Computer Algebra with D-Finite Functions

Christoph Koutschan

Johann Radon Institute for Computational and Applied Mathematics (RICAM) Austrian Academy of Sciences

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Airy function

arise in mathematical analysis and in real-world phenomena



Airy function





Bessel function

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Airy function



Bessel function





Coulomb function

- > arise in mathematical analysis and in real-world phenomena
- are solutions to certain differential equations



Airy function



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Coulomb function

- arise in mathematical analysis and in real-world phenomena
- are solutions to certain differential equations
- ▶ cannot be expressed in terms of the usual elementary functions $(\sqrt{-}, \exp, \log, \sin, \cos, \dots)$



Airy function



Bessel function





Coulomb function

Definition: A function f(x) is called **D-finite** if it satisfies a linear ordinary differential equation with polynomial coefficients:

$$p_r(x)f^{(r)}(x) + \dots + p_1(x)f'(x) + p_0(x)f(x) = 0,$$

 $p_0, \ldots, p_r \in \mathbb{K}[x]$ (not all zero).

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- a_n is P-recursive if and only if $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is D-finite.
- Equivalently, such functions/sequences are called holonomic.

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Bessel differential equation:

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Many special functions can be characterized as solutions to systems of linear differential equations and recurrences, and in fact are D-finite (holonomic).

Example: The Legendre polynomials are orthogonal polynomials w.r.t. the L^2 inner product $\int_{-1}^{1} f(x)g(x) dx$, and satisfy the ODE

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 $(x^{2} - 1)P_{n}^{(4)}(x) + 6xP_{n}^{(3)}(x) - (n - 2)(n + 3)P_{n}''(x) = 0$

$$P_n^{(4)}(x) = -\frac{6x}{x^2 - 1} P_n^{(3)}(x) + \frac{(n-2)(n+3)}{x^2 - 1} P_n''(x)$$



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 $\longrightarrow P_n(x)$ is **D-finite** w.r.t. x.

$$P_n^{(4)}(x) = -\frac{8x(n^2x^2 - n^2 + nx^2 - n + 3x^2 + 3)}{(x^2 - 1)^3} P'_n(x) + \frac{n(n+1)(n^2x^2 - n^2 + nx^2 - n + 18x^2 + 6)}{(x^2 - 1)^3} P_n(x)$$



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$$P_0(x) = 1, \quad P_1(x) = x$$

 $nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x).$



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 $\longrightarrow P_n(x)$ is **D-finite** w.r.t. n and x (of rank 2). Consider the set $\{P_{n+j}^{(i)}(x): i, j \ge 0\}$.



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Theorem (Closure Properties): If $f_n(x)$ and $g_n(x)$ are D-finite functions, then also the following expressions are D-finite:

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- $\frac{\mathrm{d}}{\mathrm{d}x}f_n(x)$
- $f_{an+b}(x)$, where $a, b \in \mathbb{Z}$
- $f_n(h(x))$, where h(x) is an algebraic function

Many Functions are D-finite

ArcCsc, KelvinBei, HypergeometricPFQ, ExpIntegralE, ArcTanh, HankelH2, AngerJ, JacobiP, ChebyshevT, AiryBi, AiryAi, Sinc, Multinomial, CatalanNumber, QBinomial, CosIntegral, ArcSech, SphericalHankelH2, HermiteH, ExpIntegralEi, Beta, AiryBiPrime, SphericalBesselJ, Binomial, ParabolicCylinderD, Erfc, EllipticK, Fibonacci, QFactorial, Cos, Hypergeometric2F1, Erf, KelvinKer, HypergeometricPFQRegularized, Log, Factorial, BesselY, Cosh, CoshIntegral, ArcTan, ArcCoth, LegendreP, LaguerreL, EllipticE, SinhIntegral, Sinh, BetaRegularized, SphericalHankelH1, ArcSin, EllipticThetaPrime, Root, LucasL, AppellF1, FresnelC, LegendreQ, ChebyshevU, GammaRegularized, Erfi, HarmonicNumber, Bessell, KelvinKei, ArithmeticGeometricMean, Exp, ArcCot, EllipticTheta, Hypergeometric0F1, EllipticPi, GegenbauerC, ArcCos, WeberE, FresnelS, EllipticF, ArcCosh, Subfactorial, QPochhammer, Gamma, StruveH, WhittakerM, ArcCsch, Hypergeometric1F1, SinIntegral, Bessel J, StruveL, ArcSec, Factorial2, KelvinBer, Bessel K, ArcSinh, HankelH1, Sqrt, PolyGamma, HypergeometricU, AiryAiPrime, Sin,

