Computer Algebra for Knot Theory

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Gentle Introduction to Knot Theory

Knot:

- embedding of a circle in the Euclidean space $\mathbb{R}^3$
- think of a knotted (closed) string
- knot complement: $\mathbb{R}^3 \setminus K$

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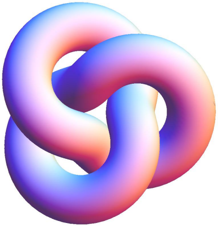
Examples:

- unknot: $\bigcirc$
- trefoil ("Kleeblattknoten"):



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Link:

- several knots
- entangled with each other

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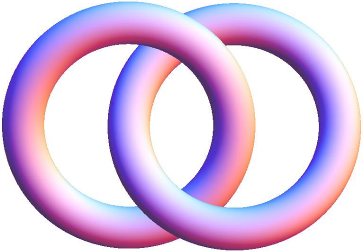
Examples:

- unlink: ◯ ◯
- Hopf link:



Equivalence of knots:

- if one can be transformed into the other
- "without cutting the string"

Knot diagram:

- planar diagram
- obtained by a projection of the knot onto the plane
- such that there are only finitely many crossings

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- such that there are only finitely many crossings

Wild knot:

- no projection with finitely many crossings is possible

Tame knot:

- there exists a projection with finitely many crossings
- from now on: consider only tame knots

Theorem (Reidemeister, 1927):
Two knot diagrams represent the same knot if and only if they can be transformed into each other by a finite sequence of Reidemeister moves.

Reidemeister moves:

- Type I: twist and untwist
- Type II: move one loop completely over another
- Type III: move a string completely over or under a crossing

Irreducible knot:

- connected sum of two knots: $K_1 \# K_2$
- a knot is irreducible if it cannot be written as connected sum of two nontrivial knots
- "unique factorization" of knots
- Rolfsen's table contains only irreducible knots

Demo:

See www.katlas.org

Gentle Introduction to Knot Theory

Fundamental problem:
Determine whether two descriptions (e.g., knot diagrams) represent the same knot.

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Knot polynomials:

- Alexander polynomial (1928)
- Jones polynomial (1984, Fields medal!)
- Kauffman polynomial
- A-polynomial
- HOMFLY polynomial

Skein relation:

- skein = “Strang”, “Strähne”
- are used to define many polynomial invariants
- three-term relation connecting the polynomials of knots which differ only locally.

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Example: Skein relation for the Jones polynomial

$$q^{-1}J(L_+) - qJ(L_-) = (q^{1/2} - q^{-1/2})J(L_0)$$

where L_+ and L_- denote a positive resp. negative crossing and L_0 no crossing. Initial condition:

$$J(\bigcirc) = 1.$$

The A-polynomial

A-polynomial of a knot:

- difficult to compute (e.g., using elimination)
- difficult to understand ("The A-polynomial of a knot parametrizes the affine variety of $\mathrm{SL}(2, \mathbb{C})$ representations of the knot complement, viewed from the boundary torus.")
- even unknown for some knots with only 9 crossings.

The Colored Jones Function

Colored Jones function: For each knot K, define

$$J_{K,n}(q) \in \mathbb{Z}[q^{\pm 1}]^{\mathbb{N}},$$

a sequence of Laurent polynomials.

Definitions:

- by the n-th parallel of a knot
- via state sums

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- via state sums

For a knot with m crossings, the state sum is an m-fold sum with q-hypergeometric summand.
$\longrightarrow$ The colored Jones function is a q-holonomic sequence!

Excursion: q-Holonomic Sequences

Notation:

- $\mathbb{K}$: field of characteristic zero
- q: indeterminate, transcendental over $\mathbb{K}$

A univariate sequence $\big(f_n(q)\big)_{n\in\mathbb{N}}$ is called **q-holonomic** if it satisfies a nontrivial linear recurrence with coefficients that are polynomials in q and q^n:

$$\sum_{j=0}^{d} c_j(q,q^n) f_{n+j}(q) = 0 \qquad (n \in \mathbb{N})$$

where d is a nonnegative integer and $c_j(u,v) \in \mathbb{K}[u,v]$ are bivariate polynomials for $j = 0,\ldots,d$ with $c_d(u,v) \neq 0$.

