Proof of Ira Gessel's Lattice Path Conjecture

Christoph Koutschan
(joint work with M. Kauers and D. Zeilberger)

Research Institute for Symbolic Computation
Johannes Kepler Universität Linz, Austria

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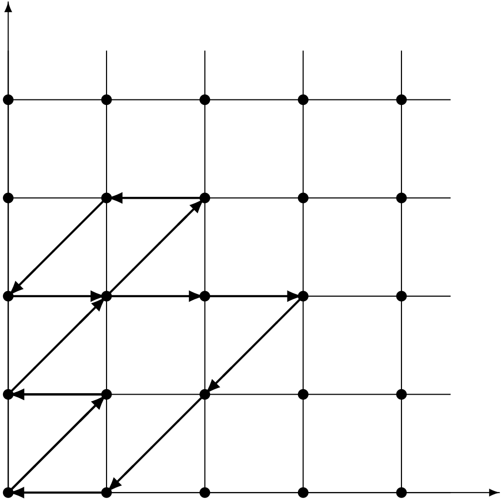
Gessel walks

- walks in the integer lattice $\mathbb{N}^2$
- start at $(0,0)$
- do not leave $\mathbb{N}^2$
- only certain steps are allowed:

$$\begin{aligned} G &:= \left\{ \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \\ &= \{\leftarrow, \rightarrow, \swarrow, \nearrow\} \end{aligned}$$

Gessel walks — Example



Definition

Let $f(n;i,j)$ denote the number of walks ("Gessel walks")

- in the integer lattice $\mathbb{N}^2$
- with exactly n steps
- starting at the origin $(0,0)$
- ending at the point (i,j)
- using only steps from G.

Gessel's conjecture

Ira Gessel (2001) conjectured that

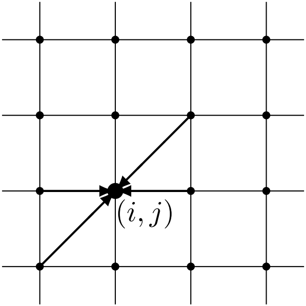
$$f(n;0,0) = \begin{cases} 16^k \frac{(5/6)_k (1/2)_k}{(2)_k (5/3)_k} & \text{if } n = 2k \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

The function $f(n;0,0)$ counts the number of closed Gessel walks.

Get ready for the proof!

Need: relations (linear recurrences with polynomial coefficients) for $f(n;i,j)$ The step set $\{\leftarrow, \rightarrow, \nearrow, \swarrow\}$ gives readily rise to the recurrence

$$
\begin{aligned}
f(n+1;i,j) = \\
&\quad f(n;i+1,j) \\
&+ \; f(n;i-1,j) \\
&+ \; f(n;i+1,j+1) \\
&+ \; f(n;i-1,j-1)
\end{aligned}
$$



More recurrences

Question: How to find more such recurrences?
Answer: With guessing!

- ansatz with unspecified coefficients
- plug in small values for n, i, j
- solve the corresponding linear system

Remark: We have to prove that the guessed recurrences are indeed correct!

Ore operators

The recurrence

$$\begin{aligned} f(n+1;i,j) &= f(n;i+1,j)+f(n;i-1,j) \\ &\quad +f(n;i+1,j+1)+f(n;i-1,j-1) \end{aligned}$$

translates to the annihilating Ore operator

$$S_n S_i S_j - S_i^2 S_j - S_j - S_i^2 S_j^2 - 1$$

in the Ore algebra $\mathbb{O} = \mathbb{Q}(i,j,n)[S_i; S_i, 0][S_j; S_j, 0][S_n; S_n, 0]$.

(S_i denotes the shift operator w.r.t. i, i.e.,

$$S_i \bullet f(n;i,j) = f(n;i+1,j).$$

In $\mathbb{O}$ it does not commute with i, namely $S_i i = (i+1)S_i$.)

