

Task: Construct a linkage that realizes a given rational motion.

1. The motion is described by a motion polynomial $P \in \mathbb{K}[t]$.
2. Factor P into linear factors.
3. Each linear factor represents an "elementary" motion (revolution, translational motion), which can be realized by a single joint.
4. A factorization of P gives rise to an open chain of links, which, among many others, realizes the desired motion.
5. Combine different factorizations in order to restrict the degrees of freedom (to 1).

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$\longrightarrow$ Show demo!

Factorization into linear factors

Let $P = Z + \eta W \in \mathbb{K}[t]$ be a normed and bounded motion polynomial of degree n.

Goal: Factor P into linear motion polynomials, i.e., $P = P_1 \cdots P_n$ with $\deg P_i = 1$.

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$$(Z + \eta\, W) \cdot (Z' + \eta\, W') = Z\, Z' + \eta\, (\overline{Z}\, W' + Z'\, W)$$

we see that $Z(t) = (t - z_1) \cdots (t - z_n)$, i.e., the z_i are precisely the complex roots of $Z(t)$.

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$\longrightarrow$ The w_i can be found by making an ansatz and solving a linear system.

Factorization into linear factors

Fix a certain permutation $(z_1, \ldots, z_n)$ of the complex roots of Z. Multiply out:

$$\prod_{i=1}^{n}(t - z_i + \eta w_i) = Z(t) + \eta \sum_{k=1}^{n} \left(\prod_{j=1}^{k-1} (t - \overline{z_j}) \right) \left(\prod_{j=k+1}^{n} (t - z_j) \right) w_k.$$

Hence we get the following condition on $w_1, \ldots, w_n$

$$W(t) = \sum_{k=1}^{n} w_k Q_k(t)$$

where the polynomials $Q_k(t) \in \mathbb{C}[t]$ are defined as above:

$$\begin{aligned}
Q_1 &= (t - z_2) \cdots (t - z_n) \\
Q_2 &= (t - \overline{z_1})(t - z_3) \cdots (t - z_n) \\
&\vdots \\
Q_k &= (t - \overline{z_1}) \cdots (t - \overline{z_{k-1}})(t - z_{k+1}) \ldots (t - z_n) \\
&\vdots \\
Q_n &= (t - \overline{z_1}) \cdots (t - \overline{z_{n-1}})
\end{aligned}$$

Characterization of factorizable polynomials

Lemma. Let $P = Z + \eta W$ be normed and let $(z_1, \ldots, z_n)$ be a fixed permutation of the roots of Z over $\mathbb{C}$. Then P admits a factorization $P = P_1 \cdots P_n$ where $P_i(t) = (t - z_i) + \eta\, w_i$ with $w_i \in \mathbb{C}$ if and only if W lies in the linear span $\langle Q_1, \ldots, Q_n \rangle_{\mathbb{C}}$.

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Lemma. Let $P = Z + \eta W \in \mathbb{K}[t]$ be a normed motion polynomial such that Z has no pair of complex conjugated roots (i.e., $Z(z) = 0 \implies Z(\overline{z}) \neq 0$). Then for every permutation $(z_1, \ldots, z_n)$ of the roots of Z, the polynomial P admits a factorization into linear factors.

Note: This condition is only sufficient, but not necessary, for the existence of a factorization.

Sufficient condition

$Z + \eta W$ admits a factorization if and only if $W \in \langle Q_1, \ldots, Q_n \rangle_{\mathbb{C}}$.

Clearly, this is always possible (for arbitrary W) if the determinant of the following matrix $M_n \in \mathbb{C}^{n \times n}$ is nonzero:

$$M_n = \begin{pmatrix} \langle t^0 \rangle Q_1 & \cdots & \langle t^0 \rangle Q_n \\ \langle t^1 \rangle Q_1 & \cdots & \langle t^1 \rangle Q_n \\ \vdots & & \vdots \\ \langle t^{n-1} \rangle Q_1 & \cdots & \langle t^{n-1} \rangle Q_n \end{pmatrix}$$

where $\langle t^i \rangle Q_k$ denotes the coefficient of t^i in Q_k.

The matrix entries can be written in terms of the elementary symmetric polynomials σ_i:

$$\langle t^i \rangle Q_k = (-1)^i \sigma_i(\boldsymbol{z}^{(k)}) \quad \text{where } \boldsymbol{z}^{(k)} := (\overline{z_1}, \ldots, \overline{z_{k-1}}, z_{k+1}, \ldots, z_n).$$

Evaluating the determinant

Lemma. Let $M_n = \big((-1)^i \sigma_i(\boldsymbol{z}^{(j)})\big)_{1 \le i,j \le n}$. Then we have

$$\det(M_n) = (-1)^{\lfloor n/2 \rfloor} \prod_{1 \le i < j \le n} \big(\overline{z_i} - z_j\big).$$

Remarks:

- The statement is very much reminiscent of the Vandermonde determinant and it can be proved in a similar fashion.
- A similar determinant evaluation is given in (Lascoux/Pragacz 2002) where the z_i appear without conjugation.
- The above formula is also a special case of a determinant evaluation that appears in (Krattenthaler 1999).

