Software Demo:
The HolonomicFunctions Package

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Ore Algebras

$\longrightarrow$ **Generalization** of the ring of differential operators

Let $\mathbb{A}$ be a ring,

- $\sigma\colon \mathbb{A} \to \mathbb{A}$ an automorphism on $\mathbb{A}$, and
- $\delta\colon \mathbb{A} \to \mathbb{A}$ be a σ-derivation, i.e.,

$$\delta(ab) = \sigma(a)\delta(b) + \delta(a)b \quad \text{for all } a, b \in \mathbb{A}.$$

Then the polynomial ring $\mathbb{O} = \mathbb{A}[\partial; \sigma, \delta] = \mathbb{A}\langle\partial\rangle$ whose non-commutative multiplication is defined by

$$\partial a = \sigma(a)\partial + \delta(a)$$

is called an **Ore algebra**.

Example: Let $\mathbb{A} = \mathbb{K}(x)$, $\sigma = \mathrm{id}$, and $\delta = \frac{\mathrm{d}}{\mathrm{d}x}$. In this case we denote $\partial = D_x$ and get $\mathbb{O} = \mathbb{K}(x)\langle D_x\rangle$.

Examples of Ore Algebras

Ore operator	∂	σ	δ
Differential operator	D_x	$\sigma = \mathrm{id}$	$\delta = \frac{\mathrm{d}}{\mathrm{d}x}$
Euler operator	θ_x	$\sigma = \mathrm{id}$	$\delta = x\frac{\mathrm{d}}{\mathrm{d}x}$
Shift operator	S_n	$\sigma(n) = n + 1$	$\delta = 0$
Difference operator	Δ_n	$\sigma(n) = n + 1$	$\delta(n) = 1$
q-Shift operator	$S_{z,q}$	$\sigma(z) = qz$	$\delta = 0$
q-Difference operator	$\Delta_{z,q}$	$\sigma(z) = qz$	$\delta(z) = (q - 1)z$

Multivariate Ore Algebras

The construction of Ore algebras can be iterated:

$$\mathbb{A}[\partial_1; \sigma_1, \delta_1] \cdots [\partial_r; \sigma_r, \delta_r] = \mathbb{A}\langle \partial_1, \ldots, \partial_r \rangle$$

In this case, one must ensure that the ∂_i's commute: $\partial_i \partial_j = \partial_j \partial_i$.

In the following, $\mathbb{A}$ is always a **rational** function field:

$$\mathbb{A} = \mathbb{K}(v_1, \ldots, v_r) = \mathbb{K}(\boldsymbol{v}).$$

Each ∂_i is related to exactly one variable, say v_i, i.e.,
$\partial_i v_j = v_j \partial_i$ for $i \neq j$; write ∂_{v_i} for ∂_i.

Summarizing, Ore algebras from now on are always of the form

$$\mathbb{O} = \mathbb{K}(v_1, \ldots, v_r)\langle \partial_{v_1}, \ldots, \partial_{v_r} \rangle = \mathbb{K}(\boldsymbol{v})\langle \boldsymbol{\partial_v} \rangle.$$

Action!

Define how operators act on functions:

Differential operator:	$D_x \bullet f(x) = \frac{\mathrm{d}}{\mathrm{d}x} f(x)$
Euler operator:	$\theta_x \bullet f(x) = x \frac{\mathrm{d}}{\mathrm{d}x} f(x)$
Shift operator:	$S_n \bullet f(n) = f(n+1)$
Difference operator:	$\Delta_n \bullet f(n) = f(n+1) - f(n)$
q-Shift operator:	$S_{z,q} \bullet f(z) = f(qz)$
q-Difference operator:	$\Delta_{z,q} \bullet f(z) = f(qz) - f(z)$

Definitions

1. The **annihilator** of a function f w.r.t. an Ore algebra $\mathbb{O}$:

$$\operatorname{ann}_{\mathbb{O}}(f) = \left\{ P \in \mathbb{O} \mid P \bullet f = 0 \right\}$$

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4. The definitions **∂-finite** and **holonomic** differ only by some technical conditions.

Worked Example 1

(This example was kindly provided by R. Vidunas.)

Consider the two differential operators P_1 and P_2 given by

$$P_1 = (ny - 2xy + y - z^2)D_y^3 + (3n - 6x - 3y + 1)zD_y^2 D_z + \dots$$

$$P_2 = (-n^2 + 3nx - ny + 2n - 2x^2 + 3xy - 3x + y - 1)z^2 D_y^3 + \dots$$

They generate a zero-dimensional ideal.

Task:

- Compute a Gröbner basis.
- Find operators only involving D_y resp. D_z.

Worked Example 1

$P_1 = (ny - 2xy + y - z^2)D_y^3 + (3nz - 6xz - 3yz + z)D_y^2 D_z + (3ny - 6xy - y - 3z^2)D_y D_z^2 + (nz - 2xz - yz - z)D_z^3 + (2n^2 - 4nx - 2ny + 3n + 4xy - 2x + 2y + 2z^2 - 3)D_y^2 + (-8nz + 8xz + 4yz + 4z)D_y D_z + (2n^2 - 4nx - 2ny - 3n + 4xy + 2x + 2y + 2z^2 + 3)D_z^2 + (-4n^2 + 4nx + 8x^2 - 4y - 4)D_y + (4nz + 4xz)D_z + (8nx - 4n - 16x^2)$

$P_2 = (-n^2z^2 + 3nxz^2 - nyz^2 + 2nz^2 - 2x^2z^2 + 3xyz^2 - 3xz^2 + yz^2 - z^2)D_y^3 + (-3n^2yz + 9nxyz - 3ny^2z + 4nyz - 6x^2yz + 6xy^2z - 7xyz + 3xz^3 + 3y^2z - yz)D_y^2 D_z + (-3n^2z^2 + 9nxz^2 - 3nyz^2 + 2nz^2 - 6x^2z^2 + 9xyz^2 - 5xz^2 + 3yz^2 + z^2)D_y D_z^2 + (-n^2yz + 3nxyz - ny^2z - 2x^2yz + 2xy^2z - xyz + xz^3 + y^2z + yz)D_z^3 + (3n^2y + 2n^2z^2 - 9nxy - 4nxz^2 + 3ny^2 + 2nyz^2 - 4ny - 3nz^2 + 6x^2y + 4x^2z^2 - 6xy^2 - 6xyz^2 + 7xy + 4xz^2 - 3y^2 - 2yz^2 + y + z^2)D_y^2 + (-4n^3z + 12n^2xz + 8n^2z - 8nx^2z - 4nxyz - 20nxz + 4ny^2z + 4nyz - 4nz + 8x^2yz + 8x^2z - 8xy^2z - 4xyz - 4xz^3 + 8xz - 4y^2z - 4yz)D_y D_z + (n^2y + 2n^2z^2 - 3nxy - 4nxz^2 + ny^2 + 2nyz^2 - nz^2 + 2x^2y + 4x^2z^2 - 2xy^2 - 6xyz^2 + xy - y^2 - 2yz^2 - y - z^2)D_z^2 + (4n^3 - 12n^2x - 8n^2 + 8nx^2 + 4nxy - 4nxz^2 + 20nx - 4ny^2 - 4ny + 4n - 8x^2y - 4x^2z^2 - 8x^2 + 8xy^2 + 4xy - 8x + 4y^2 + 4y)D_y + (4n^3z - 8n^2xz + 4n^2yz - 4n^2z - 4nx^2z - 4nxyz + 8nxz - 4nyz + 8x^3z - 8x^2yz)D_z + (-4n^3 + 8n^2x - 4n^2y + 4n^2 + 4nx^2 + 4nxy - 8nx + 4ny - 8x^3 + 8x^2y + 8x^2z^2)$

Creative Telescoping

Method for doing integrals and sums
(aka Feynman's differentiating under the integral sign)

Consider the following summation problem: $F(n) = \sum_{k=a}^{b} f(n, k)$

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Telescoping: write $f(n,k) = g(n,k+1) - g(n,k)$.

Then $F(n) = \sum_{k=a}^{b} \big(g(n,k+1) - g(n,k)\big) = g(n,b+1) - g(n,a)$.

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Creative Telescoping: write

$$c_r(n) f(n+r,k) + \cdots + c_0(n) f(n,k) = g(n,k+1) - g(n,k).$$

Summing from a to b yields a recurrence for $F(n)$:

$$c_r(n) F(n+r) + \cdots + c_0(n) F(n) = g(n,b+1) - g(n,a).$$

Creative Telescoping

Method for doing integrals and sums
(aka Feynman's differentiating under the integral sign)

Consider the following integration problem: $F(x) = \int_a^b f(x,y)\,\mathrm{d}y$

Telescoping: write $f(x,y) = \frac{\mathrm{d}}{\mathrm{d}y} g(x,y)$.

Then $F(n) = \int_a^b \left(\frac{\mathrm{d}}{\mathrm{d}y} g(x,y) \right) \mathrm{d}y \qquad = g(x,b) - g(x,a)$.

Creative Telescoping: write

$$c_r(x) \frac{\mathrm{d}^r}{\mathrm{d}x^r} f(x,y) + \cdots + c_0(x) f(x,y) \;= \frac{\mathrm{d}}{\mathrm{d}y} g(x,y).$$

Integrating from a to b yields a differential equation for $F(x)$:

$$c_r(x) \frac{\mathrm{d}^r}{\mathrm{d}x^r} F(x) + \cdots + c_0(x) F(x) \;= g(x,b) - g(x,a)$$

The Right-Hand Side

$$\begin{aligned} c_r(n)f(n+r,k) + \cdots + c_0(n)f(n,k) &= g(n,k+1) - g(n,k) \\ &= (S_k - 1) \cdot g(n,k). \end{aligned}$$

Where should we look for a suitable $g(n,k)$?
Note that there are trivial solutions like:

$$g(n,k) := \sum_{i=0}^{k-1} \big(c_r(n)f(n+r,i) + \cdots + c_0(n)f(n,i) \big)$$

The Right-Hand Side

$$c_r(n)f(n+r,k) + \cdots + c_0(n)f(n,k) = g(n,k+1) - g(n,k)$$
$$= (S_k - 1) \cdot g(n,k).$$

Where should we look for a suitable $g(n,k)$?
Note that there are trivial solutions like:

$$g(n,k) := \sum_{i=0}^{k-1} \big(c_r(n)f(n+r,i) + \cdots + c_0(n)f(n,i)\big)$$

A reasonable choice for where to look for g is $\mathbb{O} \cdot f$.

Then the task is to find $P(n, S_n) = c_r(n)S_n^r + \cdots + c_0(n)$ and $Q \in \mathbb{O}$ such that

$$\big(P - (S_k - 1)Q\big) \cdot f = 0 \quad \iff \quad P - (S_k - 1)Q \in \operatorname{ann}_{\mathbb{O}}(f).$$

Integrals

For an integral $\int_a^b f(x,y)\,\mathrm{d}y$ where the integrand f is ∂-finite, the creative telescoping problem is the following: find

$$P \in \mathbb{K}(x)\langle D_x \rangle \quad \text{and} \quad Q \in \mathbb{O} = \mathbb{K}(x,y)\langle D_x, D_y \rangle$$

such that $P - D_y Q \in \operatorname{ann}_{\mathbb{O}}(f)$.

- P is called **telescoper**.
- Q is called **certificate**.

$\longrightarrow$ Since f is ∂-finite one can restrict the search space for Q to $\mathbb{O}/\operatorname{ann}_{\mathbb{O}}(f)$.

Worked Example 2

(Example was kindly provided by A. Takemura and C. Siriteanu.)

Let $p(t,x,y)$, $t>0$, be some probability density function. However, we know only its moment generating function

$$M(s,x,y) = \int_0^\infty \mathrm{e}^{ts} p(t,x,y)\,\mathrm{d}t$$
$$= \frac{\mathrm{e}^{-y}}{(1-s)^N} \sum_{n=0}^{\infty} \frac{y^n}{n!} {}_1F_1\Bigl(N; N_\mathrm{R}+n; \frac{xs}{1-s}\Bigr).$$

Tasks:

- Compute a holonomic system for $M(s,x,y)$.
- Compute a holonomic system for $p(t,x,y)$.
- Compute the corresponding Pfaffian system.
- Do the same for $\int_0^\infty \log(1+t)\,p(t,x,y)\,\mathrm{d}t$.

Example 3

(Example is stolen from H. Hashiguchi's talk.)

Recall the pdf $f(r,\rho)$ from Hotelling (1953)

$$f(r,\rho) = \frac{n-1}{\sqrt{2\pi}} \frac{\Gamma(n)}{\Gamma\left(\frac{n-1}{2}\right)} (1-\rho^2)^{n/2} (1-r^2)^{(n-3)/2} (1-\rho r)^{-n+1/2} \times {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; n+\frac{1}{2}; \frac{1+\rho r}{2}\right)$$

Want to compute the cdf $F(r,\rho)$ defined by

$$F(r,\rho) = \int_{-1}^{r} f(x,\rho)\,\mathrm{d}x.$$

```
<<RISC`HolonomicFunctions`;
```

HolonomicFunctions Package version 1.7.1 (09-Oct-2013)
written by Christoph Koutschan


```
--> Type  ?HolonomicFunctions  for help.
```

```
?HolonomicFunctions
```

The main objective of this package is the algorithmic manipulation of ∂–finite (holonomic) functions. This includes (but is not restricted to) proving special function identities, finding recurrences, differential equations or relations of mixed type for ∂–finite functions, and computing definite sums and integrals of ∂–finite functions. Type ?DFinite to get the definition and a short introduction to ∂–finite functions.