Our guessing resulted in a set A of 68 operators:

$$
\begin{aligned}
&A = \{S_n S_i S_j - S_i^2 S_j - S_j - S_i^2 S_j^2 - 1, (i+1)(i-2j-3n-20)(i-2j-n-12) S_i S_n^3 S_j^4 - 2(i-2j-\\
&7)(2i-4j-3n-26)(i-2j-n-12) S_n^2 S_j^4 - 32(i-2j-7)(i-2j-3n-13)(n+1) S_j^4 + 16(i+\\
&1)(i^2-4ji-4ni-22i+4j^2-3n^2+44j+8jn+14n+89) S_i S_n S_j^4 - (i-n-4)(i-2j-n-12)(i-\\
&j-n-7) S_n^4 S_j^3 + (i+1)(11i^2-12ji-4ni-36i+12j^2+21n^2+104j+8jn+204n+596) S_i S_n^3 S_j^3 -\\
&4(6i^3-24ji^2+2ni^2-70i^2+32j^2 i-9n^2 i+256ji+16jni+19ni+478i-16j^3+8n^3-176j^2+\\
&6jn^2+93n^2-544j+58jn+451n-126) S_n^2 S_j^3 - 64(n+1)(2i^2-8ji-3ni-30i+8j^2-4n^2+60j+\\
&6jn+3n+96) S_j^3 + 16(i+1)(3i^2-12ji-4ni-42i+12j^2-21n^2+84j+8jn-66n+51) S_i S_n S_j^3 -\\
&(i-n-4)(5i^2-4ji-7ni-29i+4j^2+2n^2+20j+5n+16) S_n^4 S_j^2 + (i+1)(11i^2-12ji+4ni+8i+\\
&12j^2+21n^2+16j-8jn+164n+376) S_i S_n^3 S_j^2 - 4(4i^3-16ji^2+33ni^2+38i^2+16j^2 i-36n^2 i+\\
&56ji-20jni-154ni+8i+16n^3+24j^2+90n^2+120j+20j^2 n+100jn+379n+494) S_n^2 S_j^2 - 64(n+\\
&1)(3i^2-12ji-30i+12j^2-8n^2+60j-30n+51) S_j^2 + 16(i+1)(3i^2-12ji+4ni-18i+12j^2-\\
&21n^2+36j-8jn-106n-69) S_i S_n S_j^2 + (i-n-4)(j-n-2)(i-2j+n+2) S_n^4 S_j + (i+1)(i-2j+\\
&n+2)(i-2j+3n+10) S_i S_n^3 S_j + 4(2i^3-8ji^2-18ni^2-50i^2+16j^2 i+3n^2 i+64ji+16jni+3ni-\\
&14i-16j^3-8n^3-64j^2+6jn^2-63n^2+16j+58jn-161n-194) S_n^2 S_j - 64(n+1)(2i^2-8ji+3ni-\\
&10i+8j^2-4n^2+20j-6jn-27n-4) S_j + 16(i+1)(i^2-4ji+4ni+2i+4j^2-3n^2-4j-8jn-26n-\\
&31) S_i S_n S_j + 2(i-2j-3)(i-2j+n+2)(2i-4j+3n+6) S_n^2 - 32(i-2j-3)(n+1)(i-2j+3n+3), \ldots\}
\end{aligned}
$$

Zeilberger's quasi-holonomic ansatz

Note: The operators in A generate a left ideal, namely ${}_{\mathbb{O}}\langle A\rangle$, all of whose elements are annihilating $f(n;i,j)$. **Idea:** Find an operator $R \in {}_{\mathbb{O}}\langle A\rangle$ of the form

$$\begin{aligned} R(n,i,j,S_n,S_i,S_j) = P(n,S_n) &+ iQ_1(n,i,j,S_n,S_i,S_j) \\ &+ jQ_2(n,i,j,S_n,S_i,S_j) \end{aligned}$$

- $R(n,i,j,S_n,S_i,S_j)$ annihilates $f(n;i,j)$
- set $i=j=0$
- $P(n,S_n)$ annihilates $f(n;0,0)$

Problem: $R(n,i,j,S_n,S_i,S_j)$ is too big to be computed.

