Project Part 11
Certificate-Free Summation and Integration

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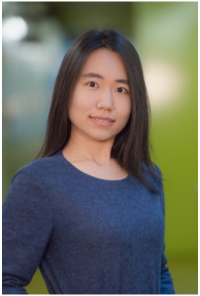
SFB Status Seminar

Project Part 11

- project joined the SFB in the second period 2017–2021
- original plan: 1 PhD Student + self-employment
- Y.A. defended in 09/2020 (co-supervised with J.-M. Maillard)

Youssef Abdelaziz

Elaine Wong



Hao Du

Ali Uncu

Selection of Topics and Achievements

- reduction-based creative telescoping
- additive decompositions in logarithmic towers
- diagonals of rational functions
- lower bounds for monochromatic Schur triples
- symbolic determinants and rhombus tilings
- enumeration of DSASMs
- partition analysis and q-series
- symbolic summation applied to quasi-Monte-Carlo methods

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Diagonals of Rational Functions

Given a rational function in n variables

$$R(x_1, \ldots, x_n) = \frac{A(x_1, \ldots, x_n)}{B(x_1, \ldots, x_n)},$$

where $A, B \in \mathbb{Q}[x_1, \ldots, x_n]$ such that $B(0, \ldots, 0) \neq 0$.

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Definition: The diagonal of R is defined through its multi-Taylor expansion around $(0, \ldots, 0)$

$$R(x_1, \ldots, x_n) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} r_{m_1,\ldots,m_n} \cdot x_1^{m_1} \cdots x_n^{m_n}$$

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as the power series in one variable:

$$\mathrm{Diag}\big(R(x_1, \ldots, x_n)\big) := \sum_{m=0}^{\infty} r_{m,\ldots,m} \cdot x^m$$

Example of a Diagonal

Consider the Taylor expansion of the bivariate rational function

$$
\begin{aligned}
f(x,y) &= \frac{1}{1-x-y-2xy} \\
&= 1 + x + y + x^2 + 4xy + y^2 + x^3 + 7x^2y + 7xy^2 + \ldots
\end{aligned}
$$

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$$\begin{array}{lllllll} = 1 & + \; y & + \; y^2 & + \; y^3 & + \; y^4 & + \; y^5 & + \dots \\ + \; x & + \; 4xy & + \; 7xy^2 & + \; 10xy^3 & + \; 13xy^4 & + \; 16xy^5 & + \dots \\ + \; x^2 & + \; 7x^2y & + \; 22x^2y^2 & + \; 46x^2y^3 & + \; 79x^2y^4 & + \; 121x^2y^5 & + \dots \\ + \; x^3 & + \; 10x^3y & + \; 46x^3y^2 & + \; 136x^3y^3 & + \; 307x^3y^4 & + \; 586x^3y^5 & + \dots \\ + \; x^4 & + \; 13x^4y & + \; 79x^4y^2 & + \; 307x^4y^3 & + \; 886x^4y^4 & + \; 2086x^4y^5 & + \dots \\ + \; x^5 & + \; 16x^5y & + \; 121x^5y^2 & + \; 586x^5y^3 & + \; 2086x^5y^4 & + \; 5944x^5y^5 & + \dots \end{array}$$

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Then the diagonal of f is

$$\mathrm{Diag}(f) = 1 + 4x + 22x^2 + 136x^3 + 886x^4 + 5944x^5 + \dots$$

Properties of Diagonals

Theorem: The diagonal $f(x)$ of every rational function is

- **globally bounded:** there exist integers $c, d \in \mathbb{N}^*$, such that $d \cdot f(cx) \in \mathbb{Z}[[x]]$ and $f(x)$ has nonzero radius of convergence.

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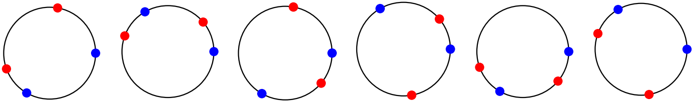
- This conjecture was first formulated in a paper in 1986 and it is still widely open.
- It doesn't say anything about the number of variables in the rational function.
- One needs at least three variables, but no explicit example requiring more than three variables is known.

