Computer Algebra with D-Finite Functions

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June 20, 2024
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ÖAW RICAM

Special Functions

- arise in mathematical analysis and in real-world phenomena

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Airy function

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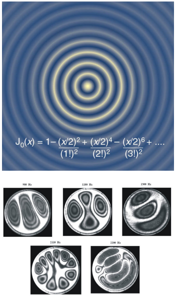

Bessel function

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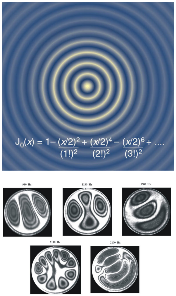

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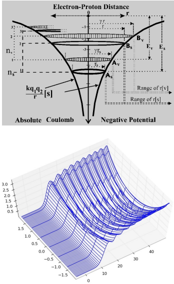

Coulomb function

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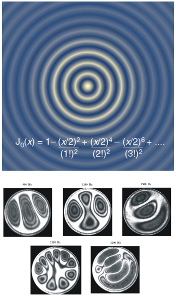

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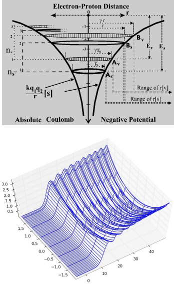

Coulomb function

Special Functions

- arise in mathematical analysis and in real-world phenomena
- are solutions to certain differential equations
- cannot be expressed in terms of the usual elementary functions ($\sqrt{\ }$, exp, log, sin, cos, ...)

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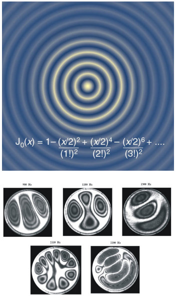

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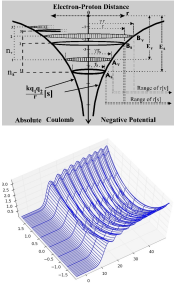

Coulomb function

D-finite Functions

Definition: A function $f(x)$ is called **D-finite** if it satisfies a linear ordinary differential equation with polynomial coefficients:

$$p_r(x)f^{(r)}(x) + \cdots + p_1(x)f'(x) + p_0(x)f(x) = 0,$$

$p_0, \ldots, p_r \in \mathbb{K}[x]$ (not all zero).

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- Equivalently, such functions/sequences are called **holonomic**.

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Example: The **Bessel function** $J_\nu(x)$ describes the vibrations of a circular membrane and other phenomena with cylindrical symmetry.

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Many special functions can be characterized as solutions to systems of linear differential equations and recurrences, and in fact are D-finite (holonomic).

Finiteness Property

Example: The **Legendre polynomials** are orthogonal polynomials w.r.t. the L^2 inner product $\int_{-1}^{1} f(x)g(x)\,\mathrm{d}x$, and satisfy the ODE

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$$(x^2-1)P_n^{(4)}(x) + 6xP_n^{(3)}(x) - (n-2)(n+3)P_n''(x) = 0$$

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$$P_n^{(4)}(x) = \frac{n^2x^2-n^2+nx^2-n+18x^2+6}{(x^2-1)^2}P_n''(x) - \frac{6(n-1)(n+2)x}{(x^2-1)^2}P_n'(x)$$

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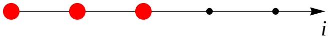

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$\longrightarrow$ $P_n(x)$ is **D-finite** w.r.t. x.

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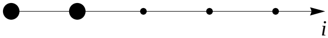

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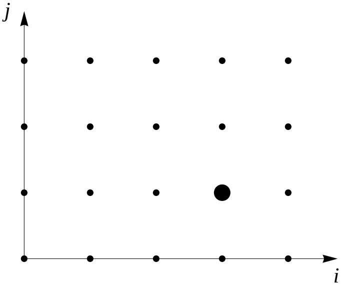

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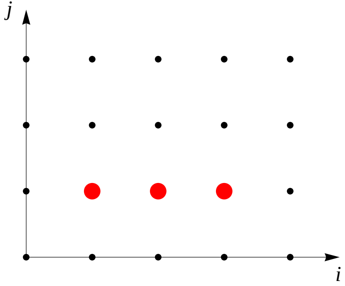

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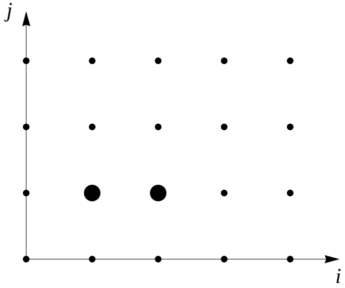

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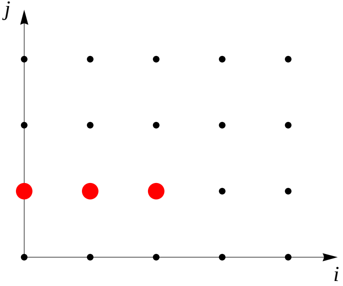

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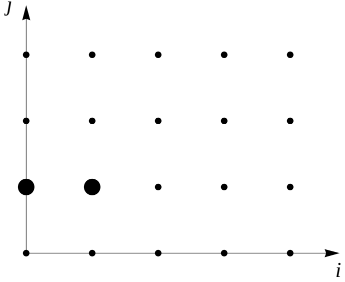

$$P_{n+1}^{(3)}(x) = \frac{(n^2x^2-n^2+3nx^2-3n+8x^2)}{(x^2-1)^2}P_{n+1}'(x) - \frac{4(n^2x+3nx+2x)}{(x^2-1)^2}P_{n+1}(x)$$

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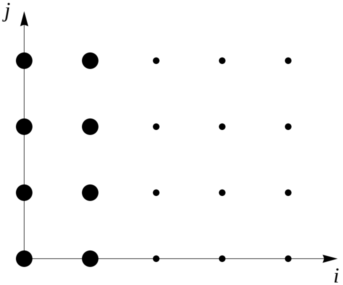

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$$P_0(x) = 1, \quad P_1(x) = x$$
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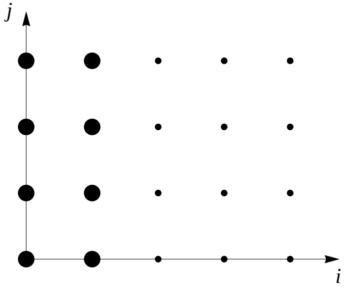