Takayama enters the game

$$R(n,i,j,S_n,S_i,S_j) = P(n,S_n) \quad + \quad iQ_1(n,i,j,S_n,S_i,S_j)$$
$$\quad + \quad jQ_2(n,i,j,S_n,S_i,S_j)$$

We use Takayama's trick:

1. substitute $i \to 0$ and $j \to 0$ for all operators $T \in A$
2. eliminate S_i and S_j

Remark: The result will be $P(n,S_n)$ as above, but Q_1 and Q_2 are not computed at all.

$\longrightarrow$ Computation becomes feasable!

How to eliminate?

Problem: After setting $i = 0$, no multiplication by S_i is allowed!

Example:

$$\begin{array}{ccccc} & & P + iQ & & \\ & \swarrow{\scriptstyle \cdot S_i} & & \searrow{\scriptstyle i = 0} & \\ S_iP + (i+1)S_iQ & & & & P \\ \downarrow{\scriptstyle i = 0} & & & & \downarrow{\scriptstyle \cdot S_i} \\ S_iP + S_iQ & & \neq & & S_iP \end{array}$$

A variant of Takayama's algorithm

Let $A = \{A_1, \ldots, A_m\}$ be a set of annihilating operators.

1. $d_i := \max_{1 \leq k \leq m} \deg_{S_i} A_k$
2. set $A := A \cup \bigcup_{k=1}^{m} \{S_i^\alpha A_k \mid 1 \leq \alpha \leq d_i - \deg_{S_i} A_k\}$
3. do the same for j
4. $A := A|_{i \to 0, j \to 0}$
5. translate the elements of A to vectors w.r.t. the basis $\{S_i^\alpha S_j^\beta \mid 0 \leq \alpha \leq d_i, 0 \leq \beta \leq d_j\}$

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3. do the same for j
4. $A := A|_{i \to 0, j \to 0}$
5. translate the elements of A to vectors w.r.t. the basis $\{S_i^\alpha S_j^\beta \,|\, 0 \leq \alpha \leq d_i, 0 \leq \beta \leq d_j\}$, e.g., $S_n S_i S_j - S_i^2 S_j - S_j - S_i^2 S_j^2 - 1$ translates to $(-1, -1, 0, 0, S_n, 0, 0, -1, -1)$
6. compute a Gröbner basis of A in this module
7. if no $(P(n, S_n), 0, \ldots, 0)$ is found, increase d

Result

The operator $P(n, S_n)$ annihilating $f(n; 0, 0)$ has

- order 32
- polynomial coefficients of degree 172
- and integer coefficients up to 385 digits.

The computation was done with CK's implementation of noncommutative Gröbner bases and Takayama, and took 7 hours.

Make the proof rigorous!

Verify that $P(n, S_n)$ also annihilates $g(n; 0, 0)$ for

$$g(n;0,0) := \begin{cases} 16^k \frac{(5/6)_k (1/2)_k}{(2)_k (5/3)_k} & \text{if } n = 2k \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Compare initial values, i.e., $f(n; 0, 0) = g(n; 0, 0)$ for $0 \leq n \leq 31$.
Make sure that the leading coefficient of $P(n, S_n)$ (and all contents that have been cancelled out during the computation) does not have positive integer roots ($=$ poles).

Doron Zeilberger's bet

"I offer a prize of one hundred (100) US-dollars for a short, self-contained, human-generated (and computer-free) proof of Gessel's conjecture, not to exceed five standard pages typed in standard font. The longer that prize would remain unclaimed, the more (empirical) evidence we would have that a proof of Gessel's conjecture is indeed beyond the scope of humankind."