(Zeilberger 1990)

The noncommutative A-polynomial

Introduce operator notation:

$$(Lf)_n(q) = f_{n+1}(q), \qquad (Mf)_n(q) = q^n f_n(q)$$

and let

$$\mathbb{W} = \mathbb{K}(q, M)\langle L\rangle/(LM - qML).$$

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Noncommutative A-polynomial:
Denoted by $A_K(q, M, L)$ for a knot K, is defined to be the (homogeneous and content-free) q-holonomic recurrence for $J_{K,n}(q)$ that has minimal order.

The AJ Conjecture

There is a close relation between the A-polynomial $A_K(M, L)$ and the recurrence (given as an operator $A_K(q, M, L) \in \mathbb{W}$) for the colored Jones function:

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For every knot K the following identity holds:

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AJ Conjecture:
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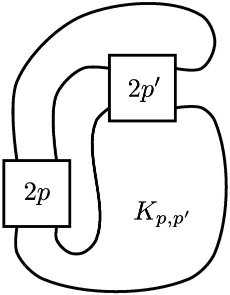
$$A_K(1, M, L) = \mathrm{poly}(M) \cdot A_K(M^{1/2}, L)$$

$\longrightarrow$ The AJ conjecture has been verified (rigorously / non-rigorously) for some knots with few crossings, by explicit computations, as well as for some special families of knots.

Double Twist Knots

One such family are the so-called **double twist knots** $K_{p,p'}$:



$\longrightarrow$ Interesting family because their A-polynomials are reducible.

Colored Jones Function of $K_{p,p'}$

Using the Habiro theory of the colored Jones function, we get

$$J_{K_{p,p'},n}(q) = \sum_{k=0}^{n-1} (-1)^k c_{p,k}(q) c_{p',k}(q) q^{-kn-\frac{k(k+3)}{2}} (q^{n-1}; q^{-1})_k (q^{n+1}; q)_k$$

where $(a;q)_n$ denotes the q-Pochhammer symbol defined as

$$(a;q)_n = \prod_{j=0}^{n-1} (1 - aq^j)$$

and where

$$c_{p,n}(q) = \sum_{k=0}^{n} (-1)^{k+n} q^{-\frac{k}{2}+\frac{k^2}{2}+\frac{3n}{2}+\frac{n^2}{2}+kp+k^2 p} \frac{(1-q^{2k+1})(q;q)_n}{(q;q)_{n-k}(q;q)_{n+k+1}}.$$

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$\longrightarrow$ Perfect application for HolonomicFunctions!

Apply HolonomicFunctions

Consider the case $p = p' = 2$, i.e., the knot $K_{2,2}$ which corresponds to the entry 7_4 in Rolfsen's table.

Result:

- Recurrence of order 5, with M-degree 24 and q-degree 65
- corresponds to 4 printed pages

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Strategy:

To prove minimality, we show that the corresponding operator is irreducible.

An Easy Sufficient Criterion for Irreducibility

Consider

$$A(q, M, L) = \sum_{j=0}^{d} a_j(q, M) L^j \in \mathbb{W}$$

with $d > 1$ and assume

- $A(1, M, L) \in \mathbb{K}(M)[L]$ is well-defined,
- irreducible,
- and $a_0(1, M) a_d(1, M) \neq 0$.

Then $A(q, M, L)$ is irreducible in $\mathbb{W}$.

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$\longrightarrow$ Unfortunately, we cannot apply this criterion, since $A(1, M, L)$ in our case is reducible (double twist knots!).