Hypergeometric Functions

The ${}_pF_q$ functions provide a natural testing ground for CC:

- Any hypergeometric ${}_pF_q$ function is D-finite.
- Criterion for global boundedness, e.g. for ${}_2F_1\left(\left[\frac{2}{9}, \frac{5}{9}\right], \left[\frac{2}{3}\right], x\right)$:



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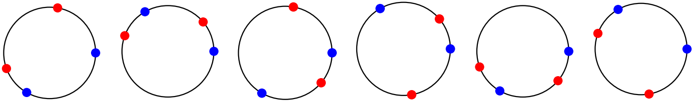

We can restrict to hypergeometric functions of the form ${}_pF_{p-1}$:

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- If $q > p - 1$ then the ${}_pF_q$ series cannot be globally bounded.

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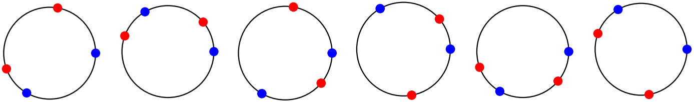

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Many globally bounded ${}_pF_{p-1}$'s can easily be shown to be diagonals:

- Christol's conjecture holds for all ${}_2F_1$'s.
- Some ${}_3F_2$'s are seen to be diagonals by Hadamard factorization.

Potential Counterexamples

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- Christol came up with an "unresolved example":

$$ {}_3F_2\left(\left[\tfrac{1}{9}, \tfrac{4}{9}, \tfrac{5}{9}\right], \left[\tfrac{1}{3}, 1\right], 27x\right) $$

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Examples: The following two functions are globally bounded:

$$ {}_3F_2\left(\left[\tfrac{2}{9}, \tfrac{5}{9}, \tfrac{8}{9}\right], \left[\tfrac{2}{3}, 1\right], 3^6x\right) = 1 + 120x + 47124x^2 + \dots $$

$$ {}_3F_2\left(\left[\tfrac{1}{9}, \tfrac{4}{9}, \tfrac{7}{9}\right], \left[\tfrac{1}{3}, 1\right], 3^6x\right) = 1 + 84x + 32760x^2 + \dots $$

but are they diagonals of rational functions???

Main Result

The hypergeometric function

$${}_3F_2\Big(\Big[\frac{3a-b}{3a}, \frac{2a-b}{3a}, \frac{a-b}{3a}\Big], \Big[\frac{a-b}{a}, 1\Big], 27x\Big).$$

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The following six-variable rational function witnesses this fact:

$$ \begin{aligned} 1 &+ \frac{au^3v\,(1-ux-uy-uz)\,(1+u)^{a-1}(1-ux-uy-uz)^{a-1}}{(1+u)^a(1-ux-uy-uz)^a-(1-ux-uy)^b(u-v)(v-w)} \\ &- \frac{av^4\,(1-vx-vy-vz)\,(1+v)^{a-1}(1-vx-vy-vz)^{a-1}}{(1+v)^a(1-vx-vy-vz)^a-(1-vx-vy)^b(u-v)(v-w)} \\ &- \frac{au^3w\,(1-ux-uy-uz)\,(1+u)^{a-1}(1-ux-uy-uz)^{a-1}}{(1+u)^a(1-ux-uy-uz)^a-(1-ux-uy)^b(u-w)(v-w)} \\ &- \frac{aw^4\,(1-wx-wy-wz)\,(1+w)^{a-1}(1-wx-wy-wz)^{a-1}}{(1+w)^a(1-wx-wy-wz)^a-(1-wx-wy)^b(u-w)(v-w)} \end{aligned} $$