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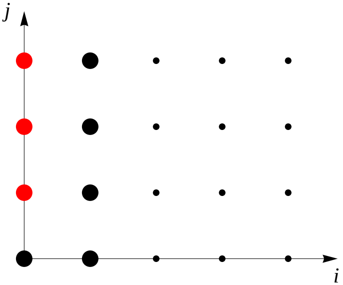

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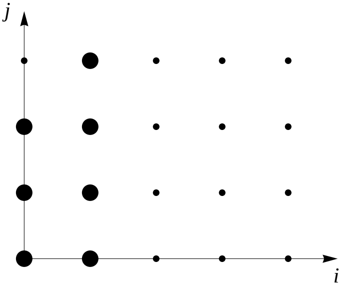

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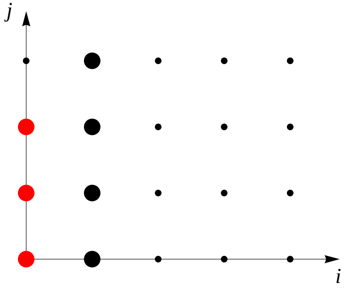

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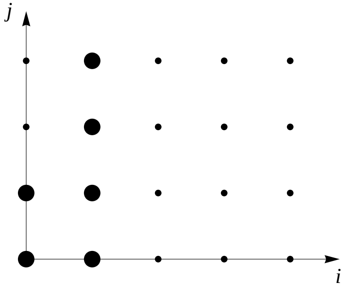

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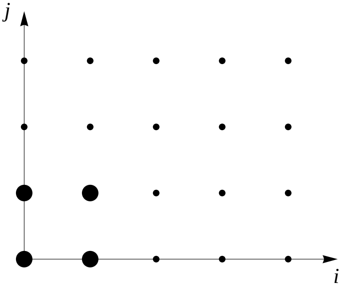

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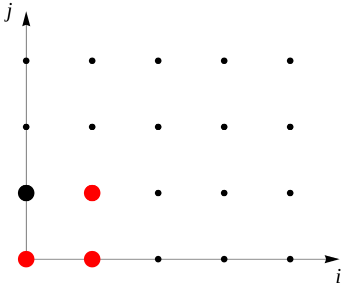

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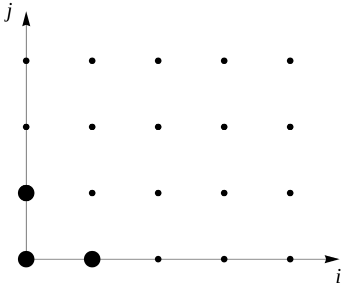
$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x).$$

Consider the set $\left\{P_{n+j}^{(i)}(x) : i, j \geqslant 0\right\}$.



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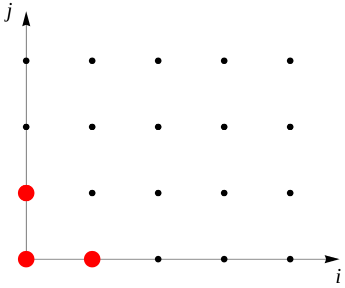

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$\longrightarrow P_n(x)$ is **D-finite** w.r.t. n and x (of rank 2).

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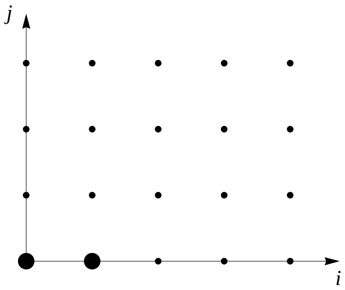

Multivariate D-finite Functions

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- $f_n(h(x))$, where $h(x)$ is an algebraic function

Many Functions are D-finite

ArcCsc, KelvinBei, HypergeometricPFQ, ExpIntegralE, ArcTanh, HankelH2, AngerJ, JacobiP, ChebyshevT, AiryBi, AiryAi, Sinc, Multinomial, CatalanNumber, QBinomial, CosIntegral, ArcSech, SphericalHankelH2, HermiteH, ExpIntegralEi, Beta, AiryBiPrime, SphericalBesselJ, Binomial, ParabolicCylinderD, Erfc, EllipticK, Fibonacci, QFactorial, Cos, Hypergeometric2F1, Erf, KelvinKer, HypergeometricPFQRegularized, Log, Factorial, BesselY, Cosh, CoshIntegral, ArcTan, ArcCoth, LegendreP, LaguerreL, EllipticE, SinhIntegral, Sinh, BetaRegularized, SphericalHankelH1, ArcSin, EllipticThetaPrime, Root, LucasL, AppellF1, FresnelC, LegendreQ, ChebyshevU, GammaRegularized, Erfi, HarmonicNumber, BesselI, KelvinKei, ArithmeticGeometricMean, Exp, ArcCot, EllipticTheta, Hypergeometric0F1, EllipticPi, GegenbauerC, ArcCos, WeberE, FresnelS, EllipticF, ArcCosh, Subfactorial, QPochhammer, Gamma, StruveH, WhittakerM, ArcCsch, Hypergeometric1F1, SinIntegral, BesselJ, StruveL, ArcSec, Factorial2, KelvinBer, BesselK, ArcSinh, HankelH1, Sqrt, PolyGamma, HypergeometricU, AiryAiPrime, Sin,

Quiz: Who is D-Finite?

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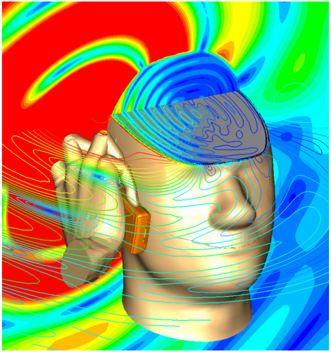
- evaluate symbolic determinants (e.g., in combinatorics)

Finite Elements



(joint work with Joachim Schöberl and Peter Paule)

Problem Setting

Simulate the propagation of electromagnetic waves according to

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \operatorname{curl} E, \quad \frac{\mathrm{d}E}{\mathrm{d}t} = -\operatorname{curl} H \qquad \text{(Maxwell)}$$

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Define basis functions (2D case):

$$\varphi_{i,j}(x, y) := (1-x)^i P_j^{(2i+1,0)}(2x-1) P_i\left(\tfrac{2y}{1-x} - 1\right)$$

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Problem: Represent the partial derivatives of $\varphi_{i,j}(x,y)$ in the basis (i.e., as linear combinations of shifts of the $\varphi_{i,j}(x,y)$ itself).

