

Symbolic Evaluation of Determinants and Rhombus Tilings of Holey Hexagons

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(joint work with Thotsaporn “Aek” Thanatipanonda)

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Plane Partitions (III): The Weak Macdonald Conjecture

George E. Andrews*

The Pennsylvania State University, University Park, Pennsylvania 16802, U.S.A.

Dedicated to the memory of Alfred Young and F.J.W. Whipple

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$$D_{0,0}(n).$$

Andrews's Result (1979)

Theorem. We have

$$D_{0,0}(n) = 2 \prod_{i=1}^{n-1} R_{0,0}(i),$$

in other words $R_{0,0}(n) = D_{0,0}(n+1)/D_{0,0}(n)$,

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$$R_{0,0}(2n) = \frac{(\mu + 2n)_n \left(\frac{\mu}{2} + 2n + \frac{1}{2}\right)_{n-1}}{(n)_n \left(\frac{\mu}{2} + n + \frac{1}{2}\right)_{n-1}},$$

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and where $(a)_n$ denotes the Pochhammer symbol

$$(a)_n := a \cdot (a+1) \cdots (a+n-1).$$

Another question is the possibility of other general determinants of this nature. At first glance

$$E_m(\mu) = \det \left(\delta_{ij} + \binom{\mu+i+j}{i+1} \right)_{0 \leq i, j \leq m-1}$$

looks interesting. Indeed it turns out that

$$E_1(\mu) = \mu + 1,$$

$$E_2(\mu) = (\mu+2)(\mu+1),$$

$$E_3(\mu) = \frac{(\mu+14)(\mu+3)(\mu+2)(\mu+1)}{12},$$

$$E_4(\mu) = \frac{(\mu+14)(\mu+9)(\mu+4)(\mu+3)(\mu+2)(\mu+1)}{72},$$

$$E_5(\mu) = \frac{(\mu+9)(\mu+5)(\mu+4)(\mu+3)(\mu+2)(\mu+1)(\mu^3+45\mu^2+722\mu+3432)}{8640}.$$

Empirically it seems reasonable to guess that

$$\frac{E_{2m}(\mu)}{E_{2m-1}(\mu)} = f_{2m, 2m}(\mu-2),$$

George Andrews (1980):
Macdonald's conjecture and
descending plane partitions

Andrews's Conjecture (1980)

We let $D_{1,1}(n)$ denote Andrews's 1980 determinant

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with $\lambda := \mu - 2$.

Conjecture. The following holds:

$$\frac{D_{1,1}(2n)}{D_{1,1}(2n-1)} = (-1)^{\frac{(n-1)(n-2)}{2}} 2^n \frac{\left(\frac{\mu}{2} + 2n + \frac{1}{2}\right)_{n-1} \left(\frac{\mu}{2} + n\right)_{\lfloor (n+1)/2 \rfloor}}{\binom{n}{n} \left(-\frac{\mu}{2} - 2n + \frac{3}{2}\right)_{\lfloor (n-1)/2 \rfloor}}$$

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with $\lambda := \mu - 2$.

Theorem. The following holds:

$$\begin{aligned} \frac{D_{1,1}(2n)}{D_{1,1}(2n-1)} &= (-1)^{\frac{(n-1)(n-2)}{2}} 2^n \frac{\left(\frac{\mu}{2} + 2n + \frac{1}{2}\right)_{n-1} \left(\frac{\mu}{2} + n\right)_{\lfloor (n+1)/2 \rfloor}}{\binom{n}{n} \left(-\frac{\mu}{2} - 2n + \frac{3}{2}\right)_{\lfloor (n-1)/2 \rfloor}} \\ &= 2^n \frac{\left(\frac{\mu}{2} + 2n + \frac{1}{2}\right)_{n-1} \left(\frac{\mu}{2} + n\right)_{\lfloor (n+1)/2 \rfloor}}{\binom{n}{n} \left(\frac{\mu}{2} + \lfloor \frac{3n}{2} \rfloor + \frac{1}{2}\right)_{\lfloor (n-1)/2 \rfloor}} \\ &= \frac{(\mu + 2n)_n \left(\frac{\mu}{2} + 2n + \frac{1}{2}\right)_{n-1}}{\binom{n}{n} \left(\frac{\mu}{2} + n + \frac{1}{2}\right)_{n-1}}. \end{aligned}$$

→ Proven in 2013 using computer algebra.

$$D_{1,1}(1) = \mu + 1$$

$$D_{1,1}(2) = (\mu + 1)(\mu + 2)$$

$$D_{1,1}(3) = \frac{1}{12}(\mu + 1)(\mu + 2)(\mu + 3)(\mu + 14)$$

$$D_{1,1}(4) = \frac{1}{72}(\mu + 1)(\mu + 2)(\mu + 3)(\mu + 4)(\mu + 9)(\mu + 14)$$

$$D_{1,1}(5) = \frac{1}{8640}(\mu + 1)(\mu + 2)(\mu + 3)(\mu + 4)(\mu + 5)(\mu + 9) \\ \times (\mu^3 + 45\mu^2 + 722\mu + 3432)$$

$$D_{1,1}(6) = \frac{1}{518400}(\mu + 1)(\mu + 2)(\mu + 3)(\mu + 4)(\mu + 5)(\mu + 6) \\ \times (\mu + 8)(\mu + 13)(\mu + 15)(\mu^3 + 45\mu^2 + 722\mu + 3432)$$

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Generalization of Andrews's Determinant

Definition: For $n, s, t \in \mathbb{Z}$, $n \geq 1$, and $\lambda := \mu - 2$ with μ being an indeterminate, we define $D_{s,t}(n)$ to be the following $(n \times n)$ -determinant:

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- ▶ monstrous conjecture for $D_{1,1}(n)$ (Christoph and Aek 2013)

Determinant with Kronecker-Delta

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where B_J^I denotes the matrix that is obtained by deleting all rows with indices in I and all columns with indices in J from the matrix

$$\left(\underbrace{\begin{pmatrix} \lambda + i + j + s + t - 2 \\ j + t - 1 \end{pmatrix}}_{b_{i,j,s,t}} \right)_{1 \leq i, j \leq n} .$$

Lindström-Gessel-Viennot Lemma

Consider 'base' and 'destination' vertices of a directed acyclic graph denoted by $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_n\}$, respectively.