The following commands serve the above objectives: Annihilator, CreativeTelescoping, HermiteTelescoping, FindCreativeTelescoping, FindRelation, FindSupport, Takayama, ApplyOreOperator, UnderTheStaircase, AnnihilatorDimension.

The closure properties of ∂–finite functions are implicitely executed in Annihilator. To execute them explicitly, use the commands DFinitePlus, DFiniteTimes, DFiniteSubstitute, DFiniteOreAction, DFiniteTimesHyper, DFiniteDE2RE, DFiniteRE2DE, DFiniteQSubstitute.

An important ingredient are Groebner bases in (noncommutative) Ore algebras: OreGroebnerBasis, OreReduce, GBEqual, FGLM.

A common subtask in the above algorithms is finding rational solutions of P–finite recurrences / differential equations or of coupled systems of such equations. The following commands address these purposes: RSolvePolynomial, RSolveRational, DSolvePolynomial, DSolveRational, QSolvePolynomial, QSolveRational, SolveOreSys, SolveCoupledSystem.

An element of an Ore algebra is called an Ore polynomial; the following commands explain the data type OrePolynomial that is introduced in this package and how to deal with it: OrePolynomial, ToOrePolynomial, OrePolynomialZeroQ, LeadingPowerProduct, LeadingExponent, LeadingCoefficient, LeadingTerm, OrePolynomialListCoefficients, NormalizeCoefficients, OrePlus, OreTimes, OrePower, ApplyOreOperator, ChangeOreAlgebra, ChangeMonomialOrder, OrePolynomialSubstitute, OrePolynomialDegree, Support.

In order to define own Ore algebras use the commands OreAlgebra, OreAlgebraGenerators, OreAlgebraOperators, OreAlgebraPolynomialVariables, OreOperators, OreOperatorQ, OreSigma, OreDelta, OreAction, Der, S, Delta, Euler, QS.

Some other functions that might be useful: Printlevel, RandomPolynomial.

If this package was useful in your scientific work, proper citation would be appreciated very much. Please use the following reference for this purpose:

```
@phdthesis{Koutschan09,
  author = {Christoph Koutschan},
  title = {Advanced Applications of the Holonomic Systems Approach},
  school = {RISC, Johannes Kepler University},
  address = {Linz, Austria},
  year = {2009}
}
```

Ore algebras

```
alg1 = OreAlgebra[Der[x]]
```

$\mathbb{K}(x)[D_x; 1, D_x]$

```
op = ToOrePolynomial[Der[x] - 1/x, alg1]
```

$D_x - \frac{1}{x}$

```
FullForm [%]
```

```
OrePolynomial [List[List[1, List[1]], List[Times [-1, Power[x, -1]], List[0]]],
  OreAlgebraObject[List[Der[x]], Expand, Function[Plus[Slot[1], Slot[2]]],
   Function[Expand[Times [Slot[1], Slot[2]]]], None], DegreeLexicographic
```

```
op = ToOrePolynomial [f '[x] - f[x] / x, f[x], alg1]
```

$D_x - \frac{1}{x}$

```
op1 = op ** op + x
```

$D_x^2 - \frac{2}{x} D_x + \left(\frac{2}{x^2} + x\right)$

```
alg2 = OreAlgebra[x, Der[x]]
```

$\mathbb{K}[x][D_x; 1, D_x]$

```
ChangeOreAlgebra[op1, alg2]
```

ChangeOreAlgebra::nopoly : The elements of the new OreAlgebra do not occur polynomially.

```
$Failed
```

```
op2 = ChangeOreAlgebra[x^2 ** op1, alg2]
```

$x^2 D_x^2 + x^3 - 2 x D_x + 2$

```
alg3 = OreAlgebra[Der[x], x]
```

$\mathbb{K}[x][D_x; 1, D_x]$

```
op3 = ChangeOreAlgebra[op2, alg3]
```

$D_x^2 x^2 + x^3 - 6 D_x x + 6$

```
op2 ** op3
```

$x^4 D_x^4 + 2 x^5 D_x^2 + x^6 + 2 x^4 D_x + 2 x^2 D_x^2 + 4 x^3 - 4 x D_x + 4$

```
ChangeOreAlgebra[%, OreAlgebra[Euler[x]]]
```

$\theta_x^4 - 6 \theta_x^3 + (13 + 2 x^3) \theta_x^2 - 12 \theta_x + (4 + 4 x^3 + x^6)$

```
OreSigma [d] := σ;
OreDelta[d] := δ;
```

```
alg = OreAlgebra[d]
```

$\mathbb{K}[d; \sigma, \delta]$

```
ToOrePolynomial [d^2 ** x, alg]
```

$\sigma[\sigma[x]] d^2 + (\delta[\sigma[x]] + \sigma[\delta[x]]) d + \delta[\delta[x]]$

```
OreAlgebra[Der[x], Euler[x], S[x], Delta[x], QS[x, q^n]]
```

$\mathbb{K}(q, x)[D_x; 1, D_x][\theta_x; 1, \theta_x][S_x; S_x, 0][\Delta_x; S_x, \Delta_x][S_{x,q}; S_{x,q}, 0]$

```
ApplyOreOperator[op2, f[x]]
```

$2 f[x] + x^3 f[x] - 2 x f'[x] + x^2 f''[x]$

```
ApplyOreOperator[S[x], f[x]]
```

$f[1 + x]$

Annihilators

```
ode = x^2 * j''[x] + x * j'[x] + (x^2 - n^2) * j[x] == 0
```

$(-n^2 + x^2)\, j[x] + x\, j'[x] + x^2\, j''[x] == 0$

```
DSolve[ode, j[x], x]
```

{{j[x] → BesselJ[n, x] C[1] + BesselY[n, x] C[2]}}

```
ann1 = ToOrePolynomial[ode, j[x]]
```

$x^2 D_x^2 + x D_x + (-n^2 + x^2)$

```
OreAlgebra[ann1]
```

$\mathbb{K}(x)[D_x; 1, D_x]$

```
ApplyOreOperator[ann1, BesselJ[n, x]] // FullSimplify
```

0

```
AnnihilatorDimension[{ann1}]
```

0

```
HolonomicRank[ops_] :=
    With[{u = UnderTheStaircase[ops]}, If[u === Infinity, Throw["Not holonomic "], Length[u]]];
```

```
HolonomicRank[{ann1}]
```

2

```
ann1 = ChangeOreAlgebra[ann1, OreAlgebra[S[n], Der[x]]]
```

$x^2 D_x^2 + x D_x + (-n^2 + x^2)$

```
OreAlgebra[ann1]
```

$\mathbb{K}(n, x)[S_n; S_n, 0][D_x; 1, D_x]$

```
AnnihilatorDimension[{ann1}]
```

1

```
ann2 = ToOrePolynomial[x * S[n]^2 - (2 n + 2) ** S[n] + x, OreAlgebra[ann1]]
```

$x S_n^2 + (-2 - 2 n) S_n + x$

```
ApplyOreOperator[ann2, BesselJ[n, x]] // FullSimplify
```

0

```
ann = {ann1, ann2}
```

$\{x^2 D_x^2 + x D_x + (-n^2 + x^2),\ x S_n^2 + (-2 - 2 n) S_n + x\}$

```
AnnihilatorDimension[ann]
```

0

```
HolonomicRank[ann]
```

4

Gröbner Bases

```
gb = OreGroebnerBasis[ann]
```

$\{x S_n + x D_x - n,\ x^2 D_x^2 + x D_x + (-n^2 + x^2)\}$

```
HolonomicRank [gb]
```

2

```
ann2
```

$x S_n^2 + (-2 - 2 n) S_n + x$

```
OreReduce[ann2, gb]
```

0

```
spoly = x ** Der[x] ^ 2 ** gb[[1]] - S[n] ** gb[[2]]
```

$x^2 D_x^3 + x S_n D_x + (2 x - n x) D_x^2 + (1 + 2 n + n^2 - x^2) S_n$

```
OreReduce[spoly, gb]
```

0

```
ApplyOreOperator[spoly, BesselJ[n, x]] // FullSimplify
```

0

Annihilators (2)

```
gb
```

$\{x S_n + x D_x - n,\ x^2 D_x^2 + x D_x + (-n^2 + x^2)\}$

```
Annihilator[BesselJ[n, x], {S[n], Der[x]}]
```

$\{x S_n + x D_x - n,\ x^2 D_x^2 + x D_x + (-n^2 + x^2)\}$

```
?Annihilator
```

Annihilator [expr, ops] computes annihilating relations for expr w.r.t. the given operator(s). It returns the Groebner basis of an annihilating ideal (with monomial order DegreeLexicographic). If expr is ∂–finite , the result will be a ∂–finite ideal. If expr is not recognized to be ∂–finite , there is still a chance to find at least some relations (in this case the ideal is not zero–dimensional which is indicated by a warning). Annihilator [expr] automatically determines for which operators relations exist. The relations are computed by executing the ∂–finite closure properties DFinitePlus , DFiniteTimes , and DFiniteSubstitute .

The expression expr can contain hypergeometric expressions, hyperexponential expressions, and algebraic expressions.

Additionally the following functions are recognized: AiryAi , AiryAiPrime , AiryBi , AiryBiPrime , AngerJ, AppellF1, ArcCos, ArcCosh, ArcCot, ArcCoth, ArcCsc, ArcCsch, ArcSec, ArcSech, ArcSin, ArcSinh, ArcTan, ArcTanh, ArithmeticGeometricMean , BellB, BernoulliB, BesselI, BesselJ, BesselK, BesselY, Beta, BetaRegularized, Binomial , CatalanNumber, ChebyshevT, ChebyshevU, Cos, Cosh, CoshIntegral, CosIntegral, EllipticE , EllipticF , EllipticK , EllipticPi , EllipticTheta , EllipticThetaPrime , Erf, Erfc, Erfi, EulerE, Exp, ExpIntegralE, ExpIntegralEi, Factorial, Factorial2, Fibonacci , FresnelC, FresnelS, Gamma , GammaRegularized , GegenbauerC, HankelH1, HankelH2, HarmonicNumber , HermiteH , Hypergeometric0F1 , Hypergeometric0F1Regularized , Hypergeometric1F1 , Hypergeometric1F1Regularized , Hypergeometric2F1 , Hypergeometric2F1Regularized , HypergeometricPFQ , HypergeometricPFQRegularized , HypergeometricU , JacobiP, KelvinBei , KelvinBer, KelvinKei , KelvinKer, LaguerreL, LegendreP, LegendreQ, LerchPhi, Log, LogGamma , LucasL, Multinomial , NevilleThetaC, ParabolicCylinderD , Pochhammer , PolyGamma , PolyLog, qBinomial , QBinomial , qBrackets, qFactorial, QFactorial, qPochhammer , QPochhammer , Root, Sin, Sinc, Sinh, SinhIntegral, SinIntegral, SphericalBesselJ, SphericalBesselY, SphericalHankelH1, SphericalHankelH2, Sqrt, StirlingS1 , StirlingS2 , StruveH, StruveL, Subfactorial, WeberE, WhittakerM, WhittakerW, Zeta.

If expr contains the commands D and ApplyOreOperator then the closure property DFiniteOreAction is performed: Note the difference between
Annihilator [D[LegendreP[n, x], x], {S[n], Der[x]}] and
expr = D[LegendreP[n, x], x]; Annihilator [expr, {S[n], Der[x]}].

Similarly , if expr contains Sum or Integrate then not Mathematica is asked to simplify the expression, but CreativeTelescoping is executed automatically on the summand (resp. integrand). For evaluating the delta part, Mathematica 's FullSimplify is used; if it fails (or if you don't trust it), you can use the option Inhomogeneous –> True, in order to obtain an inhomogeneous recurrence (resp. differential equation).

