Computer algebra for basic hypergeometric functions

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The Ismail-Zhang formula

Classical expansion formula of the plane wave in terms of ultraspherical polynomials $C_m^{(\nu)}(x)$ ("Gegenbauer polynomials"):

$$e^{irx} = \left(\frac{2}{r}\right)^{\nu} \Gamma(\nu) \sum_{m=0}^{\infty} i^m (\nu + m)\, J_{\nu+m}(r)\, C_m^{(\nu)}(x).$$

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Ismail and Zhang (1994) had found the following q-analog:

$$\mathcal{E}_q(x; i\omega) = \frac{(q;q)_\infty\, \omega^{-\nu}}{(q^\nu;q)_\infty\, (-q\omega^2;q^2)_\infty} \times \sum_{m=0}^{\infty} i^m (1 - q^{\nu+m})\, q^{m^2/4} J_{\nu+m}^{(2)}(2\omega;q)\, C_m(x;q^\nu|q),$$

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where $(a;q)_n := \prod_{k=0}^{n-1}(1 - aq^k)$ is the q-Pochhammer symbol.

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and the basic sine function $S_q(x;\omega)$ as

$$\frac{(-\omega^2;q^2)_\infty}{(-q\omega^2;q^2)_\infty} \frac{2q^{1/4}\omega}{1-q} \cos(\theta) \sum_{j=0}^{\infty} \frac{(-qe^{2i\theta};q^2)_j \, (-qe^{-2i\theta};q^2)_j}{(q^3;q^2)_j \, (q^2;q^2)_j} (-\omega^2)^j.$$

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where $J^{(2)}_{\nu+m}(2\omega;q)$ is Jackson's q-Bessel function defined by

$$J^{(2)}_\nu(z;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty} q^{(\nu+n)n}\frac{(-1)^n(z/2)^{\nu+2n}}{(q;q)_n(q^{\nu+1};q)_n},$$

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and where the continuous q-ultraspherical (q-Gegenbauer) polynomials $C_m(x;q^\nu|q)$, $x = \cos(\theta)$, are defined as

$$C_m(\cos\theta;\beta|q) := \sum_{k=0}^{m} \frac{(\beta;q)_k \, (\beta;q)_{m-k}}{(q;q)_k \, (q;q)_{m-k}} e^{i(m-2k)\theta}.$$

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A holonomic systems approach to special functions identities *

Doron ZEILBERGER
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Abstract: We observe that many special functions are solutions of so-called holonomic systems. Bernstein's deep theory of holonomic systems is then invoked to show that any identity involving sums and integrals of products of these special functions can be verified in a finite number of steps. This is partially substantiated by an algorithm that proves terminating hypergeometric series identities, and that is given both in English and in MAPLE.

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Implementations (in Mathematica, created at RISC):
GeneratingFunctions, **fastZeil**, **MultiSum**, **qZeil**, **qMultiSum**, **pqTelescope**, **HolonomicFunctions**

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5. Integrals, sums, and q-sums are treated by the method of creative telescoping.
6. The output is always given as an annihilating ideal, not as a closed form.

The basic exponential function

Recall: $\mathcal{E}_q(x; i\omega) := C_q(x;\omega) + i\, S_q(x;\omega)$ and

$$C_q(x;\omega) := \frac{(-\omega^2;q^2)_\infty}{(-q\omega^2;q^2)_\infty} \sum_{j=0}^{\infty} \frac{(-qe^{2i\theta};q^2)_j\,(-qe^{-2i\theta};q^2)_j}{(q;q^2)_j\,(q^2;q^2)_j} (-\omega^2)^j$$

is the basic cosine function.