Exterior Powers

Shifted analogue of the Wronskian:

For k sequences $f_n^{(i)}$, $i = 1, \dots, k$, it is given by

$$W\big(f^{(1)}, \dots, f^{(k)}\big)_n = \det_{\substack{0 \le j \le k-1 \\ 1 \le i \le k}} f_{n+j}^{(i)} = \begin{vmatrix} f_n^{(1)} & \cdots & f_n^{(k)} \\ \vdots & & \vdots \\ f_{n+k}^{(1)} & \cdots & f_{n+k}^{(k)} \end{vmatrix}.$$

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Exterior Powers:

- $P \in \mathbb{W}$ with $\deg_L(P) = d$
- notation: $\bigwedge^k P$ ("k-th exterior power of P")
- definition: minimal-order operator for $W\left(f^{(1)}, \ldots, f^{(k)}\right)_n$
- where $f^{(1)}, \ldots, f^{(k)}$ are assumed to be linearly independent solutions of $Pf = 0$.

Lemma

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Let $P = L^d + \sum_{j=0}^{d-1} a_j L^j \in \mathbb{W}$ *with* $a_0 \neq 0$, *let* $\left\{ f_n^{(1)}, \ldots, f_n^{(d)} \right\}$ *be a fundamental solution set of the equation* $Pf = 0$, *and let* $w = W(f^{(1)}, \ldots, f^{(d)})$. *Then* $w_{n+1} - (-1)^d a_0 w_n = 0$.

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Proof.

This is proven by an elementary calculation

$$w_{n+1} = \begin{vmatrix} f_{n+1}^{(1)} & \cdots & f_{n+1}^{(d)} \\ \vdots & & \vdots \\ f_{n+d}^{(1)} & \cdots & f_{n+d}^{(d)} \end{vmatrix} = \begin{vmatrix} f_{n+1}^{(1)} & \cdots & f_{n+1}^{(d)} \\ \vdots & & \vdots \\ f_{n+d-1}^{(1)} & \cdots & f_{n+d-1}^{(d)} \\ -a_0 f_n^{(1)} & \cdots & -a_0 f_n^{(d)} \end{vmatrix} = (-1)^d a_0 w_n$$

(use $f_{n+d}^{(i)} = -\sum_{j=0}^{d-1} a_j f_{n+j}^{(i)}$ and row operations). □

A Necessary and Sufficient Criterion for Irreducibility

Theorem

Let $P, Q, R \in \mathbb{W}$ *such that* $P = QR$ *is a factorization of* P*, and let* k *denote the order of* R*, i.e.,* $k = \deg_L(R)$*. Then* $\bigwedge^k P$ *has a linear right factor of the form* $L - a$ *for some* $a \in \mathbb{K}(q, M)$*.*

A Necessary and Sufficient Criterion for Irreducibility

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Proof.

- Let $F = \{f^{(1)}, \ldots, f^{(k)}\}$ be a fundamental solution set of R.
- By the lemma it follows that $w = W(f^{(1)}, \ldots, f^{(k)})$ satisfies a recurrence of order 1, say $w_{n+1} = a w_n, a \in \mathbb{K}(q, M)$.
- But F is also a set of linearly independent solutions of $Pf = 0$ and therefore w is contained in the solution space of $\bigwedge^k P$.
- It follows that $\bigwedge^k P$ has the right factor $L - a$.

□

Computation of Exterior Powers

As before let d denote the L-degree of P.

1. Ansatz for $\bigwedge^k P$:

$$c_\ell(q,M)w_{n+\ell} + \cdots + c_1(q,M)w_{n+1} + c_0(q,M)w_n = 0.$$

2. Replace all occurrences of w_{n+j} by the expansion of the Wronskian, e.g., for $k=2$:

$$w_{n+j} = f^{(1)}_{n+j} f^{(2)}_{n+j+1} - f^{(1)}_{n+j+1} f^{(2)}_{n+j}.$$

3. Rewrite each $f^{(i)}_{n+j}$ with $j \geq d$ as a $\mathbb{K}(q,M)$-linear combination of $f^{(i)}_n, \dots, f^{(i)}_{n+d-1}$, using the equation $Pf^{(i)} = 0$.
4. Coefficient comparison with respect to $f^{(i)}_{n+j}$, $1 \leq i \leq k$, $0 \leq j < d$, yields a linear system for $c_0, \dots, c_\ell$.