$$\operatorname{erf}(\sqrt{x+1})^2 + \exp(\sqrt{x+1})^2$$

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$\label{eq:computation} \begin{array}{l} \mbox{The Symbolic Computation Viewpoint} \\ \mbox{A D-finite function a priori is an infinite object (e.g., $\mathbb{R}^2 \to \mathbb{R}^2$).} \end{array}$

A D-finite function a priori is an infinite object (e.g., $\mathbb{R}^2 \to \mathbb{R}^2$). But it can be represented (exactly!) by a finite amount of data:

- system of functional equations
- finitely many initial values

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The **holonomic systems approach** (Zeilberger 1990) is a versatile toolbox for solving many different kinds of mathematical problems:

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- calculate integrals and summation formulas
- prove special function identities
- computations in q-calculus (e.g., quantum knot invariants)
- fast numerical evaluation of mathematical functions
- number theory (e.g., irrationality proofs)
- evaluate symbolic determinants (e.g., in combinatorics)

Application

Finite Elements



(joint work with Joachim Schöberl and Peter Paule)

Problem Setting

Simulate the propagation of electromagnetic waves according to

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \operatorname{curl} E, \quad \frac{\mathrm{d}E}{\mathrm{d}t} = -\operatorname{curl} H \tag{Maxwell}$$

where H and E are the magnetic and the electric field respectively.

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where H and E are the magnetic and the electric field respectively.

Define basis functions (2D case):

$$\varphi_{i,j}(x,y) := (1-x)^i P_j^{(2i+1,0)}(2x-1) P_i\left(\frac{2y}{1-x}-1\right)$$

using Legendre and Jacobi polynomials.

Problem Setting

Simulate the propagation of electromagnetic waves according to

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \operatorname{curl} E, \quad \frac{\mathrm{d}E}{\mathrm{d}t} = -\operatorname{curl} H \tag{Maxwell}$$

where H and E are the magnetic and the electric field respectively.

Define basis functions (2D case):

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using Legendre and Jacobi polynomials.

Problem: Represent the partial derivatives of $\varphi_{i,j}(x, y)$ in the basis (i.e., as linear combinations of shifts of the $\varphi_{i,j}(x, y)$ itself).

Solution

Ansatz: One needs a relation of the form

$$\sum_{(k,l)\in A} a_{k,l}(i,j) \frac{\mathrm{d}}{\mathrm{d}x} \varphi_{i+k,j+l}(x,y) = \sum_{(m,n)\in B} b_{m,n}(i,j) \varphi_{i+m,j+n}(x,y),$$

that is free of x and y (and similarly for $\frac{d}{dy}$).

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that is free of x and y (and similarly for $\frac{d}{dy}$).

Result: Computer algebra methods (D-finite closure properties, Gröbner bases), deliver the relation

$$\begin{split} &(2i+j+3)(2i+2j+7)\frac{\mathrm{d}}{\mathrm{d}x}\varphi_{i,j+1}(x,y)+\\ &2(2i+1)(i+j+3)\frac{\mathrm{d}}{\mathrm{d}x}\varphi_{i,j+2}(x,y)-\\ &(j+3)(2i+2j+5)\frac{\mathrm{d}}{\mathrm{d}x}\varphi_{i,j+3}(x,y)+\\ &(j+1)(2i+2j+7)\frac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+1,j}(x,y)-\\ &2(2i+3)(i+j+3)\frac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+1,j+1}(x,y)-\\ &(2i+j+5)(2i+2j+5)\frac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+1,j+2}(x,y)+\\ &2(i+j+3)(2i+2j+5)(2i+2j+7)\varphi_{i,j+2}(x,y)+\\ &2(i+j+3)(2i+2j+5)(2i+2j+7)\varphi_{i+1,j+1}(x,y)=0. \end{split}$$

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Creative Telescoping



What is Creative Telescoping?