Solution

Ansatz: One needs a relation of the form

$$\sum_{(k,l)\in A} a_{k,l}(i,j)\,\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+k,j+l}(x,y) \;=\; \sum_{(m,n)\in B} b_{m,n}(i,j)\varphi_{i+m,j+n}(x,y),$$

that is free of x and y (and similarly for $\frac{\mathrm{d}}{\mathrm{d}y}$).

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that is free of x and y (and similarly for $\frac{\mathrm{d}}{\mathrm{d}y}$).

Result: Computer algebra methods (D-finite closure properties, Gröbner bases), deliver the relation

$$\begin{aligned}
&(2i+j+3)(2i+2j+7)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i,j+1}(x,y)+\\
&2(2i+1)(i+j+3)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i,j+2}(x,y)-\\
&(j+3)(2i+2j+5)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i,j+3}(x,y)+\\
&(j+1)(2i+2j+7)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+1,j}(x,y)-\\
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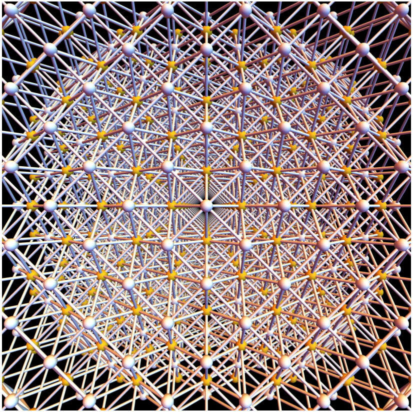
\end{aligned}$$

Creative Telescoping



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$$\underbrace{\sum_{k=1}^{\infty} \frac{1}{k(k+n)}}_{=:\, f_n} \rightsquigarrow (n+2)^2 f_{n+2} = (n+1)(2n+3) f_{n+1} - n(n+1) f_n$$

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Method for doing integrals and sums
(aka Feynman's differentiating under the integral sign)

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$$c_r(n)f(n+r,k) + \cdots + c_0(n)f(n,k) = g(n,k+1) - g(n,k).$$

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Summing from a to b yields a recurrence for $F(n)$:

$$c_r(n)F(n+r) + \cdots + c_0(n)F(n) = g(n,b+1) - g(n,a).$$

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Consider the following integration problem: $F(x) := \int_a^b f(x,y)\,\mathrm{d}y$

Telescoping: write $f(x,y) = \frac{\mathrm{d}}{\mathrm{d}y} g(x,y)$.

Then $F(n) = \int_a^b \left(\frac{\mathrm{d}}{\mathrm{d}y} g(x,y)\right) \mathrm{d}y \qquad = g(x,b) - g(x,a)$.

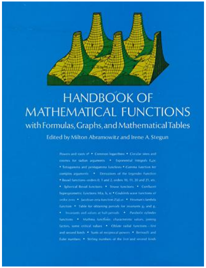
Creative Telescoping: write

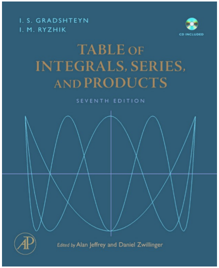
$$c_r(x)\frac{\mathrm{d}^r}{\mathrm{d}x^r} f(x,y) + \cdots + c_0(x) f(x,y) \; = \frac{\mathrm{d}}{\mathrm{d}y} g(x,y).$$

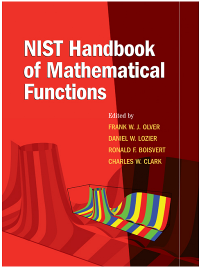
Integrating from a to b yields a differential equation for $F(x)$:

$$c_r(x)\frac{\mathrm{d}^r}{\mathrm{d}x^r} F(x) + \cdots + c_0(x) F(x) \; = g(x,b) - g(x,a)$$