Main Result

Theorem: The hypergeometric functions

$$ {}_3F_2\left(\left[\tfrac{2}{9}, \tfrac{5}{9}, \tfrac{8}{9}\right], \left[\tfrac{2}{3}, 1\right], 27x\right) \quad \text{and} \quad {}_3F_2\left(\left[\tfrac{1}{9}, \tfrac{4}{9}, \tfrac{7}{9}\right], \left[\tfrac{1}{3}, 1\right], 27x\right) $$

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More precisely, we have:

$$ {}_3F_2\left(\left[\tfrac{2}{9}, \tfrac{5}{9}, \tfrac{8}{9}\right], \left[\tfrac{2}{3}, 1\right], 27x\right) = \mathrm{Diag}\left(\frac{(1-x-y)^{1/3}}{1-x-y-z}\right), $$

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More generally, $\mathrm{Diag}\left(\frac{(1-x-y)^{a/b}}{1-x-y-z}\right)$ is shown to evaluate to

$$ {}_3F_2\left(\left[\frac{3a-b}{3a},\frac{2a-b}{3a},\frac{a-b}{3a}\right],\left[\frac{a-b}{a},1\right],27x\right). $$

Binomial Determinants

Definition: For $n \in \mathbb{N}$, for $s, t \in \mathbb{Z}$, and for μ an indeterminate, we define $D_{s,t}(n)$ to be the following $(n \times n)$-determinant:

$$D_{s,t}(n) := \det_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant n}} \left(\delta_{i+s,j+t} + \binom{\mu + i + j + s + t - 4}{j + t - 1} \right).$$

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Example:

$$D_{4,6}(5) = \begin{vmatrix} \binom{8+\mu}{6} & \binom{9+\mu}{7} & \binom{10+\mu}{8} & \binom{11+\mu}{9} & \binom{12+\mu}{10} \\ \binom{9+\mu}{6} & \binom{10+\mu}{7} & \binom{11+\mu}{8} & \binom{12+\mu}{9} & \binom{13+\mu}{10} \\ \binom{10+\mu}{6}+1 & \binom{11+\mu}{7} & \binom{12+\mu}{8} & \binom{13+\mu}{9} & \binom{14+\mu}{10} \\ \binom{11+\mu}{6} & \binom{12+\mu}{7}+1 & \binom{13+\mu}{8} & \binom{14+\mu}{9} & \binom{15+\mu}{10} \\ \binom{12+\mu}{6} & \binom{13+\mu}{7} & \binom{14+\mu}{8}+1 & \binom{15+\mu}{9} & \binom{16+\mu}{10} \end{vmatrix}$$

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History: The determinant $D_{0,0}(n)$ counts descending plane partitions and was first evaluated by George Andrews in 1979.

Conjecture for $D_{1,1}(n)$

$D_{1,1}(n) = C(n)\,F(n)\,G\big(\lfloor \tfrac{1}{2}(n+1)\rfloor\big)$, where

$$C(n) = \frac{(-1)^n+3}{2}\prod_{i=1}^{n}\frac{\lfloor \frac{i}{2}\rfloor!}{i!},$$

$$E(n) = (\mu+1)_n\left(\prod_{i=1}^{\lfloor \frac{3}{2}\lfloor \frac{1}{2}(n-1)\rfloor -2\rfloor}\Big(\mu+2i+6\Big)^{2\lfloor \frac{1}{3}(i+2)\rfloor}\right)\cdot\left(\prod_{i=1}^{\lfloor \frac{3}{2}\lfloor \frac{n}{2}\rfloor -2\rfloor}\Big(\mu+2i+2\left\lfloor \tfrac{3}{2}\left\lfloor \tfrac{n}{2}+1\right\rfloor\right\rfloor-1\Big)^{2\lfloor \frac{1}{2}\lfloor \frac{n}{2}\rfloor-\frac{1}{3}(i-1)\rfloor-1}\right),$$