Creative telescoping is a method

to deal with parametrized symbolic sums and integrals
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- to deal with parametrized symbolic sums and integrals
- that yields differential/recurrence equations for them

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Example:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \qquad \text{Bad: no parameter!}$$

 $\sum_{k=1}^{\infty} \frac{1}{k(k+n)}$

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Example:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$
 Bad: no parameter!
$$\sum_{k=1}^{\infty} \frac{1}{k(k+n)} = \frac{\gamma + \psi(n)}{n}$$

n

Creative telescoping is a method

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$$\sum_{k=1}^{\infty}rac{1}{k^2}=rac{\pi^2}{6}$$
 Bad: no parameter!

$$\underbrace{\sum_{k=1}^{\infty} \frac{1}{k(k+n)}}_{=:f_n} \rightsquigarrow (n+2)^2 f_{n+2} = (n+1)(2n+3)f_{n+1} - n(n+1)f_n$$

Method for doing integrals and sums (aka Feynman's differentiating under the integral sign)

Consider the following summation problem: $F(n) := \sum_{k=a}^{b} f(n,k)$

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Creative Telescoping: write

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Creative Telescoping: write

 $c_r(n)f(n+r,k) + \dots + c_0(n)f(n,k) = g(n,k+1) - g(n,k).$

Summing from a to b yields a recurrence for F(n):

$$c_r(n)F(n+r) + \dots + c_0(n)F(n) = g(n,b+1) - g(n,a).$$

Method for doing integrals and sums (aka Feynman's differentiating under the integral sign)

Consider the following integration problem: $F(x) := \int_a^b f(x, y) \, dy$

Telescoping: write
$$f(x, y) = \frac{d}{dy}g(x, y)$$
.
Then $F(n) = \int_{a}^{b} \left(\frac{d}{dy}g(x, y)\right) dy = g(x, b) - g(x, a)$.

Creative Telescoping: write

$$c_r(x)\frac{\mathrm{d}^r}{\mathrm{d}x^r}f(x,y) + \dots + c_0(x)f(x,y) = \frac{\mathrm{d}}{\mathrm{d}y}g(x,y).$$

Integrating from a to b yields a differential equation for F(x):

$$c_r(x)\frac{\mathrm{d}^r}{\mathrm{d}x^r}F(x) + \dots + c_0(x)F(x) = g(x,b) - g(x,a)$$

Application

Special Function Identities



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- 183 Organizate polynomials ("____) and powers TH
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7.31 Combinations of Gegenhauer polynomials C_(r) and powers

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- $\begin{bmatrix} 1 & e^{2-\frac{1}{2}} C_{\mu}(m) & m & \pi \sin (m) & \frac{1}{2} \frac{1}{2} C_{\mu}(m) + C_{\mu}(m) \\ \frac{1}{2} \frac{1}{2} C_{\mu}(m) & \frac{1}{2} C_{\mu}(m) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ (2.10)
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- $T_{2n-1}(x) = \frac{dx}{1-x} = (-1)^n \frac{1}{2} J_{2n-1}(x) \quad |x>0|$
- $T_{2n}(x) = \max \frac{dx}{1 x} = (-1)^n \frac{1}{2} A_{2n}(x) \qquad (x > 0)$ 111003

 $X^{2} = P_{2+1}(m, m)(m, d = (1)^{n+1} \frac{1}{(2n+1)(1+1)} P_{2+1}(m))$ and stiff, bilds = 0 -(1/ $\sum_{k=0}^{r}$) (r-bringed -Task P. I. San's and marker - Marker I. -.... Paul(1)mar == (1)*** = /a=j(4) (x>4) $a^2 + b^2$ (below $^{1/2}$ else $a^2 + b^2$ (below $^{1/2}$ $P_{1/2}(a) = (ad) \stackrel{1/2}{=} \mathcal{S}_{n+2}(a) \cdot \mathcal{S}_{n+2}(b)$ what has ""on what has "" Printer of "" Prints (Frid) 1. Publishedrant (n in second in all joined

10 Complete Systems of Delengend Steep Functions

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7.325 Complete System of Orthogonal Step Functions