Lindström-Gessel-Viennot Lemma

Consider 'base' and 'destination' vertices of a directed acyclic graph denoted by $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_n\}$, respectively. For each path P , let $\omega(P)$ be the product of its edge weights. Let

$$e(a, b) = \sum_{P:a \rightarrow b} \omega(P) \quad \text{and}$$
$$M = \begin{pmatrix} e(a_1, b_1) & e(a_1, b_2) & \cdots & e(a_1, b_n) \\ e(a_2, b_1) & e(a_2, b_2) & \cdots & e(a_2, b_n) \\ \vdots & \vdots & \ddots & \vdots \\ e(a_n, b_1) & e(a_n, b_2) & \cdots & e(a_n, b_n) \end{pmatrix}.$$

Lindström-Gessel-Viennot Lemma

Consider 'base' and 'destination' vertices of a directed acyclic graph denoted by $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_n\}$, respectively. For each path P , let $\omega(P)$ be the product of its edge weights. Let

$$e(a, b) = \sum_{P: a \rightarrow b} \omega(P) \quad \text{and}$$
$$M = \begin{pmatrix} e(a_1, b_1) & e(a_1, b_2) & \cdots & e(a_1, b_n) \\ e(a_2, b_1) & e(a_2, b_2) & \cdots & e(a_2, b_n) \\ \vdots & \vdots & \ddots & \vdots \\ e(a_n, b_1) & e(a_n, b_2) & \cdots & e(a_n, b_n) \end{pmatrix}.$$

Then the determinant of M is the signed sum over all n -tuples $P = (P_1, \dots, P_n)$ of non-intersecting paths from A to B :

$$\det(M) = \sum_{(P_1, \dots, P_n): A \rightarrow B} \text{sign}(\sigma(P)) \prod_{i=1}^n \omega(P_i).$$

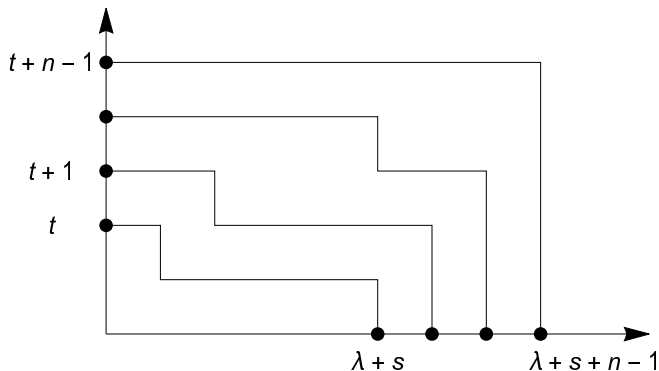
where σ denotes a permutation that is applied to B .

Lindström-Gessel-Viennot Lemma

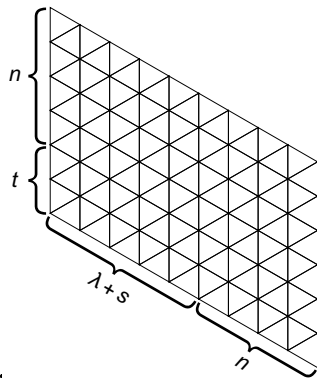
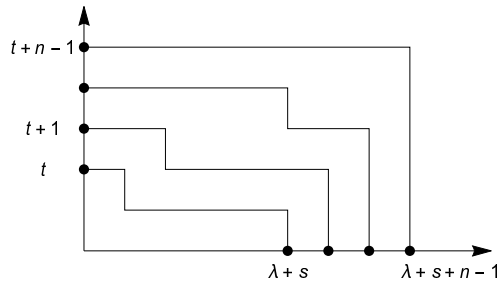
In our context, it implies that the determinant without the Kronecker-Delta

$$\det_{1 \leq i, j \leq n} \left(\binom{\lambda + i + j + s + t - 2}{j + t - 1} \right)$$

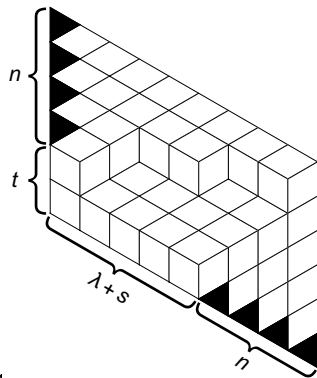
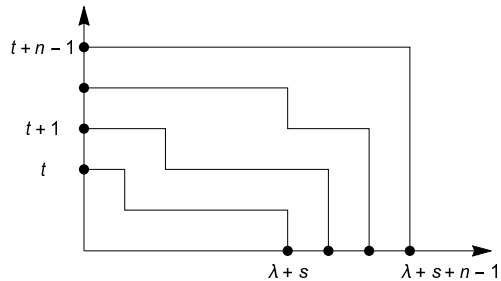
counts n -tuples of non-intersecting paths in the lattice \mathbb{N}^2 :



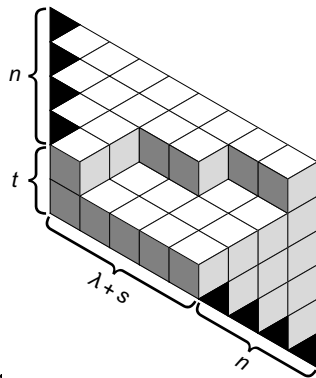
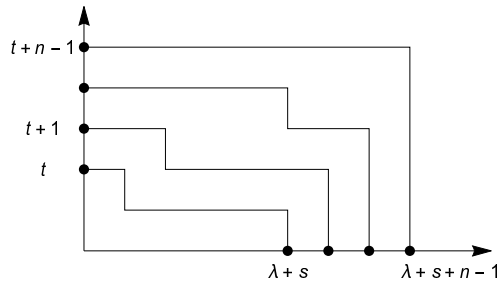
Lattice Paths \longrightarrow Rhombus Tilings



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Kronecker-Deltas on the Main Diagonal

If $s = t$, the previous formula for $D_{s,t}$ simplifies to

$$D_{s,s}(n) = \sum_{I \subseteq \{1, \dots, n\}} \det(B_I^I),$$

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Hence:

$D_{s,s}(n)$ counts all k -tuples of non-intersecting lattice paths, $k = 0, \dots, n$, and where the start and end points are given by the same k -subset.