Worked Example 1

```
op1 = ToExpression[StringReplace[
    "(3/2)*Dy^2*n-(3/2)*Dz^2*n+(1/2)*Dy^3*y-(1/2)*Dz^3*z-2*n-2*Dy-n*Dz^2*y+2*Dy*=n*x-2
      *Dy*y-2*Dy*n^2+4*Dy*x^2-Dy^2*x+Dz^2*x+Dz^2*n^2-8*x^2+2*Dy^2*x*y-Dy^2*=n*y
      -(1/2)*Dz^3*y*z-(3/2)*Dy*Dz^2*z^2-(3/2)*Dy^2+(3/2)*Dz^2+Dy^2*n^2+Dz^2*z=
      ^2-(1/2)*Dy^3*z^2+Dy^2*y+Dz^2*y+Dy^2*z^2+4*x*n-3*Dy^2*Dz*x*z+2*Dy*Dz*y*z
      +(=3/2)*Dy^2*Dz*n*z-Dy^3*x*y-(1/2)*Dy*Dz^2*y+(1/2)*Dy^2*Dz*z+(1/2)*Dy^3*
      n*y+(=1/2)*Dz^3*n*z-2*Dy^2*n*x-2*Dz^2*n*x+2*Dz*x*z+2*Dz*n*z+2*Dy*Dz*z-Dz
      ^3*x*z-3=*x*Dy*Dz^2*y+(3/2)*n*Dy*Dz^2*y-4*Dy*Dz*n*z+4*Dy*Dz*x*z-(3/2)*Dy
      ^2*Dz*y*z+2=*x*Dz^2*y", {"=" → ""}]]
```

$$-2\,Dy-\frac{3\,Dy^2}{2}+\frac{3\,Dz^2}{2}-2\,n+\frac{3\,Dy^2\,n}{2}-\frac{3\,Dz^2\,n}{2}-2\,Dy\,n^2+Dy^2\,n^2+Dz^2\,n^2-Dy^2\,x+Dz^2\,x+$$
$$4\,n\,x+2\,Dy\,n\,x-2\,Dy^2\,n\,x-2\,Dz^2\,n\,x-8\,x^2+4\,Dy\,x^2-2\,Dy\,y+Dy^2\,y+\frac{Dy^3\,y}{2}+Dz^2\,y-\frac{1}{2}Dy\,Dz^2\,y-$$
$$Dy^2\,n\,y+\frac{1}{2}Dy^3\,n\,y-Dz^2\,n\,y+\frac{3}{2}Dy\,Dz^2\,n\,y+2\,Dy^2\,x\,y-Dy^3\,x\,y+2\,Dz^2\,x\,y-3\,Dy\,Dz^2\,x\,y+2\,Dy\,Dz\,z+$$
$$\frac{1}{2}Dy^2\,Dz\,z-\frac{Dz^3\,z}{2}+2\,Dz\,n\,z-4\,Dy\,Dz\,n\,z+\frac{3}{2}Dy^2\,Dz\,n\,z+\frac{1}{2}Dz^3\,n\,z+2\,Dz\,x\,z+4\,Dy\,Dz\,x\,z-$$
$$3\,Dy^2\,Dz\,x\,z-Dz^3\,x\,z+2\,Dy\,Dz\,y\,z-\frac{3}{2}Dy^2\,Dz\,y\,z-\frac{1}{2}Dz^3\,y\,z+Dy^2\,z^2-\frac{Dy^3\,z^2}{2}+Dz^2\,z^2-\frac{3}{2}Dy\,Dz^2\,z^2$$

```
op2 = ToExpression[StringReplace[
    "-Dy^3*n^2*z^2+3*Dy^3*n*x*z^2-Dy^3*n*y*z^2-2*Dy^3*x^2*z^2+3*Dy^3*x*y*z^2-3*=Dy^2*
      Dz*n^2*y*z+9*Dy^2*Dz*n*x*y*z-3*Dy^2*Dz*n*y^2*z-6*Dy^2*Dz*x^2*y*z+6*Dy=^2
      *Dz*x*y^2*z+3*Dy^2*Dz*x*z^3-3*Dy*Dz^2*n^2*z^2+9*Dy*Dz^2*n*x*z^2-3*Dy*Dz^=
      2*n*y*z^2-6*Dy*Dz^2*x^2*z^2+9*Dy*Dz^2*x*y*z^2-Dz^3*n^2*y*z+3*Dz^3*n*x*y*z
      -=Dz^3*n*y^2*z-2*Dz^3*x^2*y*z+2*Dz^3*x*y^2*z+Dz^3*x*z^3+2*Dy^3*n*z^2-3*Dy
      ^3*=x*z^2+Dy^3*y*z^2+4*Dy^2*Dz*n*y*z-7*Dy^2*Dz*x*y*z+3*Dy^2*Dz*y^2*z+2*
      Dy^2*n^=2*z^2-4*Dy^2*n*x*z^2+2*Dy^2*n*y*z^2+4*Dy^2*x^2*z^2-6*Dy^2*x*y*z^2
      +2*Dy*Dz^=2*n*z^2-5*Dy*Dz^2*x*z^2+3*Dy*Dz^2*y*z^2-4*Dy*Dz*n^3*z+12*Dy*Dz
      *n^2*x*z-8*D=y*Dz*n*x^2*z-4*Dy*Dz*n*x*y*z+4*Dy*Dz*n*y^2*z+8*Dy*Dz*x^2*y*
      z-8*Dy*Dz*x*y^2=*z-4*Dy*Dz*x*z^3-Dz^3*x*y*z+Dz^3*y^2*z+2*Dz^2*n^2*z^2-4*
      Dz^2*n*x*z^2+2*Dz^=2*n*y*z^2+4*Dz^2*x^2*z^2-6*Dz^2*x*y*z^2-Dy^3*z^2-Dy^2*
      Dz*y*z+3*Dy^2*n^2*y-=9*Dy^2*n*x*y+3*Dy^2*n*y^2-3*Dy^2*n*z^2+6*Dy^2*x^2*y
      -6*Dy^2*x*y^2+4*Dy^2*x*=z^2-2*Dy^2*y*z^2+Dy*Dz^2*z^2+8*Dy*Dz*n^2*z-20*Dy*
      Dz*n*x*z+4*Dy*Dz*n*y*z+8*=Dy*Dz*x^2*z-4*Dy*Dz*x*y*z-4*Dy*Dz*y^2*z-4*Dy*n
      *x*z^2-4*Dy*x^2*z^2+Dz^3*y*z=+Dz^2*n^2*y-3*Dz^2*n*x*y+Dz^2*n*y^2-Dz^2*n*z
      ^2+2*Dz^2*x^2*y-2*Dz^2*x*y^2-2=*Dz^2*y*z^2+4*Dz*n^3*z-8*Dz*n^2*x*z+4*Dz*
      n^2*y*z-4*Dz*n*x^2*z-4*Dz*n*x*y*z=+8*Dz*x^3*z-8*Dz*x^2*y*z-4*Dy^2*n*y+7*
      Dy^2*x*y-3*Dy^2*y^2+Dy^2*z^2-4*Dy*Dz=*n*z+8*Dy*Dz*x*z-4*Dy*Dz*y*z+4*Dy*n
      ^3-12*Dy*n^2*x+8*Dy*n*x^2+4*Dy*n*x*y-4*=Dy*n*y^2-8*Dy*x^2*y+8*Dy*x*y^2+
      Dz^2*x*y-Dz^2*y^2-Dz^2*z^2-4*Dz*n^2*z+8*Dz*=n*x*z-4*Dz*n*y*z+8*x^2*z^2+
      Dy^2*y-8*Dy*n^2+20*Dy*n*x-4*Dy*n*y-8*Dy*x^2+4*D=y*x*y+4*Dy*y^2-Dz^2*y-4*
      n^3+8*n^2*x-4*n^2*y+4*n*x^2+4*n*x*y-8*x^3+8*x^2*y+=4*Dy*n-8*Dy*x+4*Dy*y+
      4*n^2-8*n*x+4*n*y", {"=" → ""}]]
```

$$
4\,Dy\,n + 4\,n^2 - 8\,Dy\,n^2 - 4\,n^3 + 4\,Dy\,n^3 - 8\,Dy\,x - 8\,n\,x + 20\,Dy\,n\,x + 8\,n^2\,x - 12\,Dy\,n^2\,x - 8\,Dy\,x^2 + 4\,n\,x^2 + 8\,Dy\,n\,x^2 -
8\,x^3 + 4\,Dy\,y + Dy^2\,y - Dz^2\,y + 4\,n\,y - 4\,Dy\,n\,y - 4\,Dy^2\,n\,y - 4\,n^2\,y + 3\,Dy^2\,n^2\,y + Dz^2\,n^2\,y + 4\,Dy\,x\,y + 7\,Dy^2\,x\,y +
Dz^2\,x\,y + 4\,n\,x\,y + 4\,Dy\,n\,x\,y - 9\,Dy^2\,n\,x\,y - 3\,Dz^2\,n\,x\,y + 8\,x^2\,y - 8\,Dy\,x^2\,y + 6\,Dy^2\,x^2\,y + 2\,Dz^2\,x^2\,y + 4\,Dy\,y^2 -
3\,Dy^2\,y^2 - Dz^2\,y^2 - 4\,Dy\,n\,y^2 + 3\,Dy^2\,n\,y^2 + Dz^2\,n\,y^2 + 8\,Dy\,x\,y^2 - 6\,Dy^2\,x\,y^2 - 2\,Dz^2\,x\,y^2 - 4\,Dy\,Dz\,n\,z - 4\,Dz\,n^2\,z +
8\,Dy\,Dz\,n^2\,z + 4\,Dz\,n^3\,z - 4\,Dy\,Dz\,n^3\,z + 8\,Dy\,Dz\,x\,z + 8\,Dz\,n\,x\,z - 20\,Dy\,Dz\,n\,x\,z - 8\,Dz\,n^2\,x\,z + 12\,Dy\,Dz\,n^2\,x\,z +
8\,Dy\,Dz\,x^2\,z - 4\,Dz\,n\,x^2\,z - 8\,Dy\,Dz\,n\,x^2\,z + 8\,Dz\,x^3\,z - 4\,Dy\,Dz\,y\,z - Dy^2\,Dz\,y\,z + Dz^3\,y\,z - 4\,Dz\,n\,y\,z + 4\,Dy\,Dz\,n\,y\,z +
4\,Dy^2\,Dz\,n\,y\,z + 4\,Dz\,n^2\,y\,z - 3\,Dy^2\,Dz\,n^2\,y\,z - Dz^3\,n^2\,y\,z - 4\,Dy\,Dz\,x\,y\,z - 7\,Dy^2\,Dz\,x\,y\,z - Dz^3\,x\,y\,z - 4\,Dz\,n\,x\,y\,z -
4\,Dy\,Dz\,n\,x\,y\,z + 9\,Dy^2\,Dz\,n\,x\,y\,z + 3\,Dz^3\,n\,x\,y\,z - 8\,Dz\,x^2\,y\,z + 8\,Dy\,Dz\,x^2\,y\,z - 6\,Dy^2\,Dz\,x^2\,y\,z - 2\,Dz^3\,x^2\,y\,z -
4\,Dy\,Dz\,y^2\,z + 3\,Dy^2\,Dz\,y^2\,z + Dz^3\,y^2\,z + 4\,Dy\,Dz\,n\,y^2\,z - 3\,Dy^2\,Dz\,n\,y^2\,z - Dz^3\,n\,y^2\,z - 8\,Dy\,Dz\,x\,y^2\,z + 6\,Dy^2\,Dz\,x\,y^2\,z +
2\,Dz^3\,x\,y^2\,z + Dy^2\,z^2 - Dy^3\,z^2 - Dz^2\,z^2 + Dy\,Dz^2\,z^2 - 3\,Dy^2\,n\,z^2 + 2\,Dy^3\,n\,z^2 - Dz^2\,n\,z^2 + 2\,Dy\,Dz^2\,n\,z^2 + 2\,Dy^2\,n^2\,z^2 -
Dy^3\,n^2\,z^2 + 2\,Dz^2\,n^2\,z^2 - 3\,Dy\,Dz^2\,n^2\,z^2 + 4\,Dy^2\,x\,z^2 - 3\,Dy^3\,x\,z^2 - 5\,Dy\,Dz^2\,x\,z^2 - 4\,Dy\,n\,x\,z^2 - 4\,Dy^2\,n\,x\,z^2 +
3\,Dy^3\,n\,x\,z^2 - 4\,Dz^2\,n\,x\,z^2 + 9\,Dy\,Dz^2\,n\,x\,z^2 + 8\,x^2\,z^2 - 4\,Dy\,x^2\,z^2 + 4\,Dy^2\,x^2\,z^2 - 2\,Dy^3\,x^2\,z^2 + 4\,Dz^2\,x^2\,z^2 -
6\,Dy\,Dz^2\,x^2\,z^2 - 2\,Dy^2\,y\,z^2 + Dy^3\,y\,z^2 - 2\,Dz^2\,y\,z^2 + 3\,Dy\,Dz^2\,y\,z^2 + 2\,Dy^2\,n\,y\,z^2 - Dy^3\,n\,y\,z^2 + 2\,Dz^2\,n\,y\,z^2 -
3\,Dy\,Dz^2\,n\,y\,z^2 - 6\,Dy^2\,x\,y\,z^2 + 3\,Dy^3\,x\,y\,z^2 - 6\,Dz^2\,x\,y\,z^2 + 9\,Dy\,Dz^2\,x\,y\,z^2 - 4\,Dy\,Dz\,x\,z^3 + 3\,Dy^2\,Dz\,x\,z^3 + Dz^3\,x\,z^3
$$

```
Variables[{op1, op2}]
```

```
ops = NormalizeCoefficients/@ToOrePolynomial [{op1, op2} /. {Dy→Der[y], Dz→Der[z]}]
```