 $s_{i}(c) = (-1)^{-2c}$ for $j = N \mod s_{i}(c) = (-1)^{2(n-1)/2}$ for j = k + N where cof s. Then, $s_{i}(c)$ and $s_{i}(c)$ have minimal partial J^{-1} and manifact oven and (2)-may which $s_{i}(c)$ and $s_{i}(c)$ have backward partial $s_{i}(c)$ and (2) as J. If d shows the last of the partial of J by its balance power of two lasts. As J^{-1} shows the last have backward J > J. Substant their lawest con-

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7.33 Combinations of the polynomials $C_{\alpha}(s)$ and Bound functions, Integral Gaussianser functions with respect to the index

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- "P. 1 D" [April" dr 1 [Kell" + [Americ]" ----- $\left[{}^{2} s \, P_{n} + 2 s \, h^{2} \, A_{0}(s) \, Y_{0}(s) \, ds = \frac{1}{||S|s+1|} \left[A_{n}(s) \, Y_{n}(s) + A_{n-1}(s) \, Y_{n-1}(s) \right] \right]$
- Are the Area and t

- $T_{\rm effect} = P_{\rm eff} \quad tr(A_{\rm effect})dx = \frac{1}{|h|+1} (|I_{\rm effect}| + I_{\rm event}|x|)$ -
- 1.958 (3x) P₂ (m(3x) J₂ (m(3x)) dx = x ² J₂₀₁₀(x) ----

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- $(a,c)^2 C_{a}(a,c) \frac{J(1)}{2} dc = \frac{(2+a)J(c)(J(c))}{2 \frac{M}{2}},$ = $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{M}{2} + \frac{1}{2} + \frac{M}{2} + \frac{1}{2} + \frac{$ ind Count V() de . (0, ++) (ad) Vad)
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7.32 Combinations of Gegenbauer polynomials $C_n^{ u}(x)$ and elementary functions

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1.
$$\int_{0}^{\pi} C_{n}^{\nu} (\cos \varphi) (\sin \varphi)^{2\nu} d\varphi = 0 \qquad [n = 1, 2, 3, ...] \qquad 16 / 28$$

$$\int_{-1}^{1} \left(1 - x^2\right)^{\nu - \frac{1}{2}} e^{iax} C_n^{\nu}(x) \, dx = \frac{\pi 2^{1 - \nu} i^n \, \Gamma(2\nu + n)}{n! \, \Gamma(\nu)} a^{-\nu} \, J_{\nu + n}(a)$$











 A large portion of such identities can be proven via the holonomic systems approach.



- A large portion of such identities can be proven via the holonomic systems approach.
- Algorithms are implemented in the HolonomicFunctions package.

The HolonomicFunctions Package

Example: Holonomic system, satisfied by both sides of the identity:

$$ia(n+2\nu)f'_n(a) + a(n+1)f_{n+1}(a) - in(n+2\nu)f_n(a) = 0,$$

$$a(n+1)(n+2)f_{n+2}(a) - 2i(n+1)(n+\nu+1)(n+2\nu+1)f_{n+1}(a) - a(n+2\nu)(n+2\nu+1)f_n(a) = 0.$$

The HolonomicFunctions Package

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$$\label{eq:linear} \begin{split} & \ln[42]:= Annihilator [Pi * 2^{(1-v)} * I^n * Gamma [2 v + n] / n! / Gamma [v] * a^{(-v)} * \\ & BesselJ[v + n, a], \{ Der[a], S[n] \}] // Factor \end{split}$$

Out[42]=

$$\begin{array}{l} \texttt{i} \texttt{ a } (\texttt{n}+2 \ \texttt{v}) \ \texttt{D}_{\texttt{a}}+\texttt{a} \ (\texttt{1}+\texttt{n}) \ \texttt{S}_{\texttt{n}}-\texttt{i} \ \texttt{n} \ (\texttt{n}+2 \ \texttt{v}) \ \texttt{,} \\ \texttt{a} \ (\texttt{1}+\texttt{n}) \ (\texttt{2}+\texttt{n}) \ \texttt{S}_{\texttt{n}}^2-\texttt{2} \ \texttt{i} \ (\texttt{1}+\texttt{n}) \ (\texttt{1}+\texttt{n}+\texttt{v}) \ (\texttt{1}+\texttt{n}+2 \ \texttt{v}) \ \texttt{S}_{\texttt{n}}-\texttt{a} \ (\texttt{n}+2 \ \texttt{v}) \ (\texttt{1}+\texttt{n}+2 \ \texttt{v}) \end{array} \right\}$$