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is the basic cosine function. Note that

$$\lim_{q\to 1-} C_q\big(x;\omega(1-q)/2\big) = \cos(\omega x)$$

$$\lim_{q\to 1-} S_q\big(x;\omega(1-q)/2\big) = \sin(\omega x)$$

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CreativeTelescoping delivers a telescoper T and a certificate Q

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such that $T\big(c_j(\omega)\big) + d_{j+1}(\omega) - d_j(\omega) = 0$ with $d_j(\omega) := Q\big(c_j(\omega)\big)$.

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$$T = A^2\big(q^2\omega^2+1\big)S_{\omega,q}^2 + \big(A^4q^2\omega^2 - A^2q - A^2 + q^2\omega^2\big)S_{\omega,q} + A^2q\big(q\omega^2+1\big)$$

$$Q = \frac{A^2(q^j-1)(q^j+1)\big(q^{2j}-q\big)\big(q\omega^2+1\big)}{\omega^2+1}$$

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Notation for q-shift operator: $S_{\omega,q}\big(f(\omega)\big) := f(q\omega)$

Creative telescoping

In other words, we obtain the creative telescoping relation

$$\begin{aligned} A^2\big(q^2\omega^2+1\big)c_j(q^2\omega) + \big(A^4q^2\omega^2 - A^2q - A^2 + q^2\omega^2\big)c_j(q\omega) \\ + A^2q\big(q\omega^2+1\big)c_j(\omega) = -\big(d_{j+1}(\omega) - d_j(\omega)\big) \end{aligned}$$

where $d_j(\omega) = \dfrac{A^2(q^j-1)(q^j+1)\big(q^{2j}-q\big)\big(q\omega^2+1\big)}{\omega^2+1}c_j(\omega)$.

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$$+A^2q\big(q\omega^2+1\big)c_j(\omega)=-\big(d_{j+1}(\omega)-d_j(\omega)\big)$$

where $d_j(\omega)=\dfrac{A^2(q^j-1)(q^j+1)\big(q^{2j}-q\big)\big(q\omega^2+1\big)}{\omega^2+1}c_j(\omega)$.

Summing the right-hand side from $j=0$ to $j=\infty$ gives

$$\frac{A^2(-1+q^0)(1+q^0)(q^0-q)(1+q\omega^2)}{1+\omega^2}c_0(\omega)-\frac{qA^2(1+q\omega^2)}{1+\omega^2}c_\infty(\omega)$$

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which equals zero.

Hence the telescoper T annihilates $C_q(x;\omega)=\sum_{j=0}^{\infty}c_j(\omega)$.

Annihilator for basic exp

Analogously, we find that the basic sine function $S_q(x;\omega)$ satisfies the same q-difference equation (given by the operator T):

$$\begin{aligned} &A^2\big(q^2\omega^2+1\big)S_q(x;q^2\omega) \\ &+\big(A^4q^2\omega^2-A^2q-A^2+q^2\omega^2\big)S_q(x;q\omega) \\ &+A^2q\big(q\omega^2+1\big)S_q(x;\omega)=0. \end{aligned}$$

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Hence: we have the annihilator of $\mathcal{E}_q(x;i\omega)$.

Annihilator of q-Gegenbauer

Recall the definition:

$$C_m(\cos\theta;\beta|q) := \sum_{k=0}^{m} \underbrace{\frac{(\beta;q)_k\,(\beta;q)_{m-k}}{(q;q)_k\,(q;q)_{m-k}}e^{i(m-2k)\theta}}_{\text{summand}}.$$

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By applying the command **CreativeTelescoping** to the summand, we obtain:

$$\Big\{S_{\omega,q}-1, -A\left(A^2+1\right)V(Mq-1)S_{M,q}+(V-1)\left(A^2-V\right)\left(A^2V-1\right)S_{V,q}+ A^2(V+1)\left(MV^2-1\right), (V-1)(qV-1)\left(A^2-qV\right)\left(A^2qV-1\right)S_{V,q}^2- (V-1)\left(A^4Mq^2V^2-A^4qV-A^2Mq^3V^3-A^2Mq^2V^3+A^2q+A^2+ Mq^2V^2-qV\right)S_{V,q}-A^2q\left(MV^2-1\right)\left(MqV^2-1\right)\Big\},$$

where we use the abbreviations

$$K=q^k,\quad M=q^m,\quad N=q^n,\quad V=q^\nu,\ \text{and}\quad \omega=q^w.$$

Annihilator of Jackson's q-Bessel function

Recall the definition:

$$J_\nu^{(2)}(z;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} q^{(\nu+n)n} \frac{(-1)^n (z/2)^{\nu+2n}}{(q;q)_n (q^{\nu+1};q)_n}.$$