$$
\begin{aligned}
\{ & (y+ny-2xy-z^2)\, D_y^3 + (z+3nz-6xz-3yz)\, D_y^2 D_z + (-y+3ny-6xy-3z^2)\, D_y D_z^2 + \\
& (-z+nz-2xz-yz)\, D_z^3 + (-3+3n+2n^2-2x-4nx+2y-2ny+4xy+2z^2)\, D_y^2 + \\
& (4z-8nz+8xz+4yz)\, D_y D_z + (3-3n+2n^2+2x-4nx+2y-2ny+4xy+2z^2)\, D_z^2 + \\
& (-4-4n^2+4nx+8x^2-4y)\, D_y + (4nz+4xz)\, D_z + (-4n+8nx-16x^2), \\
& (-z^2+2nz^2-n^2z^2-3xz^2+3nxz^2-2x^2z^2+yz^2-nyz^2+3xyz^2)\, D_y^3 + \\
& (-yz+4nyz-3n^2yz-7xyz+9nxyz-6x^2yz+3y^2z-3ny^2z+6xy^2z+3xz^3)\, D_y^2 D_z + \\
& (z^2+2nz^2-3n^2z^2-5xz^2+9nxz^2-6x^2z^2+3yz^2-3nyz^2+9xyz^2)\, D_y D_z^2 + \\
& (yz-n^2yz-xyz+3nxyz-2x^2yz+y^2z-ny^2z+2xy^2z+xz^3)\, D_z^3 + \\
& (y-4ny+3n^2y+7xy-9nxy+6x^2y-3y^2+3ny^2-6xy^2+z^2-3nz^2+2n^2z^2+4xz^2-4nxz^2+ \\
& \quad 4x^2z^2-2yz^2+2nyz^2-6xyz^2)\, D_y^2 + (-4nz+8n^2z-4n^3z+8xz-20nxz+12n^2xz+8x^2z- \\
& \quad 8nx^2z-4yz+4nyz-4xyz-4nxyz+8x^2yz-4y^2z+4ny^2z-8xy^2z-4xz^3)\, D_y D_z + \\
& (-y+n^2y+xy-3nxy+2x^2y-y^2+ny^2-2xy^2-z^2-nz^2+2n^2z^2-4nxz^2+4x^2z^2-2yz^2+2nyz^2-6xyz^2) \\
& \quad D_z^2 + (4n-8n^2+4n^3-8x+20nx-12n^2x-8x^2+8nx^2+4y-4ny+ \\
& \quad 4xy+4nxy-8x^2y+4y^2-4ny^2+8xy^2-4nxz^2-4x^2z^2)\, D_y + \\
& (-4n^2z+4n^3z+8nxz-8n^2xz-4nx^2z+8x^3z-4nyz+4n^2yz-4nxyz-8x^2yz)\, D_z + \\
& (4n^2-4n^3-8nx+8n^2x+4nx^2-8x^3+4ny-4n^2y+4nxy+8x^2y+8x^2z^2) \}
\end{aligned}
$$

```
Support[ops]
```

$$\{\{D_y^3, D_y^2 D_z, D_y D_z^2, D_z^3, D_y^2, D_y D_z, D_z^2, D_y, D_z, 1\}, \{D_y^3, D_y^2 D_z, D_y D_z^2, D_z^3, D_y^2, D_y D_z, D_z^2, D_y, D_z, 1\}\}$$

```
Printlevel = 5;
Timing[gb1 = OreGroebnerBasis[ops /. n→10^4];]
Printlevel = 0;
```

```
OreGroebnerBasis: Number of pairs: 1

OreGroebnerBasis: Taking {3, {3, 0}, 1, 2}

  OreReduce: LPP = {2, 1}

  OreReduce: Put leading term into remainder .

  OreReduce: LPP = {1, 2}

  OreReduce: Put leading term into remainder .

  OreReduce: LPP = {0, 3}

  OreReduce: Put leading term into remainder .

  OreReduce: LPP = {2, 0}

  OreReduce: Put leading term into remainder .

  OreReduce: LPP = {1, 1}

  OreReduce: Put leading term into remainder .

  OreReduce: LPP = {0, 2}

  OreReduce: Put leading term into remainder .

  OreReduce: LPP = {1, 0}

  OreReduce: Put leading term into remainder .

  OreReduce: LPP = {0, 1}

  OreReduce: Put leading term into remainder .

  OreReduce: LPP = {0, 0}

  OreReduce: Put leading term into remainder .
```

```
OreGroebnerBasis: Does not reduce to zero -> number  3 in the basis.
                  The lpp is {2, 1}. The ByteCount is 31040.
OreGroebnerBasis: Number  of pairs: 1
OreGroebnerBasis: Taking {4, {3, 1}, 2, 3}
  OreReduce: LPP = {2, 2}
  OreReduce: reduced with nr. 3
  OreReduce: LPP = {1, 3}
  OreReduce: Put leading term  into remainder .
  OreReduce: LPP = {0, 4}
  OreReduce: Put leading term  into remainder .
  OreReduce: LPP = {3, 0}
  OreReduce: reduced with nr. 1
  OreReduce: LPP = {2, 1}
  OreReduce: reduced with nr. 3
  OreReduce: LPP = {1, 2}
  OreReduce: Put leading term  into remainder .
  OreReduce: LPP = {0, 3}
  OreReduce: Put leading term  into remainder .
  OreReduce: LPP = {2, 0}
  OreReduce: Put leading term  into remainder .
  OreReduce: LPP = {1, 1}
  OreReduce: Put leading term  into remainder .
  OreReduce: LPP = {0, 2}
  OreReduce: Put leading term  into remainder .
  OreReduce: LPP = {1, 0}
  OreReduce: Put leading term  into remainder .
  OreReduce: LPP = {0, 1}
  OreReduce: Put leading term  into remainder .
  OreReduce: LPP = {0, 0}
  OreReduce: Put leading term  into remainder .
OreGroebnerBasis: Does not reduce to zero -> number  4 in the basis.
                  The lpp is {1, 3}. The ByteCount is 629320.
OreGroebnerBasis: Number  of pairs: 1
OreGroebnerBasis: Taking {5, {2, 3}, 3, 4}
  OreReduce: LPP = {1, 4}
  OreReduce: reduced with nr. 4
```

```
  OreReduce: LPP = {0, 5}
  OreReduce: Put leading term  into remainder .
  OreReduce: LPP = {2, 2}
  OreReduce: reduced with nr. 3
  OreReduce: LPP = {1, 3}
  OreReduce: reduced with nr. 4
  OreReduce: LPP = {0, 4}
  OreReduce: Put leading term  into remainder .
  OreReduce: LPP = {3, 0}
  OreReduce: reduced with nr. 1
  OreReduce: LPP = {2, 1}
  OreReduce: reduced with nr. 3
  OreReduce: LPP = {1, 2}
  OreReduce: Put leading term  into remainder .
  OreReduce: LPP = {0, 3}
  OreReduce: Put leading term  into remainder .
  OreReduce: LPP = {2, 0}
  OreReduce: Put leading term  into remainder .
  OreReduce: LPP = {1, 1}
  OreReduce: Put leading term  into remainder .
  OreReduce: LPP = {0, 2}
  OreReduce: Put leading term  into remainder .
  OreReduce: LPP = {1, 0}
  OreReduce: Put leading term  into remainder .
  OreReduce: LPP = {0, 1}
  OreReduce: Put leading term  into remainder .
  OreReduce: LPP = {0, 0}
  OreReduce: Put leading term  into remainder .
OreGroebnerBasis: Does not reduce to zero -> number  5 in the basis.
                  The lpp is {0, 5}. The ByteCount is 3324608.
OreGroebnerBasis: Number  of pairs: 1
OreGroebnerBasis: Taking {6, {1, 5}, 4, 5}
  OreReduce: LPP = {0, 6}
  OreReduce: reduced with nr. 5
  OreReduce: LPP = {1, 4}
  OreReduce: reduced with nr. 4
```

```
  OreReduce: LPP = {0, 5}
  OreReduce: reduced with nr. 5
  OreReduce: LPP = {2, 2}
  OreReduce: reduced with nr. 3
  OreReduce: LPP = {1, 3}
  OreReduce: reduced with nr. 4
  OreReduce: LPP = {0, 4}
  OreReduce: Put leading term  into remainder .
  OreReduce: LPP = {3, 0}
  OreReduce: reduced with nr. 1
  OreReduce: LPP = {2, 1}
  OreReduce: reduced with nr. 3
  OreReduce: LPP = {1, 2}
  OreReduce: Put leading term  into remainder .
  OreReduce: LPP = {0, 3}
  OreReduce: Put leading term  into remainder .
  OreReduce: LPP = {2, 0}
  OreReduce: Put leading term  into remainder .
  OreReduce: LPP = {1, 1}
  OreReduce: Put leading term  into remainder .
  OreReduce: LPP = {0, 2}
  OreReduce: Put leading term  into remainder .
  OreReduce: LPP = {1, 0}
  OreReduce: Put leading term  into remainder .
  OreReduce: LPP = {0, 1}
  OreReduce: Put leading term  into remainder .
  OreReduce: LPP = {0, 0}
  OreReduce: Put leading term  into remainder .
OreGroebnerBasis: Does not reduce to zero -> number  6 in the basis.
                  The lpp is {0, 4}. The ByteCount is 96800.
OreGroebnerBasis: Number  of pairs: 2
OreGroebnerBasis: Taking {7, {1, 4}, 4, 6}
  OreReduce: LPP = {0, 5}
  OreReduce: reduced with nr. 6
  OreReduce: LPP = {2, 2}
  OreReduce: reduced with nr. 3
```

```
  OreReduce: LPP = {1, 3}
  OreReduce: reduced with nr. 4
  OreReduce: LPP = {0, 4}
  OreReduce: reduced with nr. 6
  OreReduce: LPP = {3, 0}
  OreReduce: reduced with nr. 1
  OreReduce: LPP = {2, 1}
  OreReduce: reduced with nr. 3
OreGroebnerBasis: Number of pairs: 1
OreGroebnerBasis: Taking {7, {0, 5}, 5, 6}
  OreReduce: LPP = {1, 3}
  OreReduce: reduced with nr. 4
  OreReduce: LPP = {0, 4}
  OreReduce: reduced with nr. 6
  OreReduce: LPP = {2, 1}
  OreReduce: reduced with nr. 3
OreGroebnerBasis: Reducing no. 1 of 4
  OreReduce: LPP = {3, 0}
  OreReduce: Put leading term into remainder .
  OreReduce: LPP = {2, 1}
  OreReduce: reduced with nr. 2
  OreReduce: LPP = {1, 2}
  OreReduce: Put leading term into remainder .
  OreReduce: LPP = {0, 3}
  OreReduce: Put leading term into remainder .
  OreReduce: LPP = {2, 0}
  OreReduce: Put leading term into remainder .
  OreReduce: LPP = {1, 1}
  OreReduce: Put leading term into remainder .
  OreReduce: LPP = {0, 2}
  OreReduce: Put leading term into remainder .
  OreReduce: LPP = {1, 0}
  OreReduce: Put leading term into remainder .
  OreReduce: LPP = {0, 1}
  OreReduce: Put leading term into remainder .
  OreReduce: LPP = {0, 0}
```

```
  OreReduce: Put leading term  into remainder .
OreGroebnerBasis: Reducing no. 2 of 4
  OreReduce: LPP = {2, 1}
  OreReduce: Put leading term  into remainder .
  OreReduce: LPP = {1, 2}
  OreReduce: Put leading term  into remainder .
  OreReduce: LPP = {0, 3}
  OreReduce: Put leading term  into remainder .
  OreReduce: LPP = {2, 0}
  OreReduce: Put leading term  into remainder .
  OreReduce: LPP = {1, 1}
  OreReduce: Put leading term  into remainder .
  OreReduce: LPP = {0, 2}
  OreReduce: Put leading term  into remainder .
  OreReduce: LPP = {1, 0}
  OreReduce: Put leading term  into remainder .
  OreReduce: LPP = {0, 1}
  OreReduce: Put leading term  into remainder .
  OreReduce: LPP = {0, 0}
  OreReduce: Put leading term  into remainder .
OreGroebnerBasis: Reducing no. 3 of 4
  OreReduce: LPP = {1, 3}
  OreReduce: Put leading term  into remainder .
  OreReduce: LPP = {0, 4}
  OreReduce: reduced with nr. 4
  OreReduce: LPP = {1, 2}
  OreReduce: Put leading term  into remainder .
  OreReduce: LPP = {0, 3}
  OreReduce: Put leading term  into remainder .
  OreReduce: LPP = {2, 0}
  OreReduce: Put leading term  into remainder .
  OreReduce: LPP = {1, 1}
  OreReduce: Put leading term  into remainder .
  OreReduce: LPP = {0, 2}
  OreReduce: Put leading term  into remainder .
  OreReduce: LPP = {1, 0}
```

```
    OreReduce: Put leading term  into remainder .
    OreReduce: LPP = {0, 1}
    OreReduce: Put leading term  into remainder .
    OreReduce: LPP = {0, 0}
    OreReduce: Put leading term  into remainder .
  OreGroebnerBasis: Reducing no. 4 of 4
    OreReduce: LPP = {0, 4}
    OreReduce: Put leading term  into remainder .
    OreReduce: LPP = {1, 2}
    OreReduce: Put leading term  into remainder .
    OreReduce: LPP = {0, 3}
    OreReduce: Put leading term  into remainder .
    OreReduce: LPP = {2, 0}
    OreReduce: Put leading term  into remainder .
    OreReduce: LPP = {1, 1}
    OreReduce: Put leading term  into remainder .
    OreReduce: LPP = {0, 2}
    OreReduce: Put leading term  into remainder .
    OreReduce: LPP = {1, 0}
    OreReduce: Put leading term  into remainder .
    OreReduce: LPP = {0, 1}
    OreReduce: Put leading term  into remainder .
    OreReduce: LPP = {0, 0}
    OreReduce: Put leading term  into remainder .
{90.829000, Null}
```

```
MyReconst[n_Integer, b_Integer, x_] :=
   Module[{digits}, digits = Prepend[IntegerDigits[Abs[n], b], 0] //.
        {a1___, a2_, a3_ /; a3 > b / 2, a4___} :> {a1, a2 + 1, a3 - b, a4};
     Return[Sign[n] * (digits.Reverse[x ^ Range[0, Length[digits] - 1]])];];
```

```
MyReconst[6 797 200 431 965 601 240 000, 10 ^ 4, n]
```