In[43]:= CreativeTelescoping[(1 - x^2) ^ (v - 1 / 2) * Exp[I * a * x] * GegenbauerC[n, v, x], Der[x], {Der[a], S[n]}] // Factor

Out[43]=

$$\begin{split} & \left\{ \left\{ a \; \left(n+2 \; \nu \right) \; D_{a} - i \; a \; \left(1+n \right) \; S_{n} - n \; \left(n+2 \; \nu \right) \; , \right. \\ & \left. a \; \left(1+n \right) \; \left(2+n \right) \; S_{n}^{2} - 2 \; i \; \left(1+n \right) \; \left(1+n+\nu \right) \; \left(1+n+2 \; \nu \right) \; S_{n} - a \; \left(n+2 \; \nu \right) \; \left(1+n+2 \; \nu \right) \right\} \\ & \left\{ \; \left(1+n \right) \; S_{n} - x \; \left(n+2 \; \nu \right) \; , \; 2 \; i \; \left(1+n \right) \; x \; \left(1+n+\nu \right) \; S_{n} - 2 \; i \; \left(1+n+\nu \right) \; \left(n+2 \; \nu \right) \right\} \right\} \end{split}$$

$$\sum_{k=0}^{n} {\binom{n}{k}}^{2} {\binom{k+n}{k}}^{2} = \sum_{k=0}^{n} {\binom{n}{k}} {\binom{k+n}{k}} \sum_{j=0}^{k} {\binom{k}{j}}^{3}$$
(1)

$$\sum_{k=0}^{n} {\binom{n}{k}}^{2} {\binom{k+n}{k}}^{2} = \sum_{k=0}^{n} {\binom{n}{k}} {\binom{k+n}{k}} \sum_{j=0}^{k} {\binom{k}{j}}^{3}$$
(1)

$$\int_0^\infty \frac{1}{\left(x^4 + 2ax^2 + 1\right)^{m+1}} \, \mathrm{d}x = \frac{\pi P_m^{\left(m + \frac{1}{2}, -m - \frac{1}{2}\right)}(a)}{2^{m + \frac{3}{2}}(a+1)^{m + \frac{1}{2}}} \tag{2}$$

$$\sum_{k=0}^{n} {\binom{n}{k}}^{2} {\binom{k+n}{k}}^{2} = \sum_{k=0}^{n} {\binom{n}{k}} {\binom{k+n}{k}} \sum_{j=0}^{k} {\binom{k}{j}}^{3}$$
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$$\int_0^\infty \frac{1}{\left(x^4 + 2ax^2 + 1\right)^{m+1}} \, \mathrm{d}x = \frac{\pi P_m^{\left(m + \frac{1}{2}, -m - \frac{1}{2}\right)}(a)}{2^{m + \frac{3}{2}}(a+1)^{m + \frac{1}{2}}} \tag{2}$$

$$e^{-x}x^{a/2}n! L_n^a(x) = \int_0^\infty e^{-t}t^{\frac{a}{2}+n} J_a(2\sqrt{tx}) dt$$
 (3)

$$\sum_{k=0}^{n} {\binom{n}{k}}^{2} {\binom{k+n}{k}}^{2} = \sum_{k=0}^{n} {\binom{n}{k}} {\binom{k+n}{k}} \sum_{j=0}^{k} {\binom{k}{j}}^{3}$$
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$$\int_0^\infty \frac{1}{\left(x^4 + 2ax^2 + 1\right)^{m+1}} \, \mathrm{d}x = \frac{\pi P_m^{\left(m + \frac{1}{2}, -m - \frac{1}{2}\right)}(a)}{2^{m + \frac{3}{2}}(a+1)^{m + \frac{1}{2}}} \tag{2}$$

$$e^{-x}x^{a/2}n! L_n^a(x) = \int_0^\infty e^{-t}t^{\frac{a}{2}+n} J_a(2\sqrt{tx}) dt$$
 (3)