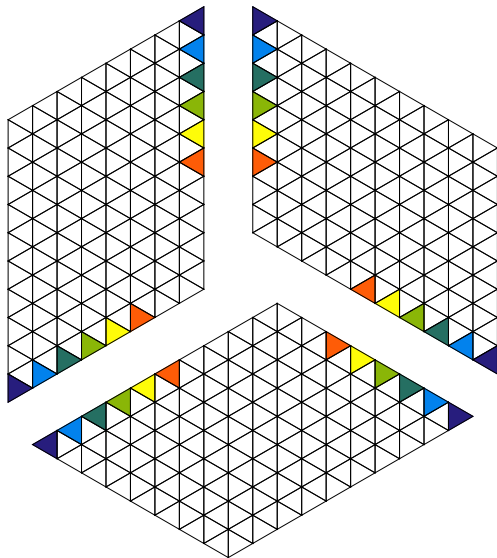
Kronecker-Deltas on the Main Diagonal

$$s = 2$$

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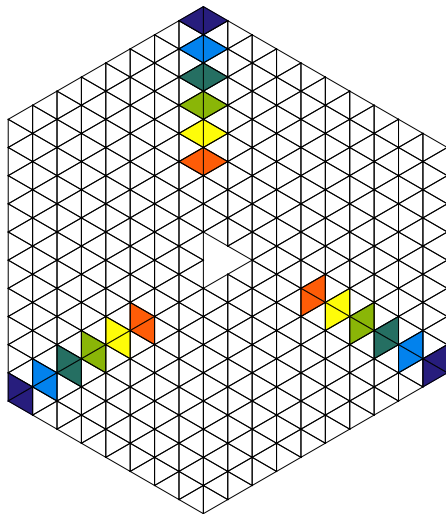
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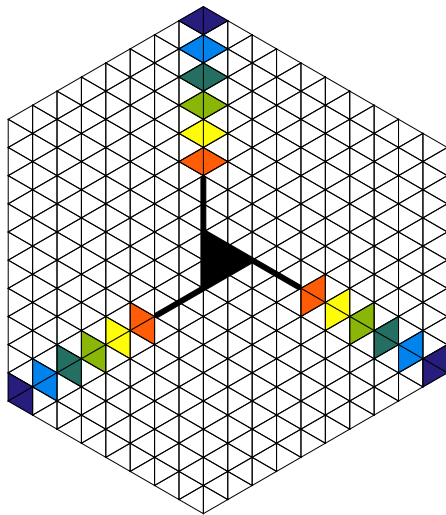
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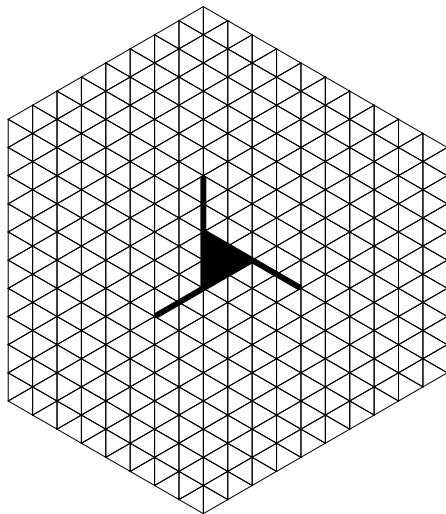
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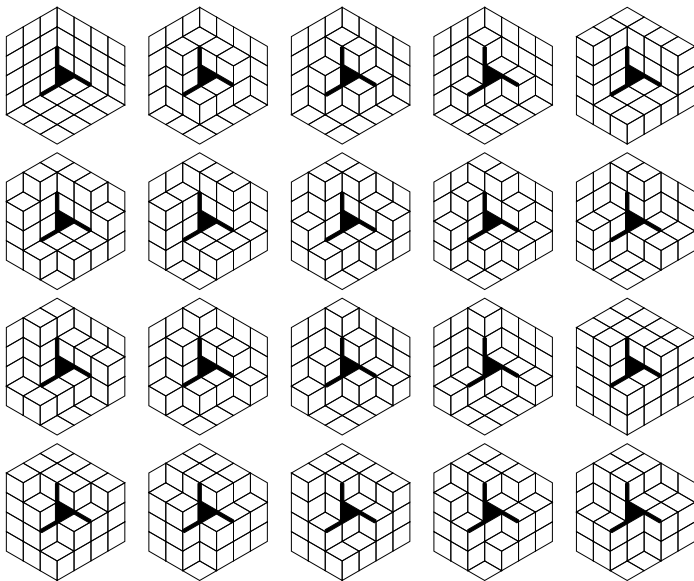
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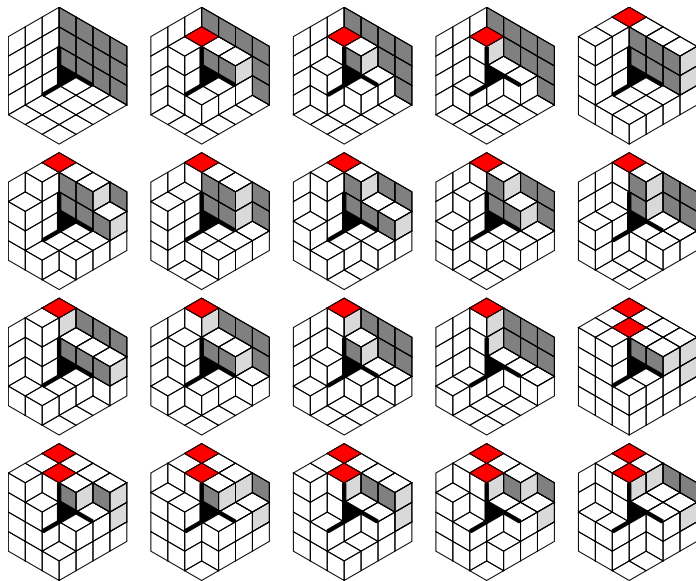
Example: For $s = t = 1$, $n = 2$, and $\lambda = 1$ we obtain

$$D_{1,1}(2)|_{\lambda \rightarrow 1} = \begin{vmatrix} 4 & 6 \\ 4 & 11 \end{vmatrix} = 20.$$

Cyclically Symmetric Rhombus Tilings of a Holey Hexagon



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$$F_m(n) = \left(\prod_{i=1}^{\lfloor \frac{1}{4}(n-1) \rfloor} (\mu + 2i + n + m)^{1-2i-m} \right) \\ \times \left(\prod_{i=1}^{\lfloor \frac{n}{4} - 1 \rfloor} (\mu - 2i + 2n - 2m + 1)^{1-2i-m} \right),$$

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$$F(n) = \begin{cases} E(n)F_0(n), & \text{if } n \text{ is even,} \\ E(n)F_1(n) \prod_{i=1}^{\frac{1}{2}(n-5)} (\mu + 2i + 2n - 1), & \text{if } n \text{ is odd,} \end{cases}$$