$$F_m(n) = \left(\prod_{i=1}^{\lfloor \frac{1}{4}(n-1)\rfloor}(\mu+2i+n+m)^{1-2i-m}\right)\cdot\left(\prod_{i=1}^{\lfloor \frac{n}{4}-1\rfloor}(\mu-2i+2n-2m+1)^{1-2i-m}\right),$$

$$F(n) = \begin{cases} E(n)F_0(n), & \text{if } n \text{ is even},\\ E(n)F_1(n)\displaystyle\prod_{i=1}^{\frac{1}{2}(n-5)}(\mu+2i+2n-1), & \text{if } n \text{ is odd},\end{cases}$$

$$\begin{aligned}T(k) &= 55296k^6+41472(\mu-1)k^5+384(30\mu^2-66\mu+53)k^4+96(\mu-1)(15\mu^2-42\mu+61)k^3\\ &\quad+4(19\mu^4-122\mu^3+419\mu^2-544\mu+72)k^2+(\mu-1)(\mu^4-14\mu^3+101\mu^2-160\mu-84)k\\ &\quad+2(\mu-3)(\mu-2)(\mu-1)(\mu+1),\end{aligned}$$

$$S_1(n) = \sum_{k=1}^{n-1}\frac{2^{6k}(\mu+8k-1)\left(\frac{1}{2}\right)_{2k-1}^2\left(\frac{1}{2}(\mu+5)\right)_{2k-3}\left(\frac{1}{2}(\mu+4k+2)\right)_{k-2}\left(\frac{1}{2}(\mu+4k+2)\right)_{2n-2k-2}}{(2k)!\left(\frac{1}{2}(\mu+6k-3)\right)_{3k+4}}\cdot T(k),$$

$$S_2(n) = \sum_{k=1}^{n-1}\frac{2^{6k}(\mu+8k+3)\left(\frac{1}{2}\right)_{2k}^2\left(\frac{1}{2}(\mu+5)\right)_{2k-2}\left(\frac{1}{2}(\mu+4k+4)\right)_{k-2}\left(\frac{1}{2}(\mu+4k+4)\right)_{2n-2k-2}}{(2k+1)!\left(\frac{1}{2}(\mu+6k+1)\right)_{3k+5}}\cdot T\big(k+\tfrac{1}{2}\big),$$

$$P_1(n) = 2^{3n-1}\frac{\left(\frac{1}{2}(\mu+6n-3)\right)_{3n-2}}{\left(\frac{1}{2}(\mu+5)\right)_{2n-3}}\left(\frac{\left(\frac{1}{2}(\mu+2)\right)_{2n-2}}{(\mu+3)^2}+\frac{\mu(\mu-1)S_1(n)}{2^{13}}\right),$$

$$P_2(n) = 2^{3n-1}\frac{\left(\frac{1}{2}(\mu+6n+1)\right)_{3n-1}}{\left(\frac{1}{2}(\mu+5)\right)_{2n-2}}\left(\frac{(\mu+14)\left(\frac{1}{2}(\mu+4)\right)_{2n-2}}{(\mu+7)(\mu+9)}+\frac{\mu(\mu-1)S_2(n)}{2^{9}}\right),$$

$$G(n) = \begin{cases} P_1\big(\frac{1}{2}(n+1)\big), & \text{if } n \text{ is odd},\\ P_2\big(\frac{n}{2}\big), & \text{if } n \text{ is even}.\end{cases}$$