Special Function Identities







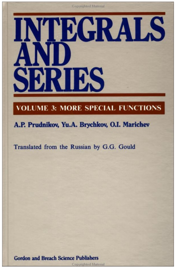


Table of Integrals by Gradshteyn and Ryzhik

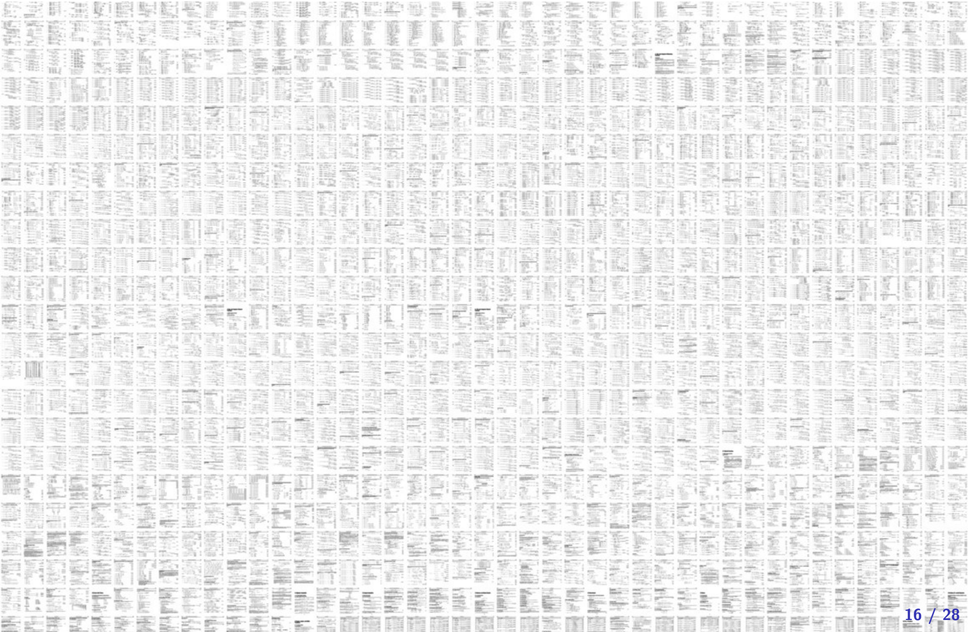


Table of Integrals by Gradshteyn and Ryzhik

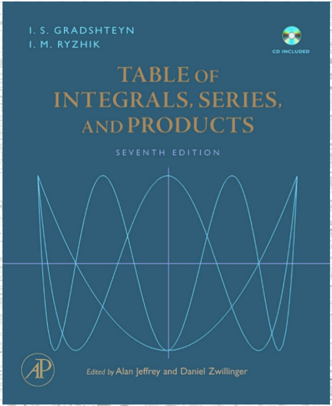


Table of Integrals by Gradshteyn and Ryzhik

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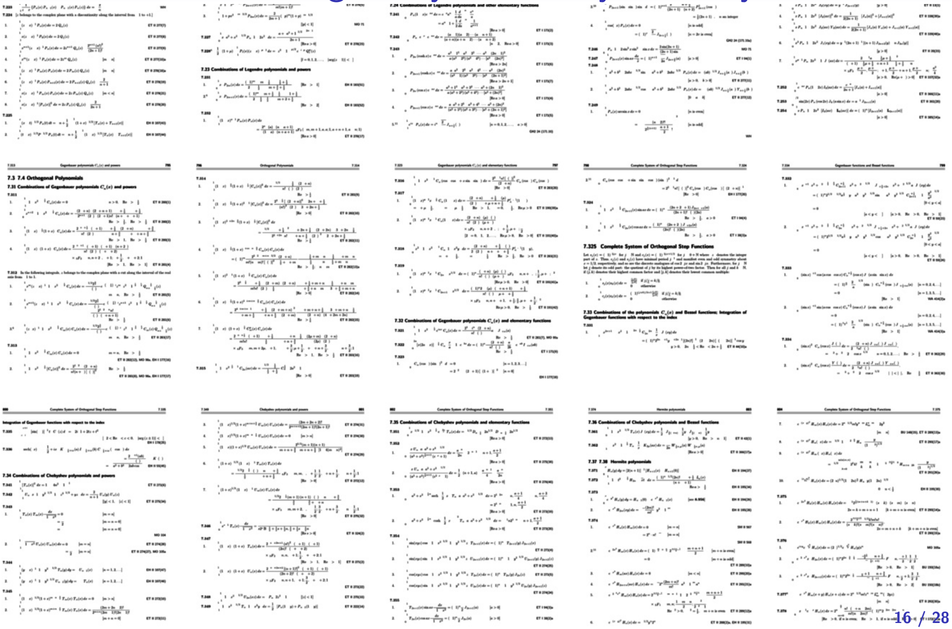


7.319

1. $$\int_0^1 (1-x)^{\mu-1} x^{\nu-1}\, C_{2n}^{\lambda}\left(\gamma x^{1/2}\right) dx = (-1)^n \frac{\Gamma(\lambda+n)\,\Gamma(\mu)\,\Gamma(\nu)}{n!\,\Gamma(\lambda)\,\Gamma(\mu+\nu)}\ {}_3F_2\left(-n, n+\lambda, \nu; \frac{1}{2}, \mu+\nu; \gamma^2\right)$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ ET II 191(41)a

2. $$\int_0^1 (1-x)^{\mu-1} x^{\nu-1}\, C_{2n+1}^{\lambda}\left(\gamma x^{1/2}\right) dx = \frac{(-1)^n 2\gamma\,\Gamma(\mu)\,\Gamma(\lambda+n+1)\,\Gamma\left(\nu+\frac{1}{2}\right)}{n!\,\Gamma(\lambda)\,\Gamma\left(\mu+\nu+\frac{1}{2}\right)} \times {}_3F_2\left(-n, n+\lambda+1, \nu+\tfrac{1}{2}; \tfrac{3}{2}, \mu+\nu+\frac{1}{2}; \gamma^2\right)$$
$\left[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right]$ ET II 191(42)

7.32 Combinations of Gegenbauer polynomials $C_n^{\nu}(x)$ and elementary functions

7.321 $$\int_{-1}^1 \left(1-x^2\right)^{\nu-\frac{1}{2}} e^{iax}\, C_n^{\nu}(x)\, dx = \frac{\pi 2^{1-\nu} i^n\, \Gamma(2\nu+n)}{n!\,\Gamma(\nu)} a^{-\nu} J_{\nu+n}(a)$$
$\left[\operatorname{Re}\nu > -\frac{1}{2}\right]$ ET II 281(7), MO 99a

7.322 $$\int_0^{2a} [x(2a-x)]^{\nu-\frac{1}{2}}\, C_n^{\nu}\left(\frac{x}{a}-1\right) e^{-bx}\, dx = (-1)^n \frac{\pi\,\Gamma(2\nu+n)}{n!\,\Gamma(\nu)} \left(\frac{a}{2b}\right)^{\nu} e^{-ab}\, I_{\nu+n}(ab)$$
$\left[\operatorname{Re}\nu > -\frac{1}{2}\right]$ ET I 171(9)

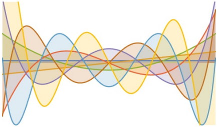
7.323

1. $$\int_0^{\pi} C_n^{\nu}(\cos\varphi)\,(\sin\varphi)^{2\nu}\, d\varphi = 0$$
$[n = 1, 2, 3, \ldots]$

Table of Integrals by Gradshteyn and Ryzhik

$$\int_{-1}^{1} \left(1-x^2\right)^{\nu-\frac{1}{2}} e^{iax}\, C_n^{\nu}(x)\, dx = \frac{\pi 2^{1-\nu} i^n\, \Gamma(2\nu+n)}{n!\, \Gamma(\nu)} a^{-\nu}\, J_{\nu+n}(a)$$

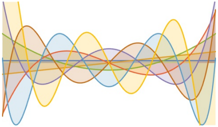
Table of Integrals by Gradshteyn and Ryzhik



Gegenbauer polynomials $C_n^{(\alpha)}(x)$

$$\int_{-1}^{1} \left(1-x^2\right)^{\nu-\frac{1}{2}} e^{iax}\, C_n^{\nu}(x)\, dx = \frac{\pi 2^{1-\nu} i^n\, \Gamma(2\nu+n)}{n!\,\Gamma(\nu)} a^{-\nu}\, J_{\nu+n}(a)$$

Table of Integrals by Gradshteyn and Ryzhik



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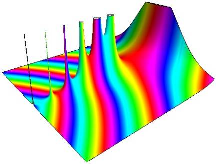
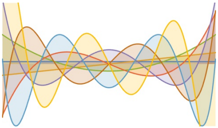


Gamma function $\Gamma(x)$

$$\int_{-1}^{1} \left(1-x^2\right)^{\nu-\frac{1}{2}} e^{iax}\, C_n^{\nu}(x)\, dx = \frac{\pi 2^{1-\nu} i^n\, \Gamma(2\nu+n)}{n!\,\Gamma(\nu)} a^{-\nu}\, J_{\nu+n}(a)$$