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$$\begin{aligned} T(k) = & 55296k^6 + 41472(\mu - 1)k^5 + 384(30\mu^2 - 66\mu + 53)k^4 \\ & + 96(\mu - 1)(15\mu^2 - 42\mu + 61)k^3 \\ & + 4(19\mu^4 - 122\mu^3 + 419\mu^2 - 544\mu + 72)k^2 \\ & + (\mu - 1)(\mu^4 - 14\mu^3 + 101\mu^2 - 160\mu - 84)k \\ & + 2(\mu - 3)(\mu - 2)(\mu - 1)(\mu + 1), \end{aligned}$$

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$$S_1(n) = \sum_{k=1}^{n-1} \left(2^{6k} (\mu + 8k - 1) \left(\frac{1}{2}\right)_{2k-1}^2 \left(\frac{1}{2}(\mu + 5)\right)_{2k-3} \right. \\ \left. \times \left(\frac{1}{2}(\mu + 4k + 2)\right)_{k-2} \left(\frac{1}{2}(\mu + 4k + 2)\right)_{2n-2k-2} T(k) \right) \\ \left/ \left((2k)! \left(\frac{1}{2}(\mu + 6k - 3)\right)_{3k+4} \right), \right.$$
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Desnanot-Jacobi-Dodgson Identity (DJD)

Theorem. Let $(m_{i,j})_{i,j \in \mathbb{Z}}$ be an infinite sequence and denote by $M_{s,t}(n)$ the determinant of the $(n \times n)$ -matrix whose upper left entry is $m_{s,t}$, more precisely the matrix $(m_{i,j})_{s \leq i < s+n, t \leq j < t+n}$. Then:

$$M_{s,t}(n)M_{s+1,t+1}(n-2) = \\ M_{s,t}(n-1)M_{s+1,t+1}(n-1) - M_{s+1,t}(n-1)M_{s,t+1}(n-1).$$

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Schematically:

The diagram illustrates the Desnanot-Jacobi-Dodgson identity using matrix diagrams. It shows the product of two matrices, where the second matrix has a shaded border, equal to the difference of two products of matrices. In the first product, the first matrix has a shaded bottom row and the second matrix has a shaded right column. In the second product, the first matrix has a shaded bottom row and the second matrix has a shaded left column. In the third product, the first matrix has a shaded left column and the second matrix has a shaded right column.

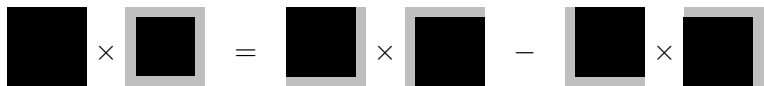
(DJD) for $D_{1,1}(n)$

$$\blacksquare \times \square = \square \times \square - \square \times \square$$

By (DJD) we obtain a recurrence equation for $D_{1,1}(n)$:

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We rewrite it slightly:

$$D_{1,1}(n) = \underbrace{\frac{D_{0,0}(n+1)}{D_{0,0}(n)}}_{= R_{0,0}(n)} D_{1,1}(n-1) + \frac{D_{1,0}(n)D_{0,1}(n)}{D_{0,0}(n)}.$$

→ Hence we need to know $D_{1,0}(n)$ and $D_{0,1}(n)$.

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- ▶ Use the holonomic systems approach to prove

$$M^{(2n)} \cdot c_n = 0, \text{ i.e., } \sum_{j=1}^{2n} M_{i,j}^{(2n)} c_{n,j} = 0 \text{ for all } i \text{ and } n.$$

Other Determinants

We obtain product formulas for $D_{1,0}(2n-1)$ and $D_{0,1}(2n-1)$, by using a variant of Zeilberger's "HOLONOMIC ANSATZ".

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$$\begin{aligned}c_{n,n} &= 1 && (n \geq 1), \\ \sum_{j=1}^n c_{n,j} a_{i,j} &= 0 && (1 \leq i < n), \\ \sum_{j=1}^n c_{n,j} a_{n,j} &= \frac{d_n}{d_{n-1}} && (n \geq 1).\end{aligned}$$

Then $\det (a_{i,j})_{1 \leq i,j \leq n} = d_n$ holds.

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Example: For $D_{0,0}(2n)$ we obtain the following holonomic system of recurrence relations for $c_{n,j}$.

Recurrence Equations for $c_{n,j}$ for $D_{0,0}(2n)$

$$\begin{aligned} & \{ (j + \mu + 2n - 3)(2\mu j^6 + 8n j^6 - 2j^6 + 3\mu^2 j^5 - 48n^2 j^5 - 12\mu j^5 - 24n j^5 + 9j^5 + \\ & \mu^3 j^4 + 48n^3 j^4 - 11\mu^2 j^4 - 84\mu n^2 j^4 + 204n^2 j^4 + 21\mu j^4 - 20\mu^2 n j^4 + 38\mu n j^4 - \\ & 10n j^4 - 11j^4 + 216n^4 j^3 - 2\mu^3 j^3 + 312\mu n^3 j^3 - 408n^3 j^3 + 7\mu^2 j^3 + 28\mu^2 n^2 j^3 + \\ & 122\mu n^2 j^3 - 198n^2 j^3 - 2\mu j^3 - 9\mu^3 n j^3 + 68\mu^2 n j^3 - 113\mu n j^3 + 78n j^3 - 3j^3 - \\ & 864n^5 j^2 - 756\mu n^4 j^2 + 432n^4 j^2 - \mu^3 j^2 - 112\mu^2 n^3 j^2 - 308\mu n^3 j^2 + 600n^3 j^2 + \\ & 11\mu^2 j^2 - 3\mu^3 n^2 j^2 - 66\mu^2 n^2 j^2 + 189\mu n^2 j^2 - 168n^2 j^2 - 23\mu j^2 - 2\mu^4 n j^2 + \\ & 15\mu^3 n j^2 - 28\mu^2 n j^2 + 33\mu n j^2 - 34n j^2 + 13j^2 + 864n^6 j + 432\mu n^5 j + 432n^5 j - \\ & 144\mu^2 n^4 j + 1116\mu n^4 j - 1104n^4 j + 2\mu^3 j - 88\mu^3 n^3 j + 384\mu^2 n^3 j - 392\mu n^3 j - \\ & 36n^3 j - 10\mu^2 j - 14\mu^4 n^2 j + 45\mu^3 n^2 j + 40\mu^2 n^2 j - 317\mu n^2 j + 270n^2 j + 14\mu j - \\ & \mu^5 n j + 3\mu^4 n j + 17\mu^3 n j - 89\mu^2 n j + 112\mu n j - 42n j - 6j + 432\mu n^6 - 864n^6 + \\ & 432\mu^2 n^5 - 1080\mu n^5 + 432n^5 + 144\mu^3 n^4 - 324\mu^2 n^4 - 156\mu n^4 + 456n^4 + 20\mu^4 n^3 - \\ & 18\mu^3 n^3 - 220\mu^2 n^3 + 470\mu n^3 - 204n^3 + \mu^5 n^2 + 3\mu^4 n^2 - 37\mu^3 n^2 + 57\mu^2 n^2 + \\ & 36\mu n^2 - 60n^2 + 2\mu^4 n - 18\mu^3 n + 54\mu^2 n - 62\mu n + 24n) \mathbf{c}_{n,j} - (j + \mu - 3)(2j + \mu - \\ & 3)(j - 2n + 1)(\mu + 4n - 1)(j^4 + 2\mu j^3 - 6j^3 + \mu^2 j^2 - 12n^2 j^2 - 9\mu j^2 - 6\mu n j^2 + \\ & 6n j^2 + 13j^2 - 3\mu^2 j - 12\mu n^2 j + 36n^2 j + 13\mu j - 6\mu^2 n j + 24\mu n j - 18n j - 12j + \\ & 2\mu^2 - 2\mu^2 n^2 + 20\mu n^2 - 24n^2 - 6\mu - \mu^3 n + 11\mu^2 n - 22\mu n + 12n + 4) \mathbf{c}_{n,j+1} + \\ & 2(2j + \mu - 2)n(2n + 1)(-j + 2n + 1)(-j + 2n + 2)(j + \mu + 2n - 1)(\mu + 4n - 3)(\mu + \\ & 4n - 1) \mathbf{c}_{n+1,j}, -(j + 1)(2j + \mu)(j - 2n)(j + \mu + 2n - 3) \mathbf{c}_{n,j} + (4j^4 + 8\mu j^3 - 8j^3 + \\ & 5\mu^2 j^2 - 8n^2 j^2 - 5\mu j^2 - 4\mu n j^2 + 12n j^2 - 8j^2 + \mu^3 j + 2\mu^2 j - 8\mu n^2 j + 8n^2 j - 15\mu j - \\ & 4\mu^2 n j + 16\mu n j - 12n j + 12j + \mu^3 - 3\mu^2 - 2\mu^2 n^2 + 16n^2 - 2\mu - \mu^3 n + 3\mu^2 n + \\ & 8\mu n - 24n + 8) \mathbf{c}_{n,j+1} - (j + \mu - 2)(2j + \mu - 2)(j - 2n + 2)(j + \mu + 2n - 1) \mathbf{c}_{n,j+2} \} \end{aligned}$$