$124\,n - 344\,n^2 + 432\,n^3 - 280\,n^4 + 68\,n^5$

```
% /. n → 10 ^ 4
```

6 797 200 431 965 601 240 000

```
gbRV = NormalizeCoefficients /@ (gb1 /. a_Integer /; Abs[a] > 10 ^ 4 / 2 :> MyReconst[a, 10 ^ 4, n]);
```

```
OreReduce[ops, gbRV]
```

{0, 0}

```
Support[gbRV]
```

$\{\{D_y^2 D_z, D_y D_z^2, D_z^3, D_y^2, D_y D_z, D_z^2, D_y, D_z, 1\}, \{D_y^3, D_y D_z^2, D_z^3, D_y^2, D_y D_z, D_z^2, D_y, D_z, 1\}, \{D_z^4, D_y D_z^2, D_z^3, D_y^2, D_y D_z, D_z^2, D_y, D_z, 1\}, \{D_y D_z^3, D_y D_z^2, D_z^3, D_y^2, D_y D_z, D_z^2, D_y, D_z, 1\}\}$

```
ByteCount[gbRV]
```

697856

```
UnderTheStaircase[gbRV]
```

$\{1, D_z, D_y, D_z^2, D_y D_z, D_y^2, D_z^3, D_y D_z^2\}$

```
Timing[opDy1 = FindRelation[gbRV /. n → 10^9, Pattern → {_, 0}][[1]];]
```

{35.717000, Null}

```
opDy = NormalizeCoefficients[2 * opDy1 /. a_Integer /; Abs[a] > 10^9/2 :→ MyReconst[a, 10^9, n]];
```

```
Support[opDy]
```

$\{D_y^8, D_y^7, D_y^6, D_y^5, D_y^4, D_y^3, D_y^2, D_y, 1\}$

```
ByteCount[opDy]
```

19635376

```
Exponent[opDy, {n, x, y, z}]
```

{19, 14, 21, 22}

```
Max[Abs[Cases[opDy, _Integer, Infinity]]]
```

325844952

Worked Example 2

```
Sum[Pochhammer[N, k] / Pochhammer[N_R + n, k] * (x * s / (1 - s))^k / k!, {k, 0, Infinity}]
```

$\text{Hypergeometric1F1}\left[N, n + N_R, -\frac{s x}{-1 + s}\right]$

```
ops = {S[k], S[n], Der[s], Der[x], Der[y]}
```

{S[k], S[n], Der[s], Der[x], Der[y]}

```
ApplyOreOperator[S[k] - (N + k), Pochhammer[N, k]] // FullSimplify
```

0

```
ann1 = Annihilator[Pochhammer[N, k], ops]
```

$\{D_y, D_x, D_s, S_n - 1, S_k + (-k - N)\}$

```
ann2 = Annihilator[1 / Pochhammer[N_R + n, k], ops]
```

$\{D_y, D_x, D_s, (k + n + N_R) S_n + (-n - N_R), (k + n + N_R) S_k - 1\}$

```
ann3 = Annihilator[(x * s / (1 - s))^k, ops]
```

$\{D_y, x D_x - k, (-s + s^2) D_s + k, S_n - 1, (-1 + s) S_k + s x\}$

```
ann4 = Annihilator[1 / k!, ops]
```

$\{D_y, D_x, D_s, S_n - 1, (1 + k) S_k - 1\}$

DFiniteTimes[ann1, ann2, ann3, ann4]

$\{D_y,\ x D_x - k,\ (-s + s^2) D_s + k,\ (k + n + N_R) S_n + (-n - N_R),$
$(-k - k^2 - n - k n + k s + k^2 s + n s + k n s - N_R - k N_R + s N_R + k s N_R) S_k + (k s x + N s x)\}$

?DFinite*

▼ RISC`HolonomicFunctions`

DFinite	DFinitePlus	DFiniteSubstitute
DFiniteDE2RE	DFiniteQSubstitute	DFiniteTimes
DFiniteOreAction	DFiniteRE2DE	DFiniteTimesHyper

annSmnd = Annihilator[Pochhammer [N, k] / Pochhammer [N_R + n, k] * (x * s / (1 - s)) ^k / k!, ops]

Annihilator::opx : Cannot handle the operators {ops}.

$Failed

ops = Rest[ops]

{S[n], Der[s], Der[x], Der[y]}

{pp, qq} = CreativeTelescoping[annSmnd , S[k] - 1, ops]

$\{\{D_y,\ (-s + s^2) D_s + x D_x,\ (n s - N s + s N_R) S_n + (n - n s + N_R - s N_R) D_x + (-n s - s N_R),$
$(-x + s x) D_x^2 + (-n + n s + s x - N_R + s N_R) D_x + N s\},$
$\{0,\ 0,\ -\frac{k(-1+s)(n+N_R)}{x},\ \frac{(-1+s)(-k+k^2+k n+k N_R)}{x}\}\}$

TableForm [Transpose[{pp, qq}]]

D_y	0
$(-s + s^2) D_s + x D_x$	0
$(n s - N s + s N_R) S_n + (n - n s + N_R - s N_R) D_x + (-n s - s N_R)$	$-\frac{k(-1+s)(n+N_R)}{x}$
$(-x + s x) D_x^2 + (-n + n s + s x - N_R + s N_R) D_x + N s$	$\frac{(-1+s)(-k+k^2+k n+k N_R)}{x}$

pp[[3]] + (S[k] - 1) ** qq[[3]]

$-\frac{(1+k)(-1+s)(n+N_R)}{x} S_k + (n s - N s + s N_R) S_n +$
$(n - n s + N_R - s N_R) D_x + \frac{1}{x}(-k n + k n s - n s x - k N_R + k s N_R - s x N_R)$

OreReduce[%, annSmnd]

0

test = ApplyOreOperator[qq[[3]],
Pochhammer [N, k] / Pochhammer [N_R + n, k] * (x * s / (1 - s)) ^k / k!]

$-\left(\left(k(-1+s)\left(\frac{s x}{1-s}\right)^k \text{Pochhammer}[N, k]\,(n + N_R)\right) \Big/ \left(x\, k!\, \text{Pochhammer}[n + N_R, k]\right)\right)$

test /. k → 0

0

Limit[test, k → Infinity]

0

```
Annihilator[
  Sum [Pochhammer  [N, k] / Pochhammer  [N_R + n, k] * (x * s / (1 - s)) ^ k / k !, {k, 0, Infinity}], ops]
```

$$\{D_y,\ (-s+s^2)\,D_s + x\,D_x,\ (n s - N s + s N_R)\,S_n + (n - n s + N_R - s N_R)\,D_x + (-n s - s N_R),\ (-x + s x)\,D_x^2 + (-n + n s + s x - N_R + s N_R)\,D_x + N s\}$$

```
pp
```

$$\{D_y,\ (-s+s^2)\,D_s + x\,D_x,\ (n s - N s + s N_R)\,S_n + (n - n s + N_R - s N_R)\,D_x + (-n s - s N_R),\ (-x + s x)\,D_x^2 + (-n + n s + s x - N_R + s N_R)\,D_x + N s\}$$

```
Annihilator[Hypergeometric1F1 [N, N_R + n, x * s / (1 - s)], ops]
```

$$\{D_y,\ (-s+s^2)\,D_s + x\,D_x,\ (n s - N s + s N_R)\,S_n + (n - n s + N_R - s N_R)\,D_x + (-n s - s N_R),\ (-x + s x)\,D_x^2 + (-n + n s + s x - N_R + s N_R)\,D_x + N s\}$$

```
ApplyOreOperator[pp, Hypergeometric1F1 [N, N_R + n, x * s / (1 - s)]] // FullSimplify
```

{0, 0, 0, 0}

```
ops = Rest[ops]
```

{Der[s], Der[x], Der[y]}

```
smnd = Exp[-y] / (1 - s) ^ N * y ^ n / n ! * Hypergeometric1F1 [N, N_R + n, x * s / (1 - s)]
```

$$\frac{1}{n!}e^{-y}(1-s)^{-N}y^n\,\text{Hypergeometric1F1}\left[N,\ n+N_R,\ \frac{s x}{1-s}\right]$$

```
Annihilator[Sum [smnd , {n, 0, Infinity}], ops]
```

Annihilator::fail : Cannot evaluate the delta part of a sum (upper bound causes problems).

$Failed

```
Annihilator[Sum [smnd , {n, 0, Infinity}], ops, Inhomogeneous → True]
```

$$\Big\{\{(-s+s^2)\,D_s + x\,D_x + N s,$$
$$(-s x y + y^2 - s y^2)\,D_y^2 + (-x N_R + s x N_R)\,D_x + (N s x - s x y + y^2 - s y^2 - s x N_R + y N_R - s y N_R)\,D_y + N s x,$$
$$(s x - y + s y)\,D_x D_y + (s x - y + s y - N_R + s N_R)\,D_x + N s\,D_y + N s,$$
$$(s x^2 - s^2 x^2 - x y + 2 s x y - s^2 x y)\,D_x^2 + (-s^2 x^2 + s x y - s^2 x y + s x N_R - s^2 x N_R)\,D_x + (-N s y + N s^2 y)\,D_y - N s^2 x\},$$
$$\Big\{\text{Hold}[\text{Limit}[0, n\to\infty]],\ \text{Hold}\Big[\text{Limit}\Big[\frac{1}{n!}e^{-y}(1-s)^{-N}y^{-1+n}\,\text{Hypergeometric1F1}\left[N,\ n+N_R,\ \frac{s x}{1-s}\right]$$
$$(n s x - n^2 s x + n N s x - n y + n^2 y + n s y - n^2 s y - n s x N_R + n y N_R - n s y N_R) - \frac{1}{n!}$$
$$e^{-y} n (1-s)^{-1-N}(-1+s)\,s x y^n\,\text{Hypergeometric1F1}^{(0,0,1)}\left[N,\ n+N_R,\ \frac{s x}{1-s}\right],\ n\to\infty\Big]\Big],$$
$$\text{Hold}\Big[\text{Limit}\Big[\frac{1}{n!}e^{-y} n N (1-s)^{-N} s\,y^{-1+n}\,\text{Hypergeometric1F1}\left[N,\ n+N_R,\ \frac{s x}{1-s}\right] + \frac{1}{n!}$$
$$e^{-y} n (1-s)^{-1-N} s^2 x\,y^{-1+n}\,\text{Hypergeometric1F1}^{(0,0,1)}\left[N,\ n+N_R,\ \frac{s x}{1-s}\right],\ n\to\infty\Big]\Big],$$
$$\text{Hold}\Big[\text{Limit}\Big[\frac{1}{n!}e^{-y} n N (1-s)^{-N}(-1+s)\,s\,y^n\,\text{Hypergeometric1F1}\left[N,\ n+N_R,\ \frac{s x}{1-s}\right] + \frac{1}{n!}$$
$$e^{-y} n (1-s)^{-1-N}(-1+s)\,s^2 x y^n\,\text{Hypergeometric1F1}^{(0,0,1)}\left[N,\ n+N_R,\ \frac{s x}{1-s}\right],\ n\to\infty\Big]\Big]\Big\}\Big\}$$

```
annM = First[CreativeTelescoping[smnd , S[n] - 1, ops]]
```

$$\{(-s+s^2)\,D_s + x\,D_x + N s,$$
$$(-s x y + y^2 - s y^2)\,D_y^2 + (-x N_R + s x N_R)\,D_x + (N s x - s x y + y^2 - s y^2 - s x N_R + y N_R - s y N_R)\,D_y + N s x,$$
$$(s x - y + s y)\,D_x D_y + (s x - y + s y - N_R + s N_R)\,D_x + N s\,D_y + N s,$$
$$(s x^2 - s^2 x^2 - x y + 2 s x y - s^2 x y)\,D_x^2 + (-s^2 x^2 + s x y - s^2 x y + s x N_R - s^2 x N_R)\,D_x + (-N s y + N s^2 y)\,D_y - N s^2 x\}$$

```
UnderTheStaircase[annM]
```

$$\{1, D_y, D_x\}$$

```
(* Eq (11) *)
OreReduce[(s (1-s)^2)**Der[s]^2-(2 (N+1) s (1-s)-(1-s) N_R+x s)**Der[s]+
    N*((N+1) s-N_R-x)-y**(1+Der[y])**(N-(1-s)**Der[s]), Together[annM], Extended→True]
```

$$\left\{0, 1, \left\{(-1+s) D_s - \frac{x}{s} D_x - \frac{y}{s} D_y + (-1+s-N s+N s^2-s x+y-s y+N_R-s N_R) \big/ ((-1+s) s),\right.\right.$$
$$\left.\left. 0, \frac{x y}{s (s x-y+s y)}, -\frac{x}{(-s+s^2) (s x-y+s y)}\right\}\right\}$$

```
(* Eq (14) *)
OreReduce[
  (y**(1+Der[y])+NR)**(Der[s]**(1-s)^2+2 (1-s))-(N*y)**(1+Der[y])**(1-s)-
    (NR*N)**(Der[s]**(1-s)+1)**Der[s]+(x*y)**(1+Der[y])**Der[y]+
    x*NR*Der[y]-(N*x)**(1+Der[y]), Together[ann], Extended→True]
```

```
(* Eq (15) *)
OreReduce[-(1-s)**(y^2**(1+Der[y])^2+(2 NR+2)**y**(1+Der[y])+NR*(NR+1))-
    (x*s)**(NR+y**Der[y]+1)**(1+Der[y])**Der[y]-
    (N*x*s)**(1+Der[y])^2, Together[ann], Extended→True]
```

```
ChangeOreAlgebra[annM, OreAlgebra[Der[s], Der[x], Der[y], s]]
```