Table of Integrals by Gradshteyn and Ryzhik



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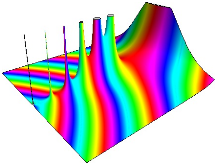
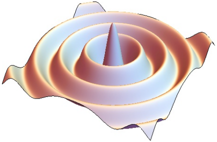


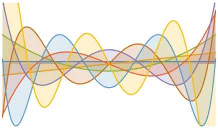
Gamma function $\Gamma(x)$



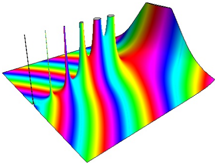
Bessel function $J_\nu(x)$

$$\int_{-1}^{1} \left(1 - x^2\right)^{\nu - \frac{1}{2}} e^{iax}\, C_n^{\nu}(x)\, dx = \frac{\pi 2^{1-\nu} i^n\, \Gamma(2\nu + n)}{n!\, \Gamma(\nu)} a^{-\nu}\, J_{\nu+n}(a)$$

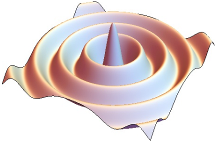
Table of Integrals by Gradshteyn and Ryzhik



Gegenbauer polynomials $C_n^{(\alpha)}(x)$



Gamma function $\Gamma(x)$

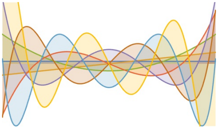


Bessel function $J_\nu(x)$

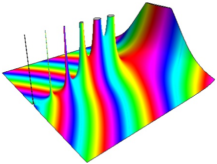
$$\int_{-1}^{1} \left(1-x^2\right)^{\nu-\frac{1}{2}} e^{iax}\, C_n^{\nu}(x)\, dx = \frac{\pi 2^{1-\nu} i^n\, \Gamma(2\nu+n)}{n!\,\Gamma(\nu)} a^{-\nu}\, J_{\nu+n}(a)$$

- A large portion of such identities can be proven via the holonomic systems approach.

Table of Integrals by Gradshteyn and Ryzhik



Gegenbauer polynomials $C_n^{(\alpha)}(x)$



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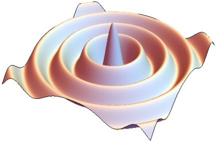

Bessel function $J_\nu(x)$

$$\int_{-1}^{1} \left(1-x^2\right)^{\nu-\frac{1}{2}} e^{iax}\, C_n^{\nu}(x)\, dx = \frac{\pi 2^{1-\nu} i^n\, \Gamma(2\nu+n)}{n!\, \Gamma(\nu)} a^{-\nu}\, J_{\nu+n}(a)$$

- A large portion of such identities can be proven via the holonomic systems approach.
- Algorithms are implemented in the HolonomicFunctions package.

Example: Holonomic system, satisfied by both sides of the identity:

$$
\begin{aligned}
&ia(n+2\nu)f_n'(a) + a(n+1)f_{n+1}(a) - in(n+2\nu)f_n(a) = 0,\\
&a(n+1)(n+2)f_{n+2}(a) - 2i(n+1)(n+\nu+1)(n+2\nu+1)f_{n+1}(a)\\
&\quad - a(n+2\nu)(n+2\nu+1)f_n(a) = 0.
\end{aligned}
$$

The HolonomicFunctions Package

Example: Holonomic system, satisfied by both sides of the identity:

$$ia(n+2\nu)f_n'(a) + a(n+1)f_{n+1}(a) - in(n+2\nu)f_n(a) = 0,$$
$$a(n+1)(n+2)f_{n+2}(a) - 2i(n+1)(n+\nu+1)(n+2\nu+1)f_{n+1}(a)$$
$$- a(n+2\nu)(n+2\nu+1)f_n(a) = 0.$$

```
In[42]:= Annihilator[Pi * 2 ^ (1 - ν) * I ^ n * Gamma[2 ν + n] / n ! / Gamma[ν] * a ^ (-ν) *
           BesselJ[ν + n, a], {Der[a], S[n]}] // Factor
Out[42]=
  {i a (n + 2 ν) D_a + a (1 + n) S_n - i n (n + 2 ν),
   a (1 + n) (2 + n) S_n^2 - 2 i (1 + n) (1 + n + ν) (1 + n + 2 ν) S_n - a (n + 2 ν) (1 + n + 2 ν)}

In[43]:= CreativeTelescoping[(1 - x ^ 2) ^ (ν - 1 / 2) * Exp[I * a * x] * GegenbauerC[n, ν, x],
          Der[x], {Der[a], S[n]}] // Factor
Out[43]=
  {{a (n + 2 ν) D_a - i a (1 + n) S_n - n (n + 2 ν),
    a (1 + n) (2 + n) S_n^2 - 2 i (1 + n) (1 + n + ν) (1 + n + 2 ν) S_n - a (n + 2 ν) (1 + n + 2 ν)},
   {(1 + n) S_n - x (n + 2 ν), 2 i (1 + n) x (1 + n + ν) S_n - 2 i (1 + n + ν) (n + 2 ν)}}
```

Special Function Identities

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{k+n}{k}^2 = \sum_{k=0}^{n} \binom{n}{k} \binom{k+n}{k} \sum_{j=0}^{k} \binom{k}{j}^3 \qquad (1)$$

Special Function Identities

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$$\int_0^\infty \frac{1}{(x^4 + 2ax^2 + 1)^{m+1}} \, \mathrm{d}x = \frac{\pi P_m^{\left(m+\frac{1}{2}, -m-\frac{1}{2}\right)}(a)}{2^{m+\frac{3}{2}} (a+1)^{m+\frac{1}{2}}} \qquad (2)$$

Special Function Identities

$$\sum_{k=0}^{n}\binom{n}{k}^2\binom{k+n}{k}^2=\sum_{k=0}^{n}\binom{n}{k}\binom{k+n}{k}\sum_{j=0}^{k}\binom{k}{j}^3 \qquad (1)$$

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$$e^{-x}x^{a/2}n!\,L_n^a(x)=\int_0^\infty e^{-t}t^{\frac{a}{2}+n}J_a\left(2\sqrt{tx}\right)\mathrm{d}t \qquad (3)$$