$$\int_{-\infty}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{H_m(x)H_n(x)r^m s^n e^{-x^2}}{m! \, n!} \, \mathrm{d}x = \sqrt{\pi} e^{2rs} \qquad (4)$$

$$\sum_{k=0}^{n} {\binom{n}{k}}^{2} {\binom{k+n}{k}}^{2} = \sum_{k=0}^{n} {\binom{n}{k}} {\binom{k+n}{k}} \sum_{j=0}^{k} {\binom{k}{j}}^{3}$$
(1)

$$\int_0^\infty \frac{1}{\left(x^4 + 2ax^2 + 1\right)^{m+1}} \, \mathrm{d}x = \frac{\pi P_m^{\left(m + \frac{1}{2}, -m - \frac{1}{2}\right)}(a)}{2^{m + \frac{3}{2}}(a+1)^{m + \frac{1}{2}}} \tag{2}$$

$$e^{-x}x^{a/2}n!L_n^a(x) = \int_0^\infty e^{-t}t^{\frac{a}{2}+n}J_a(2\sqrt{tx})\,\mathrm{d}t \tag{3}$$

$$\int_{-\infty}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{H_m(x)H_n(x)r^m s^n e^{-x^2}}{m! \, n!} \, \mathrm{d}x = \sqrt{\pi} e^{2rs} \qquad (4)$$

$$\int_{-1}^{1} (1 - x^2)^{\nu - \frac{1}{2}} e^{iax} C_n^{(\nu)}(x) \, \mathrm{d}x = \frac{\pi i^n \Gamma(n + 2\nu) J_{n+\nu}(a)}{2^{\nu - 1} a^{\nu} n! \, \Gamma(\nu)}$$
(5)

Application Determinants Count!

(joint work with Hao Du, Christian Krattenthaler, Michael Schlosser, Aek Thanatipanonda, Elaine Wong)



Application Determinants Count!

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Application Determinants Count!

(joint work with Hao Du, Christian Krattenthaler, Michael Schlosser, Aek Thanatipanonda, Elaine Wong)



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Number of rhombus tilings: 19,180,227,670,614,654,793,187,652,900
$$\det_{1 \le i,j \le n} \frac{1}{i+j-1} = \frac{1}{(2n-1)!} \prod_{k=1}^{n-1} \frac{(k!)^2}{(k+1)_{n-1}}$$

$$\det_{1\leqslant i,j\leqslant n} \frac{1}{i+j-1} = \frac{1}{(2n-1)!} \prod_{k=1}^{n-1} \frac{(k!)^2}{(k+1)_{n-1}}$$
$$\det_{0\leqslant i,j\leqslant n-1} \binom{2i+2a}{j+b} = 2^{n(n-1)/2} \prod_{k=0}^{n-1} \frac{(2k+2a)!k!}{(k+b)!(2k+2a-b)!}$$

$$\det_{1\leqslant i,j\leqslant n} \frac{1}{i+j-1} = \frac{1}{(2n-1)!} \prod_{k=1}^{n-1} \frac{(k!)^2}{(k+1)_{n-1}}$$
$$\det_{0\leqslant i,j\leqslant n-1} \binom{2i+2a}{j+b} = 2^{n(n-1)/2} \prod_{k=0}^{n-1} \frac{(2k+2a)!k!}{(k+b)!(2k+2a-b)!}$$
$$\det_{0\leqslant i,j\leqslant n-1} \sum_k \binom{i}{k} \binom{j}{k} 2^k = 2^{n(n-1)/2}$$

$$\begin{aligned} \det_{1\leqslant i,j\leqslant n} \frac{1}{i+j-1} &= \frac{1}{(2n-1)!} \prod_{k=1}^{n-1} \frac{(k!)^2}{(k+1)_{n-1}} \\ \det_{0\leqslant i,j\leqslant n-1} \binom{2i+2a}{j+b} &= 2^{n(n-1)/2} \prod_{k=0}^{n-1} \frac{(2k+2a)!k!}{(k+b)!(2k+2a-b)!} \\ \det_{0\leqslant i,j\leqslant n-1} \sum_k \binom{i}{k} \binom{j}{k} 2^k &= 2^{n(n-1)/2} \\ \det_{1\leqslant i,j\leqslant 2m+1} \left[\binom{\mu+i+j+2r}{j+2r-2} - \delta_{i,j+2r} \right] \\ &= \frac{(-1)^{m-r+1} (\mu+3) (m+r+1)_{m-r}}{2^{2m-2r+1} (\frac{\mu}{2}+r+\frac{3}{2})_{m-r+1}} \cdot \prod_{i=1}^{2m} \frac{(\mu+i+3)_{2r}}{(i)_{2r}} \end{aligned}$$