$$\{D_s s^2 - D_s s + x D_x + (-2+N) s + 1,$$
$$(-x y - y^2) D_y^2 s + x N_R D_x s + y^2 D_y^2 + (N x - x y - y^2 - x N_R - y N_R) D_y s - x N_R D_x + (y^2 + y N_R) D_y + N x s,$$
$$(x+y) D_x D_y s - y D_x D_y + (x + y + N_R) D_x s + N D_y s + (-y - N_R) D_x + N s,$$
$$(-x^2 - x y) D_x^2 s^2 + (x^2 + 2 x y) D_x^2 s + (-x^2 - x y - x N_R) D_x s^2 + N y D_y s^2 - x y D_x^2 + (x y + x N_R) D_x s - N y D_y s - N x s^2\}$$

```
Normal /@%
```

$$\{1 + (-2+N) s - s\,\mathrm{Der}[s] + s^2\,\mathrm{Der}[s] + x\,\mathrm{Der}[x], N s x + y^2\,\mathrm{Der}[y]^2 + s (-x y - y^2)\,\mathrm{Der}[y]^2 -$$
$$x\,\mathrm{Der}[x]\,N_R + s x\,\mathrm{Der}[x]\,N_R + s\,\mathrm{Der}[y]\,(N x - x y - y^2 - x N_R - y N_R) + \mathrm{Der}[y]\,(y^2 + y N_R),$$
$$N s + N s\,\mathrm{Der}[y] - y\,\mathrm{Der}[x]\,\mathrm{Der}[y] + s (x+y)\,\mathrm{Der}[x]\,\mathrm{Der}[y] + \mathrm{Der}[x]\,(-y - N_R) + s\,\mathrm{Der}[x]\,(x + y + N_R),$$
$$-N s^2 x - x y\,\mathrm{Der}[x]^2 + s^2 (-x^2 - x y)\,\mathrm{Der}[x]^2 + s (x^2 + 2 x y)\,\mathrm{Der}[x]^2 -$$
$$N s y\,\mathrm{Der}[y] + N s^2 y\,\mathrm{Der}[y] + s^2\,\mathrm{Der}[x]\,(-x^2 - x y - x N_R) + s\,\mathrm{Der}[x]\,(x y + x N_R)\}$$

```
%/.{Der[s]→t, s→-Der[t]}
```

$$\{1 - (-2+N)\,\mathrm{Der}[t] + t\,\mathrm{Der}[t] + t\,\mathrm{Der}[t]^2 + x\,\mathrm{Der}[x],$$
$$-N x\,\mathrm{Der}[t] + y^2\,\mathrm{Der}[y]^2 - (-x y - y^2)\,\mathrm{Der}[t]\,\mathrm{Der}[y]^2 - x\,\mathrm{Der}[x]\,N_R - x\,\mathrm{Der}[t]\,\mathrm{Der}[x]\,N_R -$$
$$\mathrm{Der}[t]\,\mathrm{Der}[y]\,(N x - x y - y^2 - x N_R - y N_R) + \mathrm{Der}[y]\,(y^2 + y N_R), -N\,\mathrm{Der}[t] - N\,\mathrm{Der}[t]\,\mathrm{Der}[y] -$$
$$y\,\mathrm{Der}[x]\,\mathrm{Der}[y] - (x+y)\,\mathrm{Der}[t]\,\mathrm{Der}[x]\,\mathrm{Der}[y] + \mathrm{Der}[x]\,(-y - N_R) - \mathrm{Der}[t]\,\mathrm{Der}[x]\,(x + y + N_R),$$
$$-N x\,\mathrm{Der}[t]^2 - x y\,\mathrm{Der}[x]^2 - (x^2 + 2 x y)\,\mathrm{Der}[t]\,\mathrm{Der}[x]^2 + (-x^2 - x y)\,\mathrm{Der}[t]^2\,\mathrm{Der}[x]^2 +$$
$$N y\,\mathrm{Der}[t]\,\mathrm{Der}[y] + N y\,\mathrm{Der}[t]^2\,\mathrm{Der}[y] + \mathrm{Der}[t]^2\,\mathrm{Der}[x]\,(-x^2 - x y - x N_R) - \mathrm{Der}[t]\,\mathrm{Der}[x]\,(x y + x N_R)\}$$

```
trans=ToOrePolynomial [Expand[%]]
```

$$\{t D_t^2 + (2 - N + t) D_t + x D_x + 1,$$
$$(x y + y^2) D_t D_y^2 - x N_R D_t D_x + (-N x + x y + y^2 + x N_R + y N_R) D_t D_y + y^2 D_y^2 - N x D_t - x N_R D_x + (y^2 + y N_R) D_y,$$
$$(-x - y) D_t D_x D_y + (-x - y - N_R) D_t D_x - N D_t D_y - y D_x D_y - N D_t + (-y - N_R) D_x, (-x^2 - x y) D_t^2 D_x^2 +$$
$$(-x^2 - x y - x N_R) D_t^2 D_x + N y D_t^2 D_y + (-x^2 - 2 x y) D_t D_x^2 - N x D_t^2 + (-x y - x N_R) D_t D_x + N y D_t D_y - x y D_x^2\}$$

```
LeadingPowerProduct/@trans
```

$$\{D_t^2, D_t D_y^2, D_t D_x D_y, D_t^2 D_x^2\}$$

```
annp1 = OreGroebnerBasis[trans]
```

$$\begin{aligned}
&\{x D_x D_y + y D_y^2 + x D_x + (y + N_R) D_y,\ t D_t^2 + (2 - N + t) D_t + x D_x + 1, \\
&\quad (-x^2 y^2 - 2 x y^3 - y^4) D_y^3 + t x^2 N_R D_t D_x - N t x y D_t D_y + (-x^3 N_R - x^2 y N_R) D_x^2 + \\
&\qquad (N x^2 y - 2 x y^2 + 2 N x y^2 - t x y^2 - x^2 y^2 - 2 y^3 + N y^3 - 2 x y^3 - y^4 - 2 x^2 y N_R - 4 x y^2 N_R - 2 y^3 N_R) D_y^2 - \\
&\qquad N t x y D_t + (-N x^2 N_R + t x^2 N_R - x^3 N_R - N x y N_R - 2 x^2 y N_R - x y^2 N_R) D_x + \\
&\qquad (-N x^2 - N x y + N x^2 y - 2 x y^2 + 2 N x y^2 - t x y^2 - 2 y^3 + N y^3 + x^2 N_R + N x^2 N_R + 2 N x y N_R - t x y N_R - \\
&\qquad\quad x^2 y N_R - y^2 N_R + N y^2 N_R - 2 x y^2 N_R - y^3 N_R - x^2 N_R^2 - 2 x y N_R^2 - y^2 N_R^2) D_y + (-N x^2 - N x y), \\
&\quad (-x^4 - 2 x^3 y - x^2 y^2) D_x^3 + (-t x^3 - 2 t x^2 y - t x y^2 - t x y N_R) D_t D_x + N t y^2 D_t D_y + \\
&\qquad (-x^3 - N x^3 - x^4 - 2 x^2 y - 2 N x^2 y - 2 x^3 y - x y^2 - N x y^2 - x^2 y^2 - x^3 N_R - x^2 y N_R) D_x^2 + (-2 x y^2 - 2 y^3 + t y^3) D_y^2 + \\
&\qquad (-N t x^2 - 2 N t x y) D_t + (-2 x^3 - N x^3 - 4 x^2 y - 2 N x^2 y - 2 x y^2 - N x y^2 - N x^2 N_R - N x y N_R - t x y N_R) D_x + \\
&\qquad (N x y + N y^2 - 2 x y^2 - 2 y^3 + t y^3 - 2 x y N_R - 2 y^2 N_R + t y^2 N_R) D_y + (-N x^2 - N x y), \\
&\quad (x y + y^2) D_t D_y^2 - x N_R D_t D_x + (-N x + x y + y^2 + x N_R + y N_R) D_t D_y + y^2 D_y^2 - N x D_t - x N_R D_x + (y^2 + y N_R) D_y, \\
&\quad (-x^2 - x y) D_t D_x^2 + (-x^2 - x y - x N_R) D_t D_x + N y D_t D_y + (-x^2 - x y) D_x^2 + y^2 D_y^2 - N x D_t - x N_R D_x + (y^2 + y N_R) D_y\}
\end{aligned}$$

```
HolonomicRank [annp1]
```

8

```
PrependTo[ops, Der[t]]
```

```
{Der[t], Der[s], Der[x], Der[y]}
```

```
ToOrePolynomial [annM, OreAlgebra@@ops]
```

$$\begin{aligned}
&\{(-s + s^2) D_s + x D_x + N s, \\
&\quad (-s x y + y^2 - s y^2) D_y^2 + (-x N_R + s x N_R) D_x + (N s x - s x y + y^2 - s y^2 - s x N_R + y N_R - s y N_R) D_y + N s x, \\
&\quad (s x - y + s y) D_x D_y + (s x - y + s y - N_R + s N_R) D_x + N s D_y + N s, \\
&\quad (s x^2 - s^2 x^2 - x y + 2 s x y - s^2 x y) D_x^2 + (-s^2 x^2 + s x y - s^2 x y + s x N_R - s^2 x N_R) D_x + (-N s y + N s^2 y) D_y - N s^2 x\}
\end{aligned}$$

```
HolonomicRank [%]
```

Throw::nocatch : Uncaught Throw[Not holonomic] returned to top level. ≫

```
Hold[Throw[Not holonomic ]]
```

```
ToOrePolynomial [Append[annM, Der[t]], OreAlgebra@@ops]
```

$$\begin{aligned}
&\{(-s + s^2) D_s + x D_x + N s, \\
&\quad (-s x y + y^2 - s y^2) D_y^2 + (-x N_R + s x N_R) D_x + (N s x - s x y + y^2 - s y^2 - s x N_R + y N_R - s y N_R) D_y + N s x, \\
&\quad (s x - y + s y) D_x D_y + (s x - y + s y - N_R + s N_R) D_x + N s D_y + N s, \\
&\quad (s x^2 - s^2 x^2 - x y + 2 s x y - s^2 x y) D_x^2 + (-s^2 x^2 + s x y - s^2 x y + s x N_R - s^2 x N_R) D_x + (-N s y + N s^2 y) D_y - N s^2 x,\ D_t\}
\end{aligned}$$

```
ann1 = DFiniteTimes[%, Annihilator[Exp[-s*t], ops]]
```

$$\begin{aligned}
&\{(-s + s^2) D_s + x D_x + (N s - s t + s^2 t),\ D_t + s, \\
&\quad (s x y - y^2 + s y^2) D_y^2 + (x N_R - s x N_R) D_x + (-N s x + s x y - y^2 + s y^2 + s x N_R - y N_R + s y N_R) D_y - N s x, \\
&\quad (s x - y + s y) D_x D_y + (s x - y + s y - N_R + s N_R) D_x + N s D_y + N s, \\
&\quad (-s x^2 + s^2 x^2 + x y - 2 s x y + s^2 x y) D_x^2 + (s^2 x^2 - s x y + s^2 x y - s x N_R + s^2 x N_R) D_x + (N s y - N s^2 y) D_y + N s^2 x\}
\end{aligned}$$

```
annp = First[CreativeTelescoping[ann1, Der[s]]]
```

$$\begin{aligned}
&\{(x^2 y + 2 x y^2 + y^3) D_y^2 + t x N_R D_t + (-x^2 N_R - x y N_R) D_x + \\
&\qquad (-N x^2 - N x y + t x y + x^2 y + 2 x y^2 + y^3 + x^2 N_R + 2 x y N_R + y^2 N_R) D_y + (-N x^2 - N x y + t x y + x N_R - N x N_R + t x N_R), \\
&\quad (x^2 + 2 x y + y^2) D_x D_y - t N_R D_t + (x^2 + 2 x y + y^2 + x N_R + y N_R) D_x + \\
&\qquad (N x + N y - t y) D_y + (N x + N y - t y - N_R + N N_R - t N_R), \\
&\quad (x^3 + 2 x^2 y + x y^2) D_x^2 + (t x^2 + 2 t x y + t y^2 + t y N_R) D_t + (x^3 + 2 x^2 y + x y^2 + x^2 N_R + x y N_R) D_x + \\
&\qquad (-N x y - N y^2 + t y^2) D_y + (x^2 + 2 x y - N x y + y^2 - N y^2 + t y^2 + y N_R - N y N_R + t y N_R), \\
&\quad (t x + t y) D_t D_y + (t x + t y + t N_R) D_t + (x - N x + y - N y + t y) D_y + (x - N x + y - N y + t y + N_R - N N_R + t N_R), \\
&\quad (t x^2 + t x y) D_t D_x + (-t x + t x^2 - t y + t x y + t x N_R) D_t + (x^2 - N x^2 + t x^2 + x y - N x y + t x y) D_x + t x y D_y + \\
&\qquad (-x + N x - t x + x^2 - N x^2 - y + N y - t y + x y - N x y + t x y + x N_R - N x N_R + t x N_R),\ t D_t^2 + (2 - N + t) D_t + x D_x + 1\}
\end{aligned}$$

```
UnderTheStaircase[annp]
```

$$\{1, D_y, D_x, D_t\}$$

```
Length/@{annp1, annp}
```

{6, 6}

```
OreReduce[annp1, annp]
```

{0, 0, 0, 0, 0, 0}

```
LeadingExponent/@UnderTheStaircase[annp]
```

{{0, 0, 0}, {0, 0, 1}, {0, 1, 0}, {1, 0, 0}}

```
mycfl [op_, uts_] := Total[Cases[op[[1]], {a_, LeadingExponent[#]} :→ a]] & /@uts;
PfaffianSystem [ann_, op_] :=
    With[{uts = UnderTheStaircase[ann]},
      mycfl [OreReduce[op ** #, Together[ann]], uts] & /@uts];
MatrixForm [PfaffianSystem [annp, Der[t]]]
```