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 & 4n - 1) \mathbf{c}_{n+1,j}, - (j + 1)(2j + \mu)(j - 2n)(j + \mu + 2n - 3) \mathbf{c}_{n,j} + (4j^4 + 8\mu j^3 - 8j^3 + \\
 & 5\mu^2 j^2 - 8n^2 j^2 - 5\mu j^2 - 4\mu n j^2 + 12n j^2 - 8j^2 + \mu^3 j + 2\mu^2 j - 8\mu n^2 j + 8n^2 j - 15\mu j - \\
 & 4\mu^2 n j + 16\mu n j - 12n j + 12j + \mu^3 - 3\mu^2 - 2\mu^2 n^2 + 16n^2 - 2\mu - \mu^3 n + 3\mu^2 n + \\
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 \end{aligned}$$

Back to $D_{1,1}(n)$

By (DJD) we had the recurrence

$$D_{1,1}(n) = R_{0,0}(n)D_{1,1}(n-1) + \frac{D_{1,0}(n)D_{0,1}(n)}{D_{0,0}(n)}.$$

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$$= R_{0,0}(n)D_{1,1}(n-1) + (\mu - 1) \frac{\left(\prod_{j=1}^{\frac{n-1}{2}} R_{1,0}(j)\right) \left(\prod_{j=1}^{\frac{n-1}{2}} R_{0,1}(j)\right)}{2 \prod_{j=1}^{n-1} R_{0,0}(j)}$$

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Main Result

Theorem. Let μ be an indeterminate and let ρ_k be defined as $\rho_0(a, b) = a$ and $\rho_k(a, b) = b$ for $k > 0$.

If n is an odd positive integer then

$$\begin{aligned} D_{1,1}(n) = & \sum_{k=0}^{(n+1)/2} \rho_k \left(4(\mu - 2), \frac{1}{(2k-1)!} \right) \frac{(\mu - 1)_{3k-2}}{2 \left(\frac{\mu}{2} + k - \frac{1}{2} \right)_{k-1}} \\ & \times \left(\prod_{j=1}^{k-1} \frac{(\mu + 2j + 1)_{j-1} \left(\frac{\mu}{2} + 2j + \frac{1}{2} \right)_{j-1}}{(j)_{j-1} \left(\frac{\mu}{2} + j + \frac{1}{2} \right)_{j-1}} \right)^2 \\ & \times \left(\prod_{j=k}^{(n-1)/2} \frac{(\mu + 2j)_j^2 \left(\frac{\mu}{2} + 2j - \frac{1}{2} \right)_j \left(\frac{\mu}{2} + 2j + \frac{3}{2} \right)_{j+1}}{(j)_j (j+1)_{j+1} \left(\frac{\mu}{2} + j + \frac{1}{2} \right)_j^2} \right) \end{aligned}$$

If n is an even positive integer then... [similar formula]

Off-Diagonal Kronecker-Deltas

Now let's look at the situation $s \neq t$.

General formula:

$$D_{s,t}(n) = \begin{cases} \sum_{I \subseteq \{1, \dots, n-s+t\}} (-1)^{(s-t) \cdot |I|} \det(B_{I+s-t}^I) & \text{if } s \geq t \\ \sum_{I \subseteq \{1, \dots, n-t+s\}} (-1)^{(s-t) \cdot |I|} \det(B_I^{I+s-t}) & \text{if } s \leq t, \end{cases}$$

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Remark:

For a particular (cyclically symmetric) rhombus tiling, the number of unit segments which are not crossed by a horizontal rhombus corresponds to the cardinality of the set I and hence its parity determines whether this tiling is counted with weight $+1$ or with weight -1 .