Theorem for $D_{1,1}(n)$

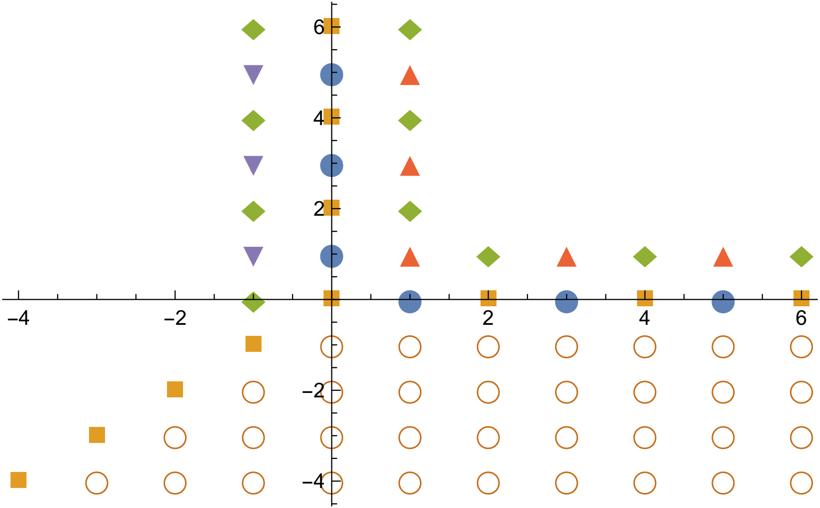
Let ρ_k be defined as $\rho_0(a,b) = a$ and $\rho_k(a,b) = b$ for $k > 0$.
If n is an odd positive integer then

$$D_{1,1}(n) = \sum_{k=0}^{(n+1)/2} \rho_k\left(4(\mu-2), \frac{1}{(2k-1)!}\right) \frac{(\mu-1)_{3k-2}}{2\left(\frac{\mu}{2}+k-\frac{1}{2}\right)_{k-1}} \left(\prod_{j=1}^{k-1} \frac{(\mu+2j+1)_{j-1}\left(\frac{\mu}{2}+2j+\frac{1}{2}\right)_{j-1}}{(j)_{j-1}\left(\frac{\mu}{2}+j+\frac{1}{2}\right)_{j-1}}\right)^2 \times \left(\prod_{j=k}^{(n-1)/2} \frac{(\mu+2j)_j^2\left(\frac{\mu}{2}+2j-\frac{1}{2}\right)_j\left(\frac{\mu}{2}+2j+\frac{3}{2}\right)_{j+1}}{(j)_j\,(j+1)_{j+1}\left(\frac{\mu}{2}+j+\frac{1}{2}\right)_j^2}\right).$$

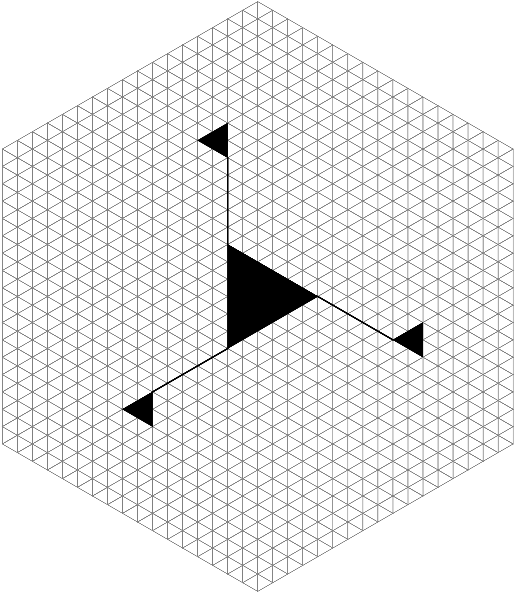
If n is an even positive integer then

$$D_{1,1}(n) = \sum_{k=0}^{n/2} \rho_k\left(4(\mu-2), \frac{1}{(2k-1)!}\right) \frac{(\mu-1)_{3k-2}}{2\left(\frac{\mu}{2}+k-\frac{1}{2}\right)_{k-1}} \left(\prod_{j=1}^{k-1} \frac{(\mu+2j+1)_{j-1}\left(\frac{\mu}{2}+2j+\frac{1}{2}\right)_{j-1}}{(j)_{j-1}\left(\frac{\mu}{2}+j+\frac{1}{2}\right)_{j-1}}\right)^2 \times \left(\prod_{j=k}^{n/2} \frac{(\mu+2j)_j\left(\frac{\mu}{2}+2j+\frac{1}{2}\right)_{j-1}}{(j)_j\left(\frac{\mu}{2}+j+\frac{1}{2}\right)_{j-1}}\right)\left(\prod_{j=k}^{n/2-1} \frac{(\mu+2j)_j\left(\frac{\mu}{2}+2j+\frac{3}{2}\right)_{j+1}}{(j+1)_{j+1}\left(\frac{\mu}{2}+j+\frac{1}{2}\right)_j}\right).$$