Condition for existence of a factorization

Proposition. Let $P = Z + \eta W \in \mathbb{K}[t]$ be a normed motion polynomial and let $(z_1, \ldots, z_n)$ be a fixed permutation of the roots of Z. Then

$$W \in \langle Q_1, \ldots, Q_n \rangle_{\mathbb{C}} \iff W \in \langle Q_1, \ldots, Q_n \rangle_{\mathbb{C}[t]}.$$

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Remarks:

- The ideal on the right-hand side is generated by a single polynomial $G := \gcd(Q_1, \ldots, Q_n)$.
- The condition on factorizability rephrases as $G \mid W$.
- Note that G depends on the permutation of the z_i.

No factorization?

Problem: What if $G \nmid W$ for any permutation z?

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Solution: Multiply P by some real polynomial $R \in \mathbb{R}[t]$!

- Note that this doesn't change the motion itself.
- W.l.o.g. assume $R = (t - z)(t - \overline{z})$ and put $P' = PR$.
- Clearly, $W' = WR$, so we add two roots to W.
- On the other hand, we can achieve $G' = G \cdot (t - z)$ or $G' = G \cdot (t - \overline{z})$. Thus we add only a single root to G.
- Repeating this process, we finally achieve $G \mid W$, as desired.

Computation of G

Definition. Let $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. A set

$$M \subseteq \left\{(i,j) : 1 \leq i < j \leq n \wedge z_i = \overline{z_j}\right\}$$

is called a **matching** of z if for all $(i_1, j_1), (i_2, j_2) \in M$ we have $i_1 \neq i_2$ and $j_1 \neq j_2$.

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Lemma. Let $Z \in \mathbb{C}[t]$ have no real roots, and let $z = (z_1, \ldots, z_n)$ be a permutation of its (not necessarily distinct) roots. Let M be a matching of z of maximal size, and let $Q_1, \ldots, Q_n$ be defined as before. Then we have

$$G := \gcd(Q_1, \ldots, Q_n) = \prod_{(i,j) \in M} (t - z_j)$$

(where the gcd is assumed to be a monic polynomial).

Some examples

Let $Z = (t-\alpha)^r (t-\overline{\alpha})^{r+1}$. In the following table we consider different permutations z of the roots of Z:

z	G	M
$(\alpha, \dots, \alpha, \overline{\alpha}, \dots, \overline{\alpha})$	$(t-\overline{\alpha})^r$	$\{(1, r+1), (2, r+2), \dots, (r, 2r)\}$
$(\overline{\alpha}, \dots, \overline{\alpha}, \alpha, \dots, \alpha)$	$(t-\alpha)^r$	$\{(1, r+2), (2, r+3), \dots, (r, 2r+1)\}$
$(\overline{\alpha}, \alpha, \overline{\alpha}, \alpha, \dots, \alpha, \overline{\alpha})$	$(t-\alpha)^r (t-\overline{\alpha})^r$	$\{(1, 2), (2, 3), \dots, (2r, 2r+1)\}$

The cases displayed above are the extreme ones:

- It is easy to see that $r \leq \deg(G) \leq 2r$.
- For any $G = (t-\alpha)^i (t-\overline{\alpha})^j$ with $0 \leq i, j \leq r$ and $i + j \geq r$ there exists a permutation z which produces this gcd G.

Connections to combinatorics

Task: Count number of factorizations.

From now on consider only a single root and its complex conjugate: $Z(t) = (t - \alpha)^r (t - \overline{\alpha})^s$. (The general case is easily obtained using the multinomial coefficient.)

Thus a permutation z of the roots of Z can be viewed as a word λ over the alphabet $\{\alpha, \overline{\alpha}\}$.

Definition:

1. Let $\overline{\lambda}$ denote the component-wise complex conjugation of λ.
2. By $\ell(\lambda)$ we denote the length of λ.
3. Let $\mu \leq \lambda$ denote the fact that μ is a subword of λ, i.e., $\mu = (\lambda_{i_1}, \ldots, \lambda_{i_k})$ for $1 \leq i_1 < \cdots < i_k \leq \ell(\lambda)$.
4. Associate steps in $\mathbb{Z}^2$ to the letters: $\alpha = \binom{1}{-1}$ and $\overline{\alpha} = \binom{1}{1}$.
5. Identify a word λ with a lattice walk that starts at $(0, 0)$.

Dyck paths

Definition: The Dyck length $D(\lambda)$ of a word λ is

$$D(\lambda) = \frac{1}{2} \max_{\substack{\mu \leq \lambda \\ \mu \text{ Dyck path}}} \ell(\mu).$$

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Proposition. The gcd $G = G_\lambda = \gcd(Q_1, \ldots, Q_{r+s})$ associated to λ is given by:

$$G_\lambda = (t - \alpha)^{D(\lambda)} (t - \overline{\alpha})^{D(\overline{\lambda})}.$$

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$\longrightarrow$ Thus the number of factorizations corresponds to the number of words λ with certain Dyck lengths $D(\lambda)$ and $D(\overline{\lambda})$.

Outlook

Work in progress:

- Make construction of linkage precise, show that the result has one degree of freedom, etc.
- Take care about cases where there are too few factorizations.
- Which class of curves can be drawn by this construction?

A similar construction can be done for 3D linkages. In that case direct isometries in SE_3 are represented by "dual quaternions".