Special Function Identities

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{k+n}{k}^2 = \sum_{k=0}^{n} \binom{n}{k} \binom{k+n}{k} \sum_{j=0}^{k} \binom{k}{j}^3 \qquad (1)$$

$$\int_0^\infty \frac{1}{(x^4+2ax^2+1)^{m+1}}\,\mathrm{d}x = \frac{\pi P_m^{\left(m+\frac{1}{2},-m-\frac{1}{2}\right)}(a)}{2^{m+\frac{3}{2}}(a+1)^{m+\frac{1}{2}}} \qquad (2)$$

$$e^{-x}x^{a/2}n!\,L_n^a(x) = \int_0^\infty e^{-t}t^{\frac{a}{2}+n}J_a\left(2\sqrt{tx}\right)\mathrm{d}t \qquad (3)$$

$$\int_{-\infty}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{H_m(x)H_n(x)r^m s^n e^{-x^2}}{m!\,n!}\,\mathrm{d}x = \sqrt{\pi}e^{2rs} \qquad (4)$$

Special Function Identities

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{k+n}{k}^2 = \sum_{k=0}^{n} \binom{n}{k} \binom{k+n}{k} \sum_{j=0}^{k} \binom{k}{j}^3 \tag{1}$$

$$\int_0^\infty \frac{1}{(x^4 + 2ax^2 + 1)^{m+1}} \, \mathrm{d}x = \frac{\pi P_m^{\left(m+\frac{1}{2}, -m-\frac{1}{2}\right)}(a)}{2^{m+\frac{3}{2}} (a+1)^{m+\frac{1}{2}}} \tag{2}$$

$$e^{-x} x^{a/2} n! \, L_n^a(x) = \int_0^\infty e^{-t} t^{\frac{a}{2}+n} J_a\left(2\sqrt{tx}\right) \mathrm{d}t \tag{3}$$

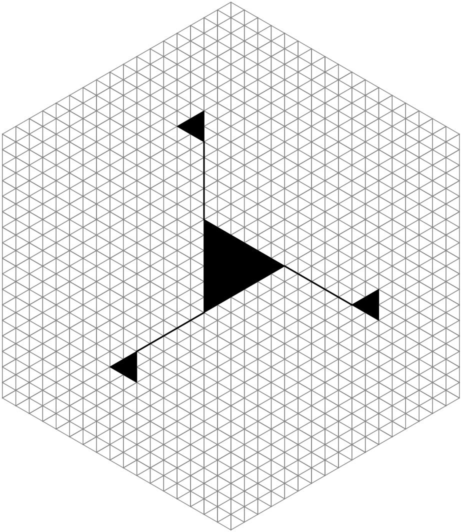
$$\int_{-\infty}^\infty \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{H_m(x) H_n(x) r^m s^n e^{-x^2}}{m! \, n!} \, \mathrm{d}x = \sqrt{\pi} e^{2rs} \tag{4}$$

$$\int_{-1}^1 \left(1 - x^2\right)^{\nu - \frac{1}{2}} e^{iax} C_n^{(\nu)}(x) \, \mathrm{d}x = \frac{\pi i^n \Gamma(n + 2\nu) J_{n+\nu}(a)}{2^{\nu-1} a^\nu \, n! \, \Gamma(\nu)} \tag{5}$$

Application

Determinants Count!

(joint work with
Hao Du,
Christian Krattenthaler,
Michael Schlosser,
Aek Thanatipanonda,
Elaine Wong)



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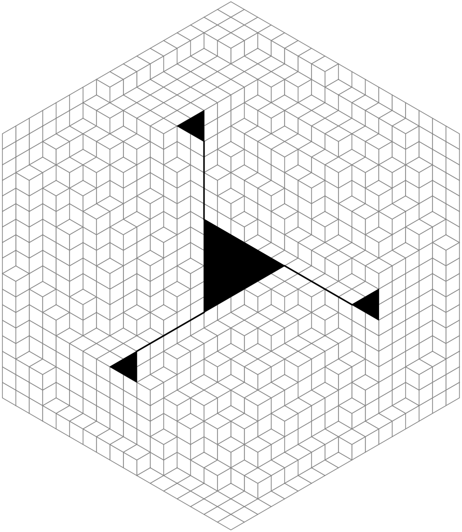

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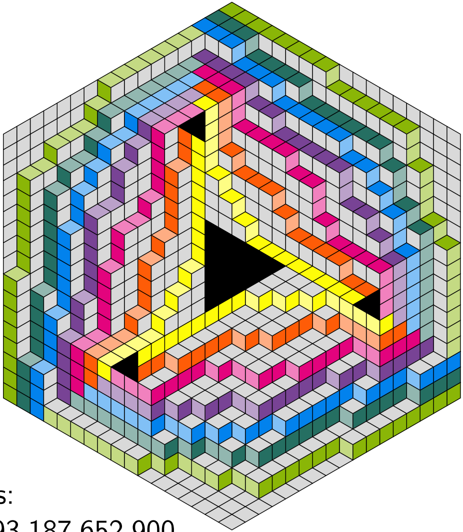

Number of rhombus tilings:
19,180,227,670,614,654,793,187,652,900

Symbolic Determinants via Holonomic Ansatz

$$\det_{1\leqslant i,j\leqslant n} \frac{1}{i+j-1} = \frac{1}{(2n-1)!}\prod_{k=1}^{n-1}\frac{(k!)^2}{(k+1)_{n-1}}$$

Symbolic Determinants via Holonomic Ansatz

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$$\det_{0\leqslant i,j\leqslant n-1} \binom{2i+2a}{j+b} = 2^{n(n-1)/2}\prod_{k=0}^{n-1}\frac{(2k+2a)!k!}{(k+b)!(2k+2a-b)!}$$

Symbolic Determinants via Holonomic Ansatz

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$$\det_{0\leqslant i,j\leqslant n-1} \sum_k \binom{i}{k}\binom{j}{k}2^k = 2^{n(n-1)/2}$$

Symbolic Determinants via Holonomic Ansatz

$$\det_{1\leqslant i,j\leqslant n} \frac{1}{i+j-1} = \frac{1}{(2n-1)!}\prod_{k=1}^{n-1}\frac{(k!)^2}{(k+1)_{n-1}}$$

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$$\det_{0\leqslant i,j\leqslant n-1} \sum_k \binom{i}{k}\binom{j}{k}2^k = 2^{n(n-1)/2}$$