Exterior Powers of P_{7_4}

Some statistics concerning P_{7_4} and its exterior powers, according to the factorization of $P_{7_4}(1, M, L)$:

	L-degree	M-degree	q-degree	ByteCount
P_{7_4}	5	24	65	463,544
$\bigwedge^2 P_{7_4}$	10	134	749	37,293,800
$\bigwedge^3 P_{7_4}$	10	183	1108	62,150,408

$\longrightarrow$ We now have to prove that $\bigwedge^2 P_{7_4}$ and $\bigwedge^3 P_{7_4}$ have no linear right factors.

qHyper

Let $P(q, M, L) = p_d(q, M)L^d + \cdots + p_0(q, M)$, $p_i \in \mathbb{K}[q, M]$.

The qHyper algorithm (Abramov+Paule+Petkovšek 1998) attempts to find a right factor $L - r(q, M)$ of P where

$$r(q, M) = z(q)\frac{a(q, M)}{b(q, M)}\frac{c(q, qM)}{c(q, M)}, \quad a, b, c \in \mathbb{K}[q, M]$$

is assumed to be in normal form, defined by the conditions

$$\begin{aligned} \gcd\big(a(q, M), b(q, q^n M)\big) &= 1 \text{ for all } n \in \mathbb{N}, \\ \gcd\big(a(q, M), c(q, M)\big) &= 1, \\ \gcd\big(b(q, M), c(q, qM)\big) &= 1. \end{aligned}$$

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It is not difficult to show that under these assumptions

$$a(q, M) \mid p_0(q, M) \qquad \text{and} \qquad b(q, M) \mid p_d(q, q^{1-d}M).$$

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$\longrightarrow$ qHyper proceeds by testing all admissible choices of a and b.

Application of qHyper

Now let's apply qHyper to $P^{(2)}(q,M,L) := \bigwedge^2 P_{7_4}$ whose trailing and leading coefficients are given by

$$p_0(q,M) = q^{162}M^{44}(M-1)\left(\prod_{i=6}^{9}(q^iM-1)\right)$$
$$\times\left(\prod_{i=6}^{10}(q^iM+1)(q^{2i+1}M^2-1)\right)F_1(q,M)$$

$$p_{10}(q,q^{-9}M) = q^{-397}(q^2M-1)\left(\prod_{i=4}^{7}(M-q^i)\right)$$
$$\times\left(\prod_{i=4}^{8}(M+q^i)(M^2-q^{2i+1})\right)F_2(q,M)$$

where F_1 and F_2 are large irreducible polynomials, related by $q^{280}F_1(q,M) = F_2(q,q^{10}M)$.

Application of qHyper

$$p_0(q, M) = q^{162} M^{44} (M-1) \left(\prod_{i=6}^{9} (q^i M - 1) \right) \times \left(\prod_{i=6}^{10} (q^i M + 1)(q^{2i+1} M^2 - 1) \right) F_1(q, M)$$

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$\longrightarrow$ A blind application of qHyper would result in $45 \cdot 2^{16} \cdot 2^{16} = 193\,273\,528\,320$ possible choices for a and b.

Confine the Number of qHyper's Test Cases

We exploit two facts:

Fact 1: Study the image under $q = 1$:

$$P^{(2)}(1, M, L) = R_1(M) \cdot (L - M^4) \cdot Q_1(M, L) \cdot Q_2(M, L)$$

where Q_1 and Q_2 are irreducible of L-degree 3 and 6, respectively. Thus we need only to test pairs (a, b) which satisfy the condition

$$(*) \qquad a(1, M) = M^4 b(1, M).$$

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Fact 2: a and b must fulfill the gcd condition:

$$\gcd(a(q, M), b(q, q^n M)) = 1 \text{ for all } n \in \mathbb{N}.$$

$\longrightarrow$ These two facts allow to exclude most of the admissible choices for a and b.