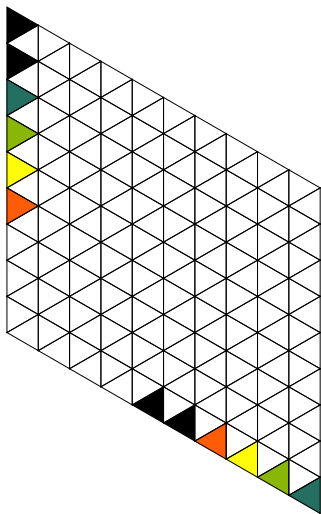
Off-Diagonal Kronecker-Deltas

$$s = 1$$

$$t = 3$$

$$n = 6$$

$$\lambda = 3$$



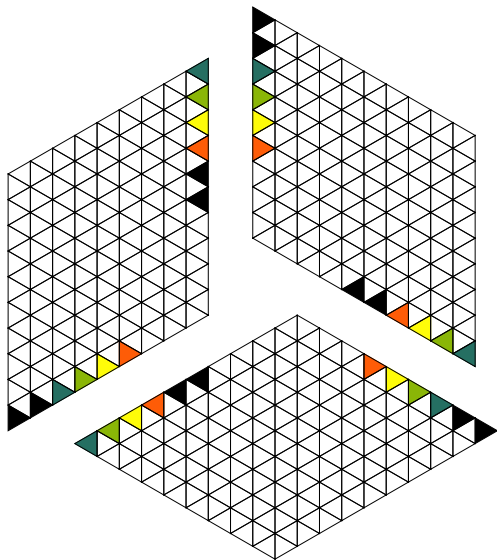
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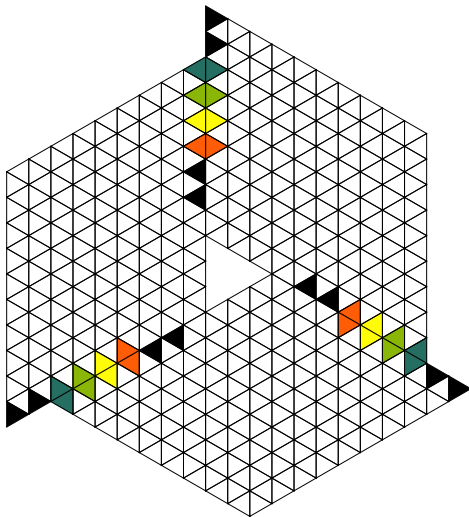
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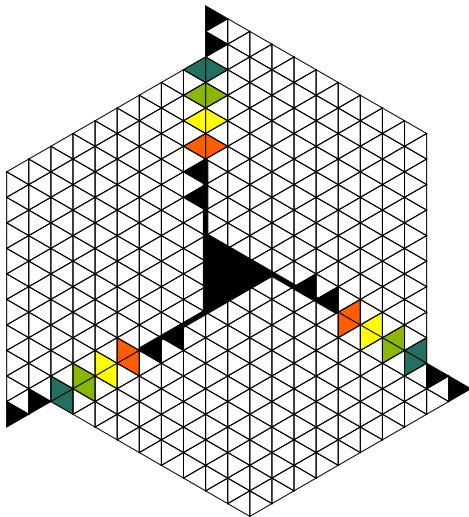
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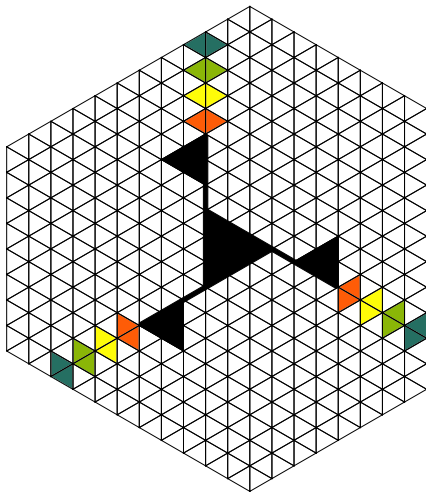
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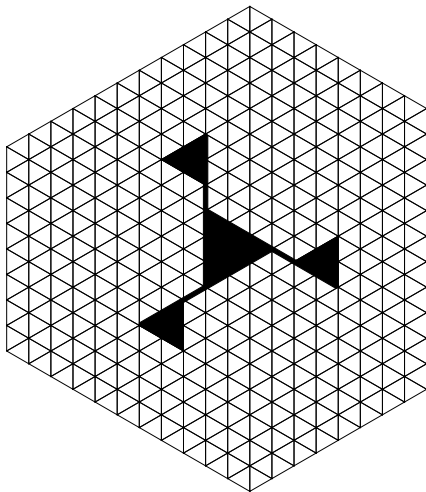
Off-Diagonal Kronecker-Deltas

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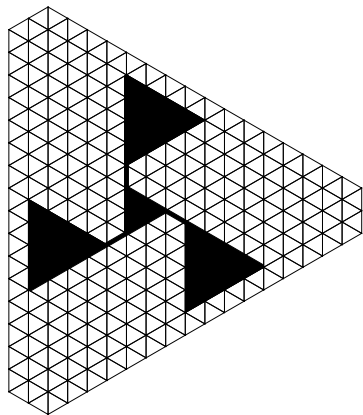
$$n = 6$$

$$\lambda = 3$$

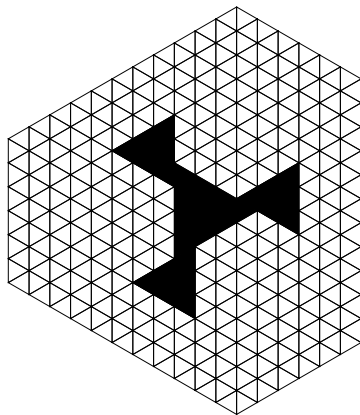


Off-Diagonal Kronecker-Deltas

Example: Shapes for different choices parameters.



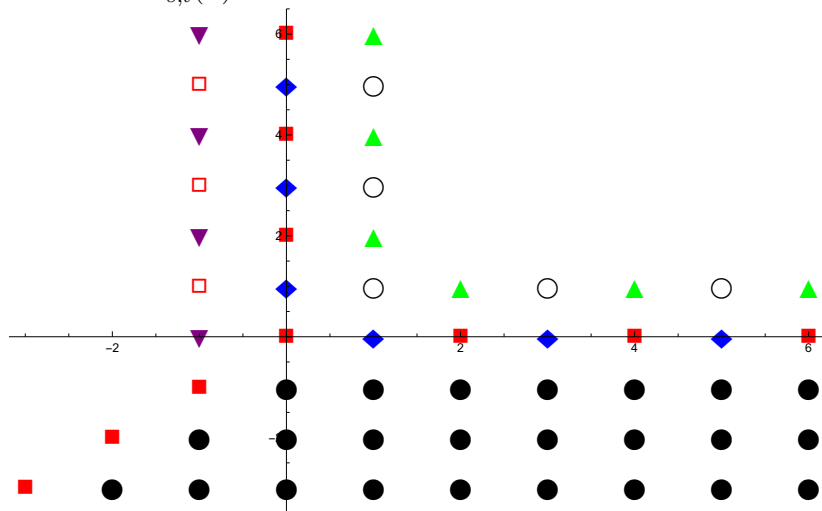
$$s = 5, t = 1, n = 5, \lambda = 2$$



$$s = -1, t = 2, n = 6, \lambda = 4$$

More Results

We find closed-form evaluations of some infinite 1-dimensional families of $D_{s,t}(n)$.