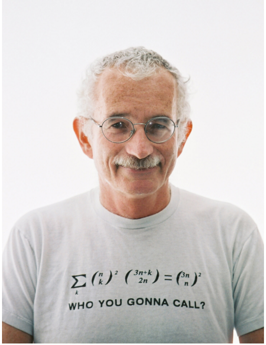
$$\det_{1\leqslant i,j\leqslant 2m+1}\left[\binom{\mu+i+j+2r}{j+2r-2}-\delta_{i,j+2r}\right]$$

$$= \frac{(-1)^{m-r+1}(\mu+3)(m+r+1)_{m-r}}{2^{2m-2r+1}\left(\frac{\mu}{2}+r+\frac{3}{2}\right)_{m-r+1}}\cdot\prod_{i=1}^{2m}\frac{(\mu+i+3)_{2r}}{(i)_{2r}}$$

$$\times\prod_{i=1}^{m-r}\frac{(\mu+2i+6r+3)_i^2\left(\frac{\mu}{2}+2i+3r+2\right)_{i-1}^2}{(i)_i^2\left(\frac{\mu}{2}+i+3r+2\right)_{i-1}^2}.$$

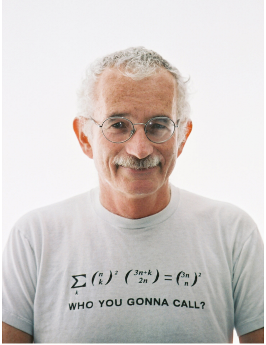
The Holonomic Ansatz

The Holonomic Ansatz



The Holonomic Ansatz

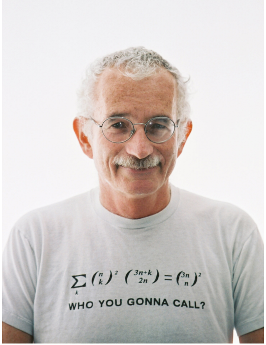
Problem: Prove a determinantal identity of the form $\det\limits_{1\leqslant i,j\leqslant n}(a_{i,j}) = b_n$



The Holonomic Ansatz

Problem: Prove a determinantal identity of the form $\det\limits_{1 \leqslant i,j \leqslant n} (a_{i,j}) = b_n$, where

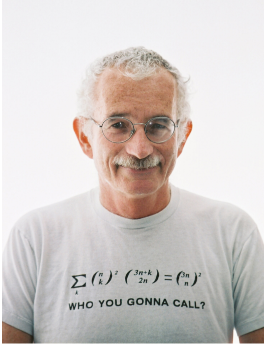
- $a_{i,j}$ is a holonomic sequence



The Holonomic Ansatz

Problem: Prove a determinantal identity of the form $\det\limits_{1\leqslant i,j\leqslant n}(a_{i,j}) = b_n$, where

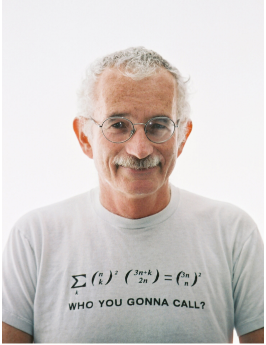
- $a_{i,j}$ is a holonomic sequence
- that does not depend on n



The Holonomic Ansatz

Problem: Prove a determinantal identity of the form $\det_{1 \leqslant i,j \leqslant n} (a_{i,j}) = b_n$, where

- $a_{i,j}$ is a holonomic sequence
- that does not depend on n, and
- b_n is a closed form ($b_n \neq 0$ for all n).



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- that does not depend on n, and
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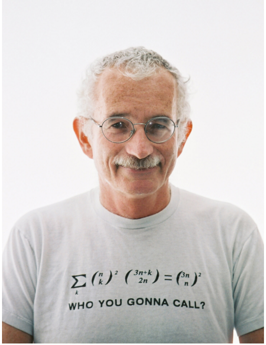
$$\mathcal{A}_n = \left(\begin{array}{ccc|c} & \mathcal{A}_{n-1} & & \\ \hline a_{n,1} & \cdots & a_{n,n-1} & a_{n,n} \end{array}\right)$$



Laplace expansion:

$$\det(\mathcal{A}_n) = a_{n,1}\mathrm{Cof}_{n,1} + \cdots + a_{n,n-1}\mathrm{Cof}_{n,n-1} + a_{n,n}\det(\mathcal{A}_{n-1})$$

The Holonomic Ansatz

Problem: Prove a determinantal identity of the form $\det\limits_{1\leqslant i,j\leqslant n}(a_{i,j}) = b_n$, where

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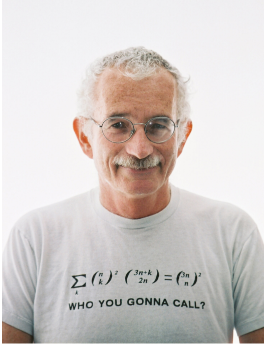

Laplace expansion:

$$\frac{\det(\mathcal{A}_n)}{\det(\mathcal{A}_{n-1})} = a_{n,1}\frac{\mathrm{Cof}_{n,1}}{\det(\mathcal{A}_{n-1})} + \cdots + a_{n,n-1}\frac{\mathrm{Cof}_{n,n-1}}{\det(\mathcal{A}_{n-1})} + a_{n,n}$$

The Holonomic Ansatz

Problem: Prove a determinantal identity of the form $\det_{1 \leqslant i,j \leqslant n}(a_{i,j}) = b_n$, where

- $a_{i,j}$ is a holonomic sequence
- that does not depend on n, and
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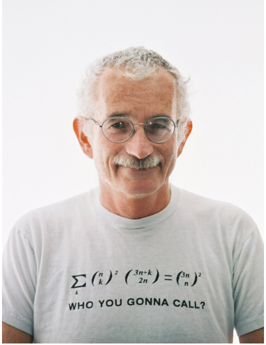

Laplace expansion:

$$\frac{\det(\mathcal{A}_n)}{\det(\mathcal{A}_{n-1})} = a_{n,1}c_{n,1} + \cdots + a_{n,n-1}c_{n,n-1} + a_{n,n}c_{n,n}$$

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Problem: Prove a determinantal identity of the form $\det\limits_{1\leqslant i,j\leqslant n}(a_{i,j}) = b_n$, where

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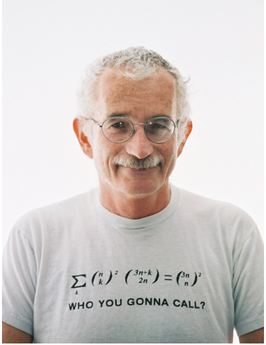

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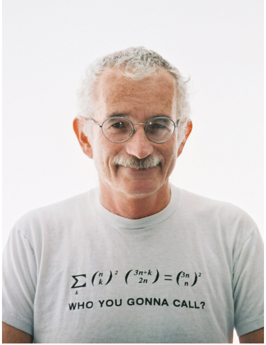