$$\times \prod_{i=1}^{m-r} \frac{\left(\mu + 2i + 6r + 3\right)_{i}^{2} \left(\frac{\mu}{2} + 2i + 3r + 2\right)_{i-1}^{2}}{\left(i\right)_{i}^{2} \left(\frac{\mu}{2} + i + 3r + 2\right)_{i-1}^{2}}.$$
20 / 28



Problem: Prove a determinantal identity of

the form $\det_{1\leqslant i,j\leqslant n}(a_{i,j})=b_n$



Problem: Prove a determinantal identity of

the form $\det_{1\leqslant i,j\leqslant n}(a_{i,j})=b_n$, where

• $a_{i,j}$ is a holonomic sequence



Problem: Prove a determinantal identity of

the form $\det_{1\leqslant i,j\leqslant n}(a_{i,j})=b_n$, where

- $a_{i,j}$ is a holonomic sequence
- \blacktriangleright that does not depend on n



Problem: Prove a determinantal identity of

the form $\det_{1\leqslant i,j\leqslant n}(a_{i,j})=b_n$, where

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Conjecture (Di Francesco's determinant for 20V configurations):

$$\det_{0 \le i,j < n} \left(2^i \binom{i+2j+1}{2j+1} - \binom{i-1}{2j+1} \right) = 2 \prod_{i=1}^n \frac{2^{i-1} (4i-2)!}{(n+2i-1)!}$$

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Theorem (Di Francesco's determinant for 20V configurations):

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(join work with M. Neumüller and S. Radu)

We consider inequalities of the form

 $||v_n||_{X(\Omega)} \leqslant c(h,n) \, ||v_n||_{Y(\Omega)} \qquad \text{for all } v_n \in V_n$

- $\blacktriangleright \ \Omega \subset \mathbb{R}^d, d \in \mathbb{N}$
- \blacktriangleright V: some infinite-dimensional space of functions defined on Ω
- $\blacktriangleright \ ||\cdot||_{X(\Omega)}, \ ||\cdot||_{Y(\Omega)}$: norms that are used in numerical methods
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Transform the problem to a reference element $\hat{\Omega}$:

$$\hat{c}(n) = \sup_{v_n \in \hat{V}_n} \frac{||v_n||_{X(\hat{\Omega})}}{||v_n||_{Y(\hat{\Omega})}} = \sqrt{\sup_{v_n \in \hat{V}_n} \frac{(v_n, v_n)_{X(\hat{\Omega})}}{(v_n, v_n)_{Y(\hat{\Omega})}}}$$

Inverse Inequalities

Here we consider the reference domain $\hat{\Omega}=(-1,1)^2$ with

$$(u,v)_{X(\hat{\Omega})} = \int_{\hat{\Omega}} \partial_x u(x,y) \partial_x v(x,y) \, \mathrm{d}x \, \mathrm{d}y,$$
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for $u,v\in \hat{V}_n,$ where \hat{V}_n is the space of polynomials of degree less than n, i.e.

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The desired "constant" $\hat{c}(n)$ can be found as the largest λ_n solving the generalized eigenvalue problem

$$B_n \vec{x}_n = \lambda_n A_n \vec{x}_n,$$

where A_n and B_n are certain $n \times n$ matrices.

$$a_{i,j} := \frac{1 - (-1)^{i+j-1}}{i+j-1}, \quad b_{i,j} := (i-1)(j-1)\frac{1 - (-1)^{i+j-3}}{i+j-3}$$