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Analogously to the q-Gegenbauer polynomial, we compute the annihilator of $J_\nu^{(2)}(z;q)$ by the creative telescoping method (where we use the fact that the sum has natural boundaries):

$$\big\{(-V\omega-\omega)S_{V,q}+(q\omega^4+q\omega^2+\omega^2+1)S_{\omega,q}+(\omega^2-V), S_{M,q}-1,$$
$$(q^5V\omega^4+q^3V\omega^2+q^2V\omega^2+V)S_{\omega,q}^2+(q^2V\omega^2+qV\omega^2-V^2-1)S_{\omega,q}+V\big\}$$

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Recall the definition:

$$J_\nu^{(2)}(z;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} q^{(\nu+n)n} \frac{(-1)^n (z/2)^{\nu+2n}}{(q;q)_n (q^{\nu+1};q)_n}.$$

Analogously to the q-Gegenbauer polynomial, we compute the annihilator of $J_\nu^{(2)}(z;q)$ by the creative telescoping method (where we use the fact that the sum has natural boundaries):

$$\{(-V\omega-\omega)S_{V,q}+(q\omega^4+q\omega^2+\omega^2+1)S_{\omega,q}+(\omega^2-V), S_{M,q}-1,$$
$$(q^5V\omega^4+q^3V\omega^2+q^2V\omega^2+V)S_{\omega,q}^2+(q^2V\omega^2+qV\omega^2-V^2-1)S_{\omega,q}+V\}$$

Note that we already included $S_{M,q}$ in the list of operator symbols, for later use.

Continue with the Ismail-Zhang formula

$$\mathcal{E}_q(x; i\omega) = \frac{(q;q)_\infty \, \omega^{-\nu}}{(q^\nu;q)_\infty \, (-q\omega^2;q^2)_\infty} \times \sum_{m=0}^{\infty} i^m (1-q^{\nu+m}) \, q^{m^2/4} J^{(2)}_{\nu+m}(2\omega;q) \, C_m(x;q^\nu|q)$$

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We proceed by computing an annihilator for the factor

$$h_1(\omega, m, \nu) := i^m (1-q^{\nu+m}).$$

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The **Annihilator** command delivers the following output:

$$\{S_{\omega,q}-1, (MV-1)S_{V,q}+(1-MqV), (MV-1)S_{M,q}+(i-iMqV)\}.$$

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Note that in fact it is trivial to compute the generators of this ideal, just consider the quotients

$$\frac{h_1(q\omega,m,\nu)}{h_1(\omega,m,\nu)}, \quad \frac{h_1(\omega,m+1,\nu)}{h_1(\omega,m,\nu)}, \quad \text{and} \quad \frac{h_1(\omega,m,\nu+1)}{h_1(\omega,m,\nu)}.$$

The annihilator of $q^{m^2/4}$

When trying to compute the annihilating ideal of

$$h_2(m) := q^{m^2/4}$$

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Hence, we figure out the minimal-order annihilator by hand, and convert it into the same Ore algebra as the previous annihilators:

$$\{S_{V,q} - 1, S_{\omega,q} - 1, S_{M,q}^2 - Mq\}.$$

Combine h_1 and h_2

We have computed annihilators for both h_1 and h_2

$$h_1(m, \nu) = i^m(1 - q^{\nu+m})$$
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In order to obtain an annihilator for $h_1 \cdot h_2$, we combine them using the **DFiniteTimes** and obtain:

$$\left\{S_{\omega,q}-1, (MV-1)S_{V,q}+(1-MqV), (MV-1)S_{M,q}^2+(M^2q^3V-Mq)\right\}$$