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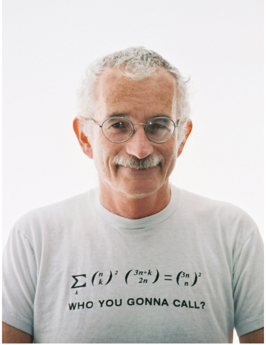

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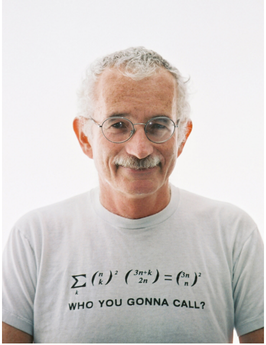

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$$0 = \sum_{j=1}^{n} a_{i,j} c_{n,j} \quad (1 \leqslant i < n), \quad c_{n,n} = 1$$

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Conjecture (Di Francesco's determinant for 20V configurations):

$$\det_{0 \leqslant i,j < n} \left(2^i \binom{i+2j+1}{2j+1} - \binom{i-1}{2j+1} \right) = 2 \prod_{i=1}^{n} \frac{2^{i-1}\,(4i-2)!}{(n+2i-1)!}$$

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Inverse Inequalities

(join work with M. Neumüller and S. Radu)

We consider inequalities of the form

$$||v_n||_{X(\Omega)} \leqslant c(h,n)\,||v_n||_{Y(\Omega)} \qquad \text{for all } v_n \in V_n$$

- $\Omega \subset \mathbb{R}^d, d \in \mathbb{N}$
- V: some infinite-dimensional space of functions defined on Ω
- $||\cdot||_{X(\Omega)}$, $||\cdot||_{Y(\Omega)}$: norms that are used in numerical methods
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Transform the problem to a reference element $\hat{\Omega}$:

$$\hat{c}(n) = \sup_{v_n \in \hat{V}_n} \frac{||v_n||_{X(\hat{\Omega})}}{||v_n||_{Y(\hat{\Omega})}} = \sqrt{\sup_{v_n \in \hat{V}_n} \frac{(v_n, v_n)_{X(\hat{\Omega})}}{(v_n, v_n)_{Y(\hat{\Omega})}}}$$

Inverse Inequalities

Here we consider the reference domain $\hat{\Omega} = (-1,1)^2$ with

$$(u,v)_{X(\hat{\Omega})} = \int_{\hat{\Omega}} \partial_x u(x,y) \partial_x v(x,y) \, \mathrm{d}x \, \mathrm{d}y,$$
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for $u, v \in \hat{V}_n$, where $\hat{V}_n$ is the space of polynomials of degree less than n, i.e.

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The desired “constant” $\hat{c}(n)$ can be found as the largest λ_n solving the generalized eigenvalue problem

$$B_n \vec{x}_n = \lambda_n A_n \vec{x}_n,$$

where A_n and B_n are certain $n \times n$ matrices.

The Matrices

$$a_{i,j} := \frac{1-(-1)^{i+j-1}}{i+j-1}, \quad b_{i,j} := (i-1)(j-1)\frac{1-(-1)^{i+j-3}}{i+j-3}$$

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Hence we get: $\det(B_n - \lambda A_n) = 2^n \det\left(A^{(1)}_{\lceil n/2 \rceil}\right) \cdot \det\left(A^{(0)}_{\lfloor n/2 \rfloor}\right)$.

The Determinant

By a variation of the holonomic ansatz we prove:

Theorem.

$$\det A_n^{(0)} = \underbrace{\left(-\tfrac{1}{2}\right)^n \prod_{i=1}^{n} \frac{\big((i-1)!\big)^2}{\left(i+\frac{1}{2}\right)_n}}_{\text{“hyperholonomic” part}} \underbrace{\sum_{j=0}^{n} (-4)^{j-n} \frac{(2n-2j+1)_{2n}}{(2j)!} \lambda^j}_{\text{holonomic part}},$$

$$\det A_n^{(1)} = \underbrace{\left(-\tfrac{1}{2}\right)^n \prod_{i=1}^{n} \frac{\big((i-1)!\big)^2}{\left(i-1+\frac{1}{2}\right)_n}}_{\text{“hyperholonomic” part}} \underbrace{\sum_{j=0}^{n-1} \frac{(2n-2j-1)_{2n-1}}{(-4)^{n-j-1}(2j+1)!} \lambda^j}_{\text{holonomic part}}.$$

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We use this explicit evaluation to estimate the largest eigenvalue.

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$$b_2(n) := m_1(n)\left(\frac{1}{3} + \left(r_1(n) + \sqrt{r_2(n)}\right)^{1/3} + \left(r_1(n) - \sqrt{r_2(n)}\right)^{1/3}\right),$$

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where m_1, r_1, and r_2 are given by

$$m_1(n) := \frac{n(n-1)(n+1)(n+2)}{8},$$

$$r_1(n) := \frac{2(n^8 + 4n^7 + 8n^6 + \cdots - 4733n^2 - 5130n + 16200)}{135n^2(n-1)^2(n+1)^2(n+2)^2},$$

$$r_2(n) := \frac{(n-2)(n-3)(n+4)(n+3)(7n^{12} + 42n^{11} + \dots)}{145800n^4(n-1)^4(n+1)^4(n+2)^4}.$$

Further Reading

- **Survey article:** *Creative telescoping for holonomic functions.* DOI: 10.1007/978-3-7091-1616-6_7, arXiv:1307.4554.
- **PhD thesis:** *Advanced applications of the holonomic systems approach* (RISC, Johannes Kepler University, Linz, Austria).
- **Software package:** *HolonomicFunctions (user's guide)*. https://risc.jku.at/sw/holonomicfunctions/
- **Electromagnetic waves application:** *Method, device and computer program product for determining an electromagnetic near field of a field excitation source for an electrical system* (with J. Schöberl and P. Paule), Patents EP2378444 and US8868382.
- **Combinatorial determinants:** *Binomial determinants for tiling problems yield to the holonomic ansatz* (with H. Du, T. Thanatipanonda, E. Wong), DOI: 10.1016/j.ejc.2021.103437.
- **20V determinants:** *Determinant evaluations inspired by Di Francesco's determinant for twenty-vertex configurations* (with C. Krattenthaler and M. Schlosser), arXiv:2401.08481.