Structure of Leading and Trailing Coefficient

$$p_0(q,M) = q^{162}M^{44}(M-1)\Big(\prod_{i=6}^{9}(q^iM-1)\Big) \times \Big(\prod_{i=6}^{10}(q^iM+1)(q^{2i+1}M^2-1)\Big)F_1(q,M)$$

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	$p_0(q,M)$	$p_{10}(q,q^{-9}M)$
q^iM-1	0, 6, 7, 8, 9	-7, -6, -5, -4, 2
q^iM+1	6, 7, 8, 9, 10	-8, -7, -6, -5, -4
q^iM^2-1	13, 15, 17, 19, 21	-17, -15, -13, -11, -9

Linear and quadratic factors of the leading and trailing coefficients; each cell contains the values of i of the corresponding factors.

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3. The most simple admissible choice is $a(q,M) = M^4$ and $b(q,M) = 1$.
4. Because of the gcd condition, a cancellation can almost never take place among factors which are equivalent under the substitution $q = 1$. This is reflected by the fact that the entries in the first column of the table are (row-wise) larger than those in the second column, e.g., $(q^6M + 1) \mid a(q,M)$ and $(q^{-4}M + 1) \mid b(q,M)$ violates the gcd condition.

Which Combinations to Test

5. The only exception is that $(M-1) \mid a(q,M)$ cancels with $(q^2M-1) \mid b(q,M)$ in $a(1,M)/b(1,M)$. In that case, the gcd condition excludes further factors of the form q^iM-1, and together with (*) we see that no other factors at all can occur. This gives the choice $a(q,M)=M^4(M-1)$ and $b(q,M)=q^2M-1$.

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6. We may assume that $a(q, M)$ contains some of the quadratic factors q^iM^2-1. For $q=1$ they factor as $(M-1)(M+1)$ and therefore can be canceled with corresponding pairs of linear factors in $b(q, M)$. The gcd condition forces $a(q, M)$ to be free of linear factors and $b(q, M)$ to be free of quadratic factors. Thus we obtain $\sum_{m=1}^{5} \binom{5}{m}^3 = 2251$ possible choices.

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$\longrightarrow$ Summing up, we have to test 4504 cases only!

Results for Double Twist Knots

$K_{2,2} = 7_4$:

- rigorous computation of $A(q, M, L)$
- rigorous proof that it is of minimal order

$K_{3,3}$:

- rigorous computation of $A(q, M, L)$
- (q, M, L)-degree $= (458, 74, 11)$
- minimality proof out of scope (requires $\bigwedge^5 P$ and $\bigwedge^6 P$)

$K_{4,4}$:

- $A(q, M, L)$ guessed
- (q, M, L)-degree $= (2045, 184, 19)$

$K_{5,5}$:

- $A(q, M, L)$ guessed
- (q, M, L)-degree $= (6922, 396, 29)$, ByteCount $=$ 8GB

Palindromicity

We say that an operator $P \in \mathbb{K}(q)\langle M^{\pm 1}, L^{\pm 1}\rangle/(LM - qML)$ is palindromic if and only if there exist integers $a, b \in \mathbb{Z}$ such that

$$P(q, M, L) = (-1)^a q^{bm/2} M^m L^b P(q, M^{-1}, L^{-1}) L^{\ell - b}$$

where $m = \deg_M(P) + \mathrm{ldeg}_M(P)$ and $\ell = \deg_L(P) + \mathrm{ldeg}_L(P)$.

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$\longrightarrow$ All operators here are palindromic! Exploit this for guessing!