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$$B_6 - \lambda A_6| = \begin{vmatrix} -2\lambda & 0 & -\frac{2}{3}\lambda & 0 & -\frac{2}{5}\lambda & 0 \\ 0 & 2 - \frac{2}{3}\lambda & 0 & 2 - \frac{2}{5}\lambda & 0 \\ 0 & 2 - \frac{2}{3}\lambda & 0 & 2 - \frac{2}{5}\lambda & 0 & 2 - \frac{2}{7}\lambda \\ -\frac{2}{3}\lambda & 0 & \frac{8}{3} - \frac{2}{5}\lambda & 0 & \frac{16}{5} - \frac{2}{7}\lambda & 0 \\ 0 & 2 - \frac{2}{5}\lambda & 0 & \frac{18}{5} - \frac{2}{7}\lambda & 0 & \frac{30}{7} - \frac{2}{9}\lambda \\ -\frac{2}{5}\lambda & 0 & \frac{16}{5} - \frac{2}{7}\lambda & 0 & \frac{32}{7} - \frac{2}{9}\lambda & 0 \\ 0 & 2 - \frac{2}{7}\lambda & 0 & \frac{30}{7} - \frac{2}{9}\lambda & 0 & \frac{50}{9} - \frac{2}{11}\lambda \end{vmatrix}$$

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Hence we get: $\det(B_n - \lambda A_n) = 2^n \det\left(A_{\lceil n/2 \rceil}^{(1)}\right) \cdot \det\left(A_{\lfloor n/2 \rfloor}^{(0)}\right).$

The Determinant

By a variation of the holonomic ansatz we prove:

Theorem.

$$\det A_n^{(0)} = \underbrace{\left(-\frac{1}{2}\right)^n \prod_{i=1}^n \frac{\left((i-1)!\right)^2}{\left(i+\frac{1}{2}\right)_n}}_{\text{"hyperholonomic" part}} \underbrace{\sum_{j=0}^n (-4)^{j-n} \frac{(2n-2j+1)_{2n}}{(2j)!} \lambda^j,}_{\text{holonomic part}} \det A_n^{(1)} = \underbrace{\left(-\frac{1}{2}\right)^n \prod_{i=1}^n \frac{\left((i-1)!\right)^2}{\left(i-1+\frac{1}{2}\right)_n}}_{\text{"hyperholonomic" part}} \underbrace{\sum_{j=0}^{n-1} \frac{(2n-2j-1)_{2n-1}}{(-4)^{n-j-1}(2j+1)!} \lambda^j.}_{\text{holonomic part}}$$

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We use this explicit evaluation to estimate the largest eigenvalue.

Final Result

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$$b_2(n) := m_1(n) \left(\frac{1}{3} + \left(r_1(n) + \sqrt{r_2(n)} \right)^{1/3} + \left(r_1(n) - \sqrt{r_2(n)} \right)^{1/3} \right),$$

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where m_1 , r_1 , and r_2 are given by

$$m_1(n) := \frac{n(n-1)(n+1)(n+2)}{8},$$

$$r_1(n) := \frac{2(n^8 + 4n^7 + 8n^6 + \dots - 4733n^2 - 5130n + 16200)}{135n^2(n-1)^2(n+1)^2(n+2)^2},$$

$$r_2(n) := \frac{(n-2)(n-3)(n+4)(n+3)(7n^{12} + 42n^{11} + \dots)}{145800n^4(n-1)^4(n+1)^4(n+2)^4}.$$

Further Reading

- Survey article: Creative telescoping for holonomic functions. DOI: 10.1007/978-3-7091-1616-6_7, arXiv:1307.4554.
- PhD thesis: Advanced applications of the holonomic systems approach (RISC, Johannes Kepler University, Linz, Austria).
- Software package: HolonomicFunctions (user's guide). https://risc.jku.at/sw/holonomicfunctions/
- Electromagnetic waves application: Method, device and computer program product for determining an electromagnetic near field of a field excitation source for an electrical system (with J. Schöberl and P. Paule), Patents EP2378444 and US8868382.
- Combinatorial determinants: Binomial determinants for tiling problems yield to the holonomic ansatz (with H. Du, T. Thanatipanonda, E. Wong), DOI: 10.1016/j.ejc.2021.103437.
- 20V determinants: Determinant evaluations inspired by Di Francesco's determinant for twenty-vertex configurations (with C. Krattenthaler and M. Schlosser), arXiv:2401.08481.