Annihilator for the summand

Recall the summand on the RHS of the Ismail-Zhang formula:

$$i^m(1-q^{\nu+m})\,q^{m^2/4}J^{(2)}_{\nu+m}(2\omega;q)\,C_m(x;q^\nu|q).$$

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Then we combine everyting using again **DFiniteTimes**:

$$\begin{gathered}\{(1{+}q^2\omega^2{+}q^3\omega^2{+}q^5\omega^4)MVS^2_{\omega,q}{+}(qMV\omega^2{+}q^2MV\omega^2{-}M^2V^2{-}1)S_{\omega,q}{+}MV,\\ (-A^2MV\omega^2{+}\ldots{-}q^5A^2M^5V^9\omega^2)S^2_{V,q}{+}(A^2MV\omega{+}\ldots{+}q^6A^2M^5V^8\omega^5)S_{V,q}S_{\omega,q}{+}\\ (-qA^2M^2V^2\omega-\ldots+q^5A^2M^5V^8\omega^3)S_{V,q}+(-A^2+\ldots+q^7A^2M^6V^8\omega^4)S_{\omega,q}+\\ (A^2MV-\ldots+q^6A^2M^6V^8\omega^2),\\ (-A^2V^2\omega^2{+}\ldots{+}q^4A^2M^5V^5\omega^2)S^2_{M,q}{+}(qA^2MV\omega{-}\ldots{-}q^4A^2M^4V^7\omega^5)S_{V,q}S_{\omega,q}{+}\\ (-q^2A^2M^2V^2\omega{+}\ldots{-}q^3A^2M^4V^7\omega^3)S_{V,q}{+}(-A^2{+}\ldots{-}q^6A^2M^5V^7\omega^4)S_{\omega,q}{+}\\ (A^2MV-\ldots-q^5A^2M^5V^7\omega^2)\}\end{gathered}$$

(full output fills about two pages)

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This means that the certificate is of the form

$$\begin{aligned} r_1 + r_2 \cdot S_{\omega,q} + r_3 \cdot S_{M,q} + r_4 \cdot S_{V,q} + r_5 \cdot S_{\omega,q} S_{M,q} \\ + r_6 \cdot S_{\omega,q} S_{V,q} + r_7 \cdot S_{M,q} S_{V,q} + r_8 \cdot S_{\omega,q} S_{M,q} S_{V,q} \end{aligned}$$

with $c_i \in \mathbb{Q}(\omega, M, V)$.

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Chyzak's algorithm determines the denominators of $r_1, \dots, r_8$ by solving a coupled system q-difference equations of dimension 8.

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Then everything works fine (the computation takes only 18 seconds), and the telescopers are

$$(V-1)S_{V,q}+\omega,(-q^5A^2\omega^4-q^3A^2\omega^2-q^2A^2\omega^2-A^2)S_{\omega,q}^2+\\(-q^2A^4V\omega^2+qA^2V+A^2V-q^2V\omega^2)S_{\omega,q}-qA^2V^2$$

(the certificate is large).

Annihilators for both sides

This does not quite match with the annihilator of the left-hand side, but we are still missing the factor in front of the sum:

$$\mathcal{E}_q(x; i\omega) = \frac{(q;q)_\infty \, \omega^{-\nu}}{(q^\nu;q)_\infty \, (-q\omega^2;q^2)_\infty} \times \sum_{m=0}^{\infty} i^m (1-q^{\nu+m}) \, q^{m^2/4} J^{(2)}_{\nu+m}(2\omega;q) \, C_m(x;q^\nu|q)$$

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The proof is completed by comparing two initial values.