Example of an Infinite Family (A): Red Squares

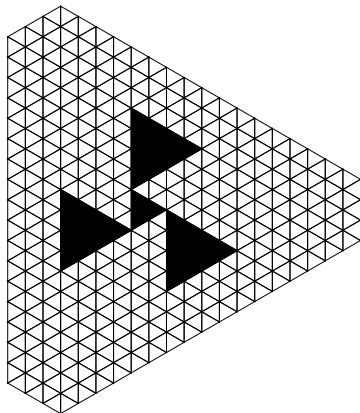
Family A: can be reduced to the base case $D_{0,0}(n)$:

$$D_{2r,0}(n) = D_{0,0}(n - 2r) \Big|_{\mu \rightarrow \mu + 6r}$$

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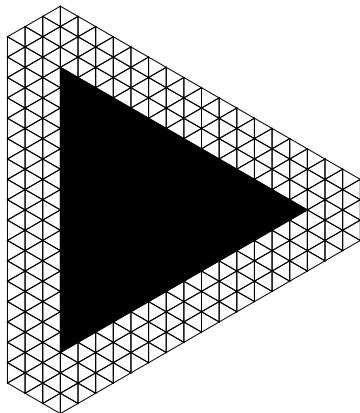
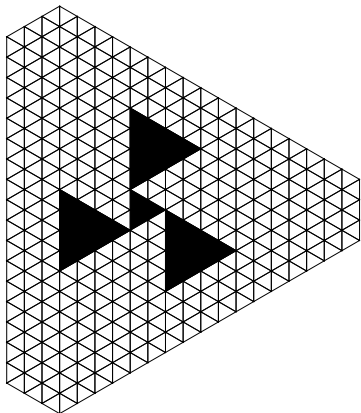
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Example of an Infinite Family (B): Blue Diamonds

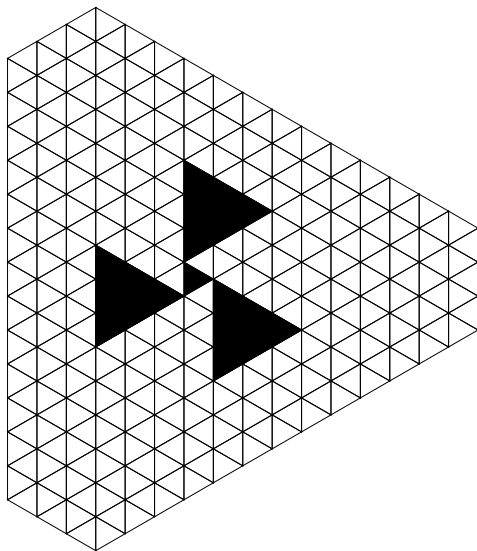
Family B: If $n \geq 2r$ is an even number, then $D_{2r-1,0}(n) = 0$.

$$s = 3$$

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$$n = 6$$

$$\lambda = 1$$



Example of an Infinite Family (B): Blue Diamonds

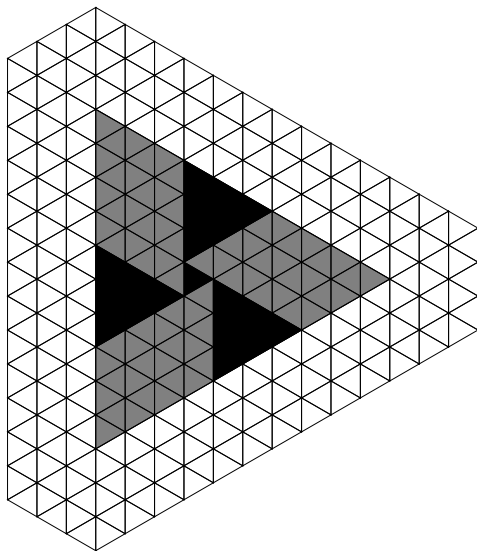
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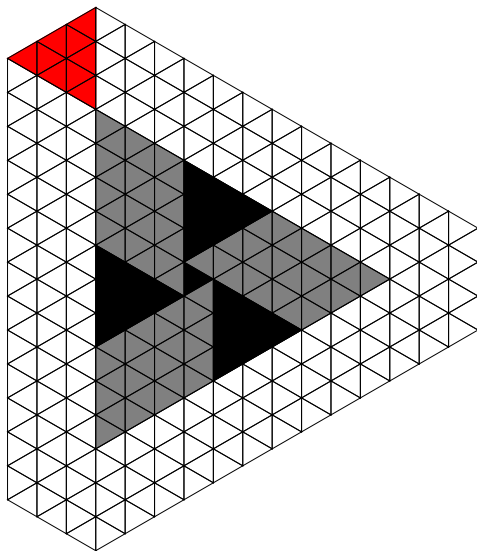
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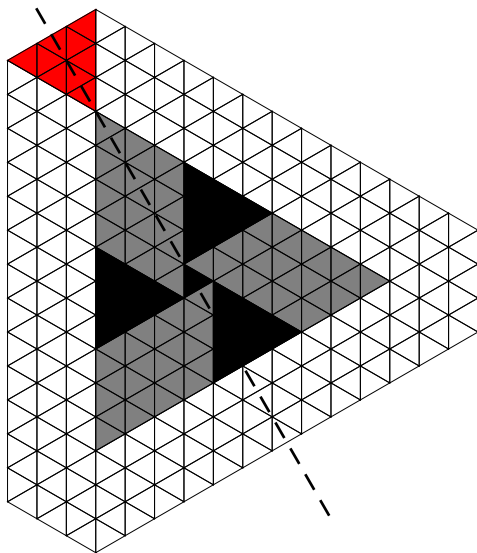
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Example of an Infinite Family (B): Blue Diamonds