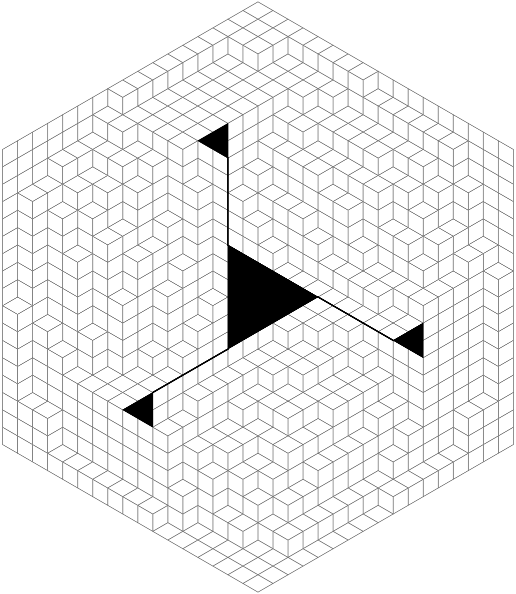
Families of Determinants



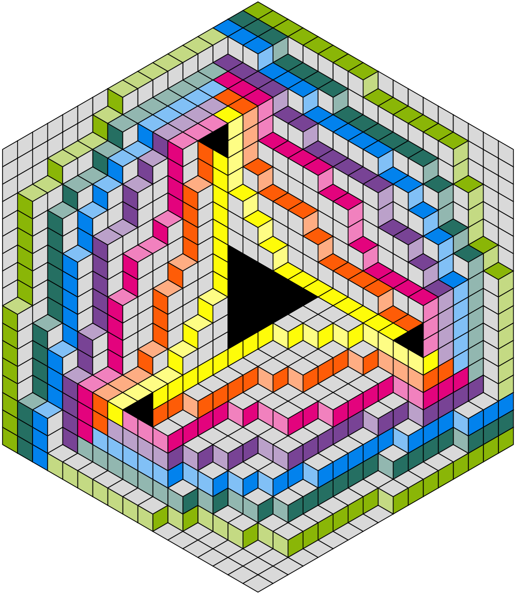
Combinatorial Interpretation: Holey Hexagon



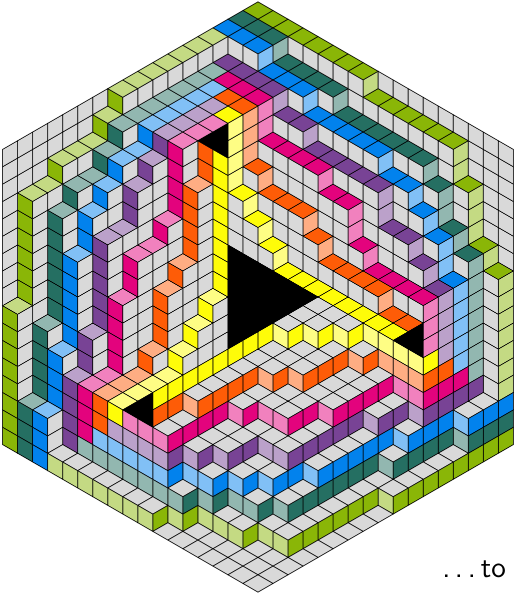
Combinatorial Interpretation: Holey Hexagon



Combinatorial Interpretation: Holey Hexagon



Combinatorial Interpretation: Holey Hexagon



. . . to be continued!