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ \frac{-x+Nx-y+Ny-ty-N_R+NN_R-tN_R}{t(x+y)} & \frac{-x+Nx-y+Ny-ty}{t(x+y)} & 0 & \frac{-x-y-N_R}{x+y} \\ \frac{x-Nx+tx-x^2+Nx^2+y-Ny+ty-xy+Nxy-txy-xN_R+NxN_R-txN_R}{tx(x+y)} & -\frac{y}{x+y} & \frac{-1+N-t}{t} & \frac{x-x^2+y-xy-xN_R}{x(x+y)} \\ -\frac{1}{t} & 0 & -\frac{x}{t} & \frac{-2+N-t}{t} \end{pmatrix}$$

```
MatrixForm [PfaffianSystem [annp, Der[x]]]
```

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ \frac{-Nx-Ny+ty+N_R-NN_R+tN_R}{(x+y)^2} & \frac{-Nx-Ny+ty}{(x+y)^2} & \frac{-x-y-N_R}{x+y} & \frac{tN_R}{(x+y)^2} \\ \frac{-x^2-2xy+Nxy-y^2+Ny^2-ty^2-yN_R+NyN_R-tyN_R}{x(x+y)^2} & \frac{Nxy+Ny^2-ty^2}{x(x+y)^2} & \frac{-x-y-N_R}{x+y} & \frac{-tx^2-2txy-ty^2-tyN_R}{x(x+y)^2} \\ \frac{x-Nx+tx-x^2+Nx^2+y-Ny+ty-xy+Nxy-txy-xN_R+NxN_R-txN_R}{tx(x+y)} & -\frac{y}{x+y} & \frac{-1+N-t}{t} & \frac{x-x^2+y-xy-xN_R}{x(x+y)} \end{pmatrix}$$

```
MatrixForm [PfaffianSystem [annp, Der[y]]]
```

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{Nx^2+Nxy-txy-xN_R+NxN_R-txN_R}{y(x+y)^2} & \frac{Nx^2+Nxy-txy-x^2y-2xy^2-y^3-x^2N_R-2xyN_R-y^2N_R}{y(x+y)^2} & \frac{xN_R}{y(x+y)} & -\frac{txN_R}{y(x+y)^2} \\ \frac{-Nx-Ny+ty+N_R-NN_R+tN_R}{(x+y)^2} & \frac{-Nx-Ny+ty}{(x+y)^2} & \frac{-x-y-N_R}{x+y} & \frac{tN_R}{(x+y)^2} \\ \frac{-x+Nx-y+Ny-ty-N_R+NN_R-tN_R}{t(x+y)} & \frac{-x+Nx-y+Ny-ty}{t(x+y)} & 0 & \frac{-x-y-N_R}{x+y} \end{pmatrix}$$

```
annplog = DFiniteTimes[annp, Annihilator[Log[t + 1], {Der[t], Der[x], Der[y]}]]
```

$$\{x D_x D_y + y D_y^2 + x D_x + (y + N_R) D_y,\ x^2 N_R D_x^2 + (-x^2 y - 2 x y^2 - y^3 - y^2 N_R) D_y^2 + (2 x^2 N_R + x y N_R + x N_R^2) D_x +$$
$$(N x^2 + N x y - t x y - x^2 y - 2 x y^2 - y^3 - x^2 N_R - 2 x y N_R - 2 y^2 N_R - y N_R^2) D_y + (N x^2 + N x y - t x y + N x N_R - t x N_R),$$
$$(x^2 y + 2 x y^2 + y^3) D_y^3 + (x^2 - N x^2 + 4 x y - N x y + t x y + 2 x^2 y + 3 y^2 + 4 x y^2 + 2 y^3 + x^2 N_R + 4 x y N_R + 3 y^2 N_R) D_y^2 +$$
$$(-x N_R - x N_R^2) D_x + (-N x + t x + x^2 - 2 N x^2 + 4 x y - 2 N x y + 2 t x y + x^2 y + 3 y^2 + 2 x y^2 + y^3 + 2 x N_R - N x N_R + t x N_R +$$
$$x^2 N_R + 2 y N_R + 4 x y N_R + 3 y^2 N_R + 2 x N_R^2 + 2 y N_R^2) D_y + (-N x + t x - N x^2 - N x y + t x y - N x N_R + t x N_R),$$
$$(2 t x^2 y + 2 t^2 x^2 y + 4 t x y^2 + 4 t^2 x y^2 + 2 t y^3 + 2 t^2 y^3) D_t D_y^2 + (t^2 x N_R + t^3 x N_R) D_t^2 +$$
$$(-2 t x^2 N_R - 2 t^2 x^2 N_R - 2 t x y N_R - 2 t^2 x y N_R) D_t D_x +$$
$$(-2 N t x^2 - 2 N t^2 x^2 - 2 N t x y + 2 t^2 x y - 2 N t^2 x y + 2 t^3 x y + 2 t x^2 y + 2 t^2 x^2 y + 4 t x y^2 + 4 t^2 x y^2 +$$
$$2 t y^3 + 2 t^2 y^3 + 2 t x^2 N_R + 2 t^2 x^2 N_R + 4 t x y N_R + 4 t^2 x y N_R + 2 t y^2 N_R + 2 t^2 y^2 N_R) D_t D_y +$$
$$(-N x^2 y + 2 t x^2 y - N t x^2 y + t^2 x^2 y - 2 N x y^2 + 4 t x y^2 - 2 N t x y^2 + 2 t^2 x y^2 - N y^3 + 2 t y^3 - N t y^3 + t^2 y^3) D_y^2 +$$
$$(-2 N t x^2 - 2 N t^2 x^2 - 2 N t x y + 2 t^2 x y - 2 N t^2 x y + 2 t^3 x y + 2 t x N_R - 2 N t x N_R + 5 t^2 x N_R - 2 N t^2 x N_R +$$
$$2 t^3 x N_R) D_t + (N x^2 N_R - 3 t x^2 N_R + N t x^2 N_R - 2 t^2 x^2 N_R + N x y N_R - 2 t x y N_R + N t x y N_R - t^2 x y N_R) D_x +$$
$$(N^2 x^2 - 2 N t x^2 + N^2 t x^2 - N t^2 x^2 + N^2 x y + 2 t x y - 3 N t x y + N^2 t x y + 4 t^2 x y - 2 N t^2 x y +$$
$$t^3 x y - N x^2 y + 2 t x^2 y - N t x^2 y + t^2 x^2 y - 2 N x y^2 + 4 t x y^2 - 2 N t x y^2 + 2 t^2 x y^2 -$$
$$N y^3 + 2 t y^3 - N t y^3 + t^2 y^3 - N x^2 N_R + 2 t x^2 N_R - N t x^2 N_R + t^2 x^2 N_R - 2 N x y N_R +$$
$$4 t x y N_R - 2 N t x y N_R + 2 t^2 x y N_R - N y^2 N_R + 2 t y^2 N_R - N t y^2 N_R + t^2 y^2 N_R) D_y +$$
$$(N^2 x^2 - 2 N t x^2 + N^2 t x^2 - N t^2 x^2 + N^2 x y + 2 t x y - 3 N t x y + N^2 t x y + 4 t^2 x y - 2 N t^2 x y +$$
$$t^3 x y - N x N_R + N^2 x N_R + 3 t x N_R - 4 N t x N_R + N^2 t x N_R + 4 t^2 x N_R - 2 N t^2 x N_R + t^3 x N_R),$$
$$(t^2 x + t^3 x + t^2 y + t^3 y) D_t^2 D_y + (t^2 x + t^3 x + t^2 y + t^3 y + t^2 N_R + t^3 N_R) D_t^2 +$$
$$(2 t x - 2 N t x + 3 t^2 x - 2 N t^2 x + 2 t y - 2 N t y + 5 t^2 y - 2 N t^2 y + 2 t^3 y) D_t D_y +$$
$$(t x y + t^2 x y + t y^2 + t^2 y^2) D_y^2 + (2 t x - 2 N t x + 3 t^2 x - 2 N t^2 x + 2 t y -$$