Theorem. Let μ be an indeterminate, and let r and n be positive integers. If n is an odd number, then

$$D_{2r-1,0}(n) = \prod_{i=r}^{(n-1)/2} (-R_{2r-1,0}(i)),$$

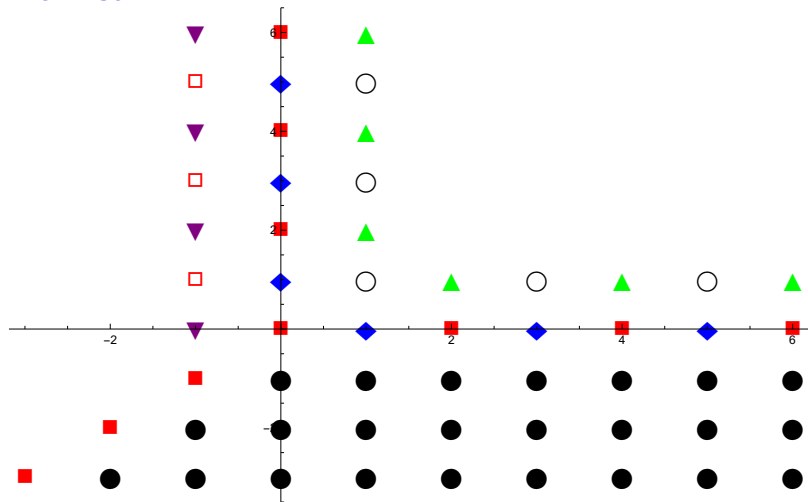
where $R_{2r-1,0}(n) =$

$$\frac{(\mu + 2n + 4r - 4)_{n-r+1} (\mu + 2n + 4r - 3)_{n-r} \left(\frac{\mu}{2} + 2n + r - \frac{1}{2}\right)_{n-r}^2}{(n-r+1)_{n-r+1} (n-r+1)_{n-r} \left(\frac{\mu}{2} + n + 2r - \frac{3}{2}\right)_{n-r}^2}$$

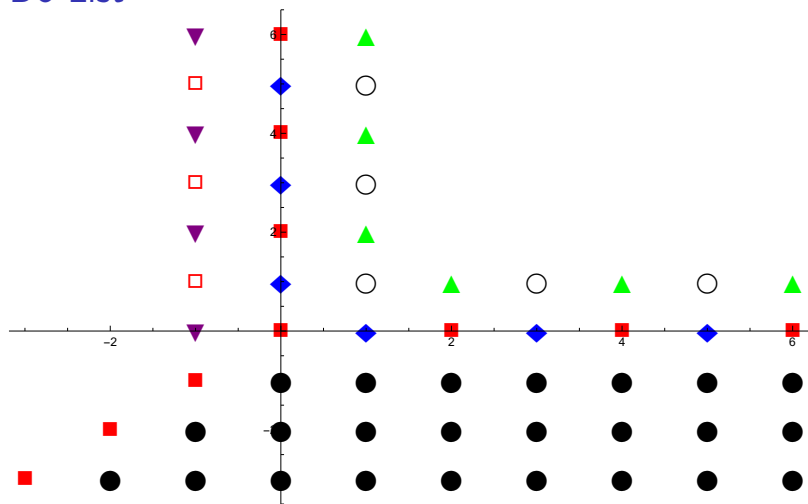
i.e., $R_{2r-1,0}(n) = D_{2r-1,0}(2n+1)/D_{2r-1,0}(2n-1)$ for $n \geq r$.
If $n \geq 2r$ is an even number, then $D_{2r-1,0}(n) = 0$. Moreover,

$$D_{0,2r-1}(n) = \left(\prod_{i=0}^{n-1} \frac{(\mu + i - 1)_{2r-1}}{(i+1)_{2r-1}} \right) \cdot D_{2r-1,0}(n).$$

To Do List

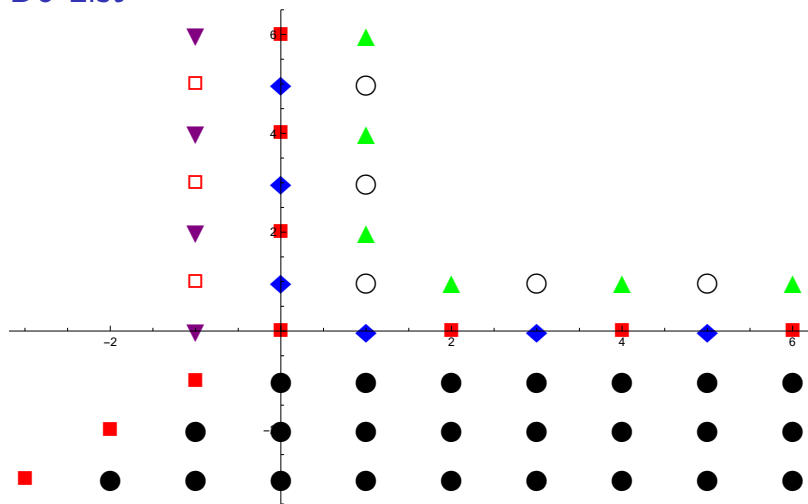


To Do List



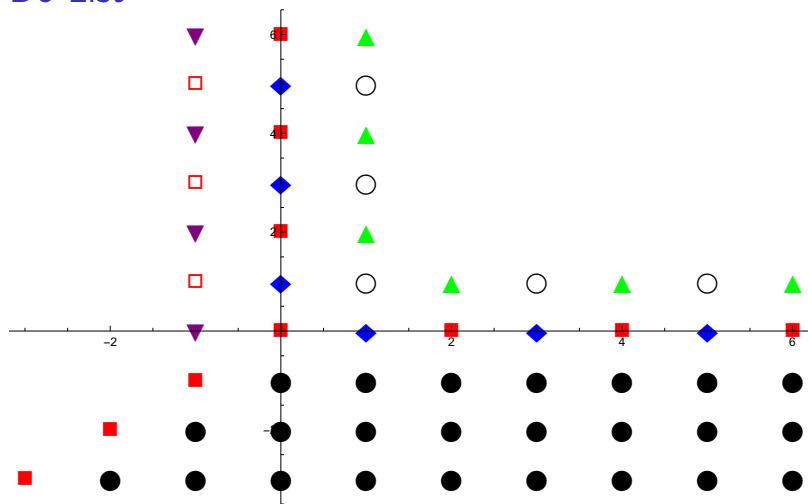
✓ Done: Red Squares and Blue Diamonds (Combinatorics)

To Do List



- ✓ Done: Red Squares and Blue Diamonds (Combinatorics)
- ✓ Follows: Hollow Squares and Hollow Circles (DJD)

To Do List



- ✓ Done: Red Squares and Blue Diamonds (Combinatorics)
- ✓ Follows: Hollow Squares and Hollow Circles (DJD)
- ✓ To Do: Purple and Green Triangles

Reference

Christoph Koutschan and Thotsaporn Thanatipanonda:
*A Curious Family of Binomial Determinants That Count Rhombus
Tilings of a Holey Hexagon*

- ▶ arxiv:1709.02616
- ▶ <http://www.koutschan.de/data/det2/>