$2 N t y + 5 t^2 y - 2 N t^2 y + 2 t^3 y + 2 t N_R - 2 N t N_R + 5 t^2 N_R - 2 N t^2 N_R + 2 t^3 N_R) D_t +$
$(-t x N_R - t^2 x N_R) D_x + (-N x + N^2 x + t x - 3 N t x + N^2 t x - N t^2 x - N y + N^2 y + 3 t y - 4 N t y + N^2 t y +$
$4 t^2 y - 2 N t^2 y + t^3 y + t x y + t^2 x y + t y^2 + t^2 y^2 + t x N_R + t^2 x N_R + t y N_R + t^2 y N_R) D_y +$
$(-N x + N^2 x + t x - 3 N t x + N^2 t x - N t^2 x - N y + N^2 y + 3 t y - 4 N t y + N^2 t y + 4 t^2 y - 2 N t^2 y + t^3 y - N N_R +$
$N^2 N_R + 3 t N_R - 4 N t N_R + N^2 t N_R + 4 t^2 N_R - 2 N t^2 N_R + t^3 N_R), (t^2 x^2 N_R + t^3 x^2 N_R + t^2 x y N_R + t^3 x y N_R) D_t^2 D_x +$
$(-t^2 x N_R - t^3 x N_R + t^2 x^2 N_R + t^3 x^2 N_R - t^2 y N_R - t^3 y N_R + t^2 x y N_R + t^3 x y N_R + t^2 x N_R^2 + t^3 x N_R^2) D_t^2 +$
$(2 t x^2 N_R - 2 N t x^2 N_R + 5 t^2 x^2 N_R - 2 N t^2 x^2 N_R + 2 t^3 x^2 N_R + 2 t x y N_R - 2 N t x y N_R +$
$5 t^2 x y N_R - 2 N t^2 x y N_R + 2 t^3 x y N_R) D_t D_x + (2 t^2 x y N_R + 2 t^3 x y N_R) D_t D_y +$
$(-t x^3 y - t^2 x^3 y - 3 t x^2 y^2 - 3 t^2 x^2 y^2 - 3 t x y^3 - 3 t^2 x y^3 - t y^4 - t^2 y^4 - t x y^2 N_R - t^2 x y^2 N_R - t y^3 N_R - t^2 y^3 N_R)$
$D_y^2 + (-2 t x N_R + 2 N t x N_R - 5 t^2 x N_R + 2 N t^2 x N_R - 2 t^3 x N_R + 2 t x^2 N_R - 2 N t x^2 N_R + 3 t^2 x^2 N_R -$
$2 N t^2 x^2 N_R - 2 t y N_R + 2 N t y N_R - 5 t^2 y N_R + 2 N t^2 y N_R - 2 t^3 y N_R + 2 t x y N_R - 2 N t x y N_R +$
$5 t^2 x y N_R - 2 N t^2 x y N_R + 2 t^3 x y N_R + 2 t x N_R^2 - 2 N t x N_R^2 + 5 t^2 x N_R^2 - 2 N t^2 x N_R^2 + 2 t^3 x N_R^2) D_t +$
$(-N x^2 N_R + N^2 x^2 N_R + 3 t x^2 N_R - 4 N t x^2 N_R + N^2 t x^2 N_R + 4 t^2 x^2 N_R - 2 N t^2 x^2 N_R + t^3 x^2 N_R + t x^3 N_R +$
$t^2 x^3 N_R - N x y N_R + N^2 x y N_R + 3 t x y N_R - 4 N t x y N_R + N^2 t x y N_R + 4 t^2 x y N_R - 2 N t^2 x y N_R +$
$t^3 x y N_R + 2 t x^2 y N_R + 2 t^2 x^2 y N_R + t x y^2 N_R + t^2 x y^2 N_R + t x y N_R^2 + t^2 x y N_R^2) D_x +$
$(N t x^3 + N t^2 x^3 + 2 N t x^2 y - t^2 x^2 y + 2 N t^2 x^2 y - t^3 x^2 y - t x^3 y - t^2 x^3 y + N t x y^2 - t^2 x y^2 +$
$N t^2 x y^2 - t^3 x y^2 - 3 t x^2 y^2 - 3 t^2 x^2 y^2 - 3 t x y^3 - 3 t^2 x y^3 - t y^4 - t^2 y^4 - t x^3 N_R -$
$t^2 x^3 N_R + 2 t x y N_R - N t x y N_R + 4 t^2 x y N_R - N t^2 x y N_R + t^3 x y N_R - 3 t x^2 y N_R - 3 t^2 x^2 y N_R -$
$4 t x y^2 N_R - 4 t^2 x y^2 N_R - 2 t y^3 N_R - 2 t^2 y^3 N_R - t x y N_R^2 - t^2 x y N_R^2 - t y^2 N_R^2 - t^2 y^2 N_R^2) D_y +$
$(N t x^3 + N t^2 x^3 + 2 N t x^2 y - t^2 x^2 y + 2 N t^2 x^2 y - t^3 x^2 y + N t x y^2 - t^2 x y^2 + N t^2 x y^2 - t^3 x y^2 + N x N_R -$
$N^2 x N_R - 3 t x N_R + 4 N t x N_R - N^2 t x N_R - 4 t^2 x N_R + 2 N t^2 x N_R - t^3 x N_R - N x^2 N_R + N^2 x^2 N_R + t x^2 N_R - 2 N t x^2 N_R +$
$N^2 t x^2 N_R - t^2 x^2 N_R - t^3 x^2 N_R + N y N_R - N^2 y N_R - 3 t y N_R + 4 N t y N_R - N^2 t y N_R - 4 t^2 y N_R + 2 N t^2 y N_R -$
$t^3 y N_R - N x y N_R + N^2 x y N_R + 3 t x y N_R - 3 N t x y N_R + N^2 t x y N_R + 3 t^2 x y N_R - N t^2 x y N_R - N x N_R^2 + N^2 x N_R^2 +$
$3 t x N_R^2 - 4 N t x N_R^2 + N^2 t x N_R^2 + 4 t^2 x N_R^2 - 2 N t^2 x N_R^2 + t^3 x N_R^2), (2 t^3 x N_R + 4 t^4 x N_R + 2 t^5 x N_R) D_t^3 +$
$(6 t^2 x N_R - 3 N t^2 x N_R + 18 t^3 x N_R - 6 N t^3 x N_R + 15 t^4 x N_R - 3 N t^4 x N_R + 3 t^5 x N_R) D_t^2 +$
$(6 t^2 x^2 N_R + 12 t^3 x^2 N_R + 6 t^4 x^2 N_R) D_t D_x +$
$(2 N x^2 y - N^2 x^2 y + 4 t x^2 y + 6 N t x^2 y - 2 N^2 t x^2 y + 6 t^2 x^2 y + 6 N t^2 x^2 y - N^2 t^2 x^2 y + 2 t^3 x^2 y + 2 N t^3 x^2 y -$
$t^4 x^2 y - 4 t x^3 y - 8 t^2 x^3 y - 4 t^3 x^3 y + 4 N x y^2 - 2 N^2 x y^2 + 8 t x y^2 + 12 N t x y^2 - 4 N^2 t x y^2 + 12 t^2 x y^2 +$
$12 N t^2 x y^2 - 2 N^2 t^2 x y^2 + 4 t^3 x y^2 + 4 N t^3 x y^2 - 2 t^4 x y^2 - 8 t x^2 y^2 - 16 t^2 x^2 y^2 - 8 t^3 x^2 y^2 + 2 N y^3 -$
$N^2 y^3 + 4 t y^3 + 6 N t y^3 - 2 N^2 t y^3 + 6 t^2 y^3 + 6 N t^2 y^3 - N^2 t^2 y^3 + 2 t^3 y^3 + 2 N t^3 y^3 - t^4 y^3 - 4 t x y^3 -$
$8 t^2 x y^3 - 4 t^3 x y^3 - 4 t x^2 y N_R - 8 t^2 x^2 y N_R - 4 t^3 x^2 y N_R - 4 t x y^2 N_R - 8 t^2 x y^2 N_R - 4 t^3 x y^2 N_R) D_y^2 +$
$(12 t^2 x N_R - 3 N t^2 x N_R + 20 t^3 x N_R - 3 N t^3 x N_R + 9 t^4 x N_R) D_t +$
$(-2 N x^2 N_R + N^2 x^2 N_R - 11 N t x^2 N_R + 2 N^2 t x^2 N_R + 10 t^2 x^2 N_R - 16 N t^2 x^2 N_R + N^2 t^2 x^2 N_R + 15 t^3 x^2 N_R -$
$7 N t^3 x^2 N_R + 6 t^4 x^2 N_R + 4 t x^3 N_R + 8 t^2 x^3 N_R + 4 t^3 x^3 N_R - 2 N x y N_R + N^2 x y N_R - 4 t x y N_R -$
$6 N t x y N_R + 2 N^2 t x y N_R - 6 t^2 x y N_R - 6 N t^2 x y N_R + N^2 t^2 x y N_R - 2 t^3 x y N_R - 2 N t^3 x y N_R +$
$t^4 x y N_R + 4 t x^2 y N_R + 8 t^2 x^2 y N_R + 4 t^3 x^2 y N_R + 4 t x^2 N_R^2 + 8 t^2 x^2 N_R^2 + 4 t^3 x^2 N_R^2) D_x +$
$(-2 N^2 x^2 + N^3 x^2 - 4 N t x^2 - 6 N^2 t x^2 + 2 N^3 t x^2 - 6 N t^2 x^2 - 6 N^2 t^2 x^2 + N^3 t^2 x^2 - 2 N t^3 x^2 - 2 N^2 t^3 x^2 +$
$N t^4 x^2 + 4 N t x^3 + 8 N t^2 x^3 + 4 N t^3 x^3 - 2 N^2 x y + N^3 x y - 2 N t x y - 7 N^2 t x y + 2 N^3 t x y + 4 t^2 x y -$
$8 N^2 t^2 x y + N^3 t^2 x y + 6 t^3 x y + 4 N t^3 x y - 3 N^2 t^3 x y + 2 t^4 x y + 3 N t^4 x y - t^5 x y + 2 N x^2 y - N^2 x^2 y +$
$4 t x^2 y + 10 N t x^2 y - 2 N^2 t x^2 y + 2 t^2 x^2 y + 14 N t^2 x^2 y - N^2 t^2 x^2 y - 6 t^3 x^2 y + 6 N t^3 x^2 y - 5 t^4 x^2 y -$
$4 t x^3 y - 8 t^2 x^3 y - 4 t^3 x^3 y + 4 N x y^2 - 2 N^2 x y^2 + 8 t x y^2 + 12 N t x y^2 - 4 N^2 t x y^2 + 12 t^2 x y^2 +$
$12 N t^2 x y^2 - 2 N^2 t^2 x y^2 + 4 t^3 x y^2 + 4 N t^3 x y^2 - 2 t^4 x y^2 - 8 t x^2 y^2 - 16 t^2 x^2 y^2 - 8 t^3 x^2 y^2 + 2 N y^3 -$
$N^2 y^3 + 4 t y^3 + 6 N t y^3 - 2 N^2 t y^3 + 6 t^2 y^3 + 6 N t^2 y^3 - N^2 t^2 y^3 + 2 t^3 y^3 + 2 N t^3 y^3 - t^4 y^3 - 4 t x y^3 -$
$8 t^2 x y^3 - 4 t^3 x y^3 + 2 N x^2 N_R - N^2 x^2 N_R + 4 t x^2 N_R + 10 N t x^2 N_R - 2 N^2 t x^2 N_R + 6 t^2 x^2 N_R + 14 N t^2 x^2 N_R -$
$N^2 t^2 x^2 N_R + 2 t^3 x^2 N_R + 6 N t^3 x^2 N_R - t^4 x^2 N_R - 4 t x^3 N_R - 8 t^2 x^3 N_R - 4 t^3 x^3 N_R + 4 N x y N_R - 2 N^2 x y N_R +$
$8 t x y N_R + 12 N t x y N_R - 4 N^2 t x y N_R + 12 t^2 x y N_R + 12 N t^2 x y N_R - 2 N^2 t^2 x y N_R + 4 t^3 x y N_R + 4 N t^3 x y N_R -$
$2 t^4 x y N_R - 12 t x^2 y N_R - 24 t^2 x^2 y N_R - 12 t^3 x^2 y N_R + 2 N y^2 N_R - N^2 y^2 N_R + 4 t y^2 N_R + 6 N t y^2 N_R -$
$2 N^2 t y^2 N_R + 6 t^2 y^2 N_R + 6 N t^2 y^2 N_R - N^2 t^2 y^2 N_R + 2 t^3 y^2 N_R + 2 N t^3 y^2 N_R - t^4 y^2 N_R - 8 t x y^2 N_R -$
$16 t^2 x y^2 N_R - 8 t^3 x y^2 N_R - 4 t x^2 N_R^2 - 8 t^2 x^2 N_R^2 - 4 t^3 x^2 N_R^2 - 4 t x y N_R^2 - 8 t^2 x y N_R^2 - 4 t^3 x y N_R^2) D_y +$
$(-2 N^2 x^2 + N^3 x^2 - 4 N t x^2 - 6 N^2 t x^2 + 2 N^3 t x^2 - 6 N t^2 x^2 - 6 N^2 t^2 x^2 + N^3 t^2 x^2 - 2 N t^3 x^2 - 2 N^2 t^3 x^2 +$
$N t^4 x^2 + 4 N t x^3 + 8 N t^2 x^3 + 4 N t^3 x^3 - 2 N^2 x y + N^3 x y - 2 N t x y - 7 N^2 t x y + 2 N^3 t x y + 4 t^2 x y -$
$8 N^2 t^2 x y + N^3 t^2 x y + 6 t^3 x y + 4 N t^3 x y - 3 N^2 t^3 x y + 2 t^4 x y + 3 N t^4 x y - t^5 x y + 4 N t x^2 y - 4 t^2 x^2 y +$
$8 N t^2 x^2 y - 8 t^3 x^2 y + 4 N t^3 x^2 y - 4 t^4 x^2 y + 2 N x N_R - 3 N^2 x N_R + N^3 x N_R + 7 N t x N_R - 9 N^2 t x N_R +$
$2 N^3 t x N_R + 2 t^2 x N_R + 12 N t^2 x N_R - 9 N^2 t^2 x N_R + N^3 t^2 x N_R + t^3 x N_R + 9 N t^3 x N_R - 3 N^2 t^3 x N_R - 2 t^4 x N_R +$
$3 N t^4 x N_R - t^5 x N_R + 4 N t x^2 N_R - 4 t^2 x^2 N_R + 8 N t^2 x^2 N_R - 8 t^3 x^2 N_R + 4 N t^3 x^2 N_R - 4 t^4 x^2 N_R)\}$

HolonomicRank[annplog]

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```
CreativeTelescoping[annplog, Der[t]]
```

$Aborted

```
Timing[{annc, cert} = FindCreativeTelescoping[annplog, Der[t]];]
```

{9.841000, Null}

```
ByteCount /@ {annc, cert}
```

{36816, 274584}

```
annc
```

$$\{x D_x D_y + y D_y^2 + x D_x + (y + N_R) D_y,\ (-x^2 y^2 - 2 x y^3 - y^4) D_y^4 + x^3 N_R D_x^3 + (-3 x^2 y + N x^2 y - 8 x y^2 + N x y^2 - x^2 y^2 - 6 y^3 - 2 x y^3 - y^4 - 2 x^2 y N_R - 6 x y^2 N_R - 3 y^3 N_R) D_y^3 + (2 x^3 N_R + x^2 y N_R) D_x^2 + (-x^2 + N x^2 - 4 x y + 2 N x y - 3 x^2 y + N x^2 y - 6 y^2 - 8 x y^2 + N x y^2 - 6 y^3 - 2 x^2 N_R + N x^2 N_R - 10 x y N_R + 2 N x y N_R - 9 y^2 N_R - 2 x y^2 N_R - y^3 N_R - x^2 N_R^2 - 6 x y N_R^2 - 3 y^2 N_R^2) D_y^2 + (N x^2 N_R + x^3 N_R - 2 x y N_R + 2 x^2 y N_R + x y^2 N_R - 2 x^2 N_R^2 - x y N_R^2) D_x + (-x^2 + N x^2 - 4 x y + 2 N x y - 6 y^2 + N x N_R - x^2 N_R - 2 y N_R - 6 x y N_R + N x y N_R + x^2 y N_R - 5 y^2 N_R + 2 x y^2 N_R + y^3 N_R - 2 x N_R^2 + N x N_R^2 + x^2 N_R^2 - 3 y N_R^2 - 2 x N_R^3 - y N_R^3) D_y,\ (-x^5 - 2 x^4 y - x^3 y^2) D_x^4 + (-2 x^4 - 2 x^5 - 4 x^3 y - 4 x^4 y - 2 x^2 y^2 - 2 x^3 y^2 - x^4 N_R - 2 x^3 y N_R) D_x^3 + (x^3 y^2 + 3 x y^3 + N x y^3 + 2 x^2 y^3 + 4 y^4 + N y^4 + x y^4 + x y^3 N_R) D_y^3 + (-4 x^4 - N x^4 - x^5 - 8 x^3 y - 2 N x^3 y - 2 x^4 y - 4 x^2 y^2 - N x^2 y^2 - x^3 y^2 - x^3 y N_R) D_x^2 + (x^3 y - N x^3 y + 5 x y^2 + N x y^2 + 2 x^2 y^2 - 2 N x^2 y^2 + x^3 y^2 + 8 y^3 + 2 N y^3 + 4 x y^3 + 2 x^2 y^3 + 4 y^4 + N y^4 + x y^4 + 2 x^3 y N_R + 7 x y^2 N_R + N x y^2 N_R + 4 x^2 y^2 N_R + 8 y^3 N_R + 2 N y^3 N_R + 3 x y^3 N_R + 2 x y^2 N_R^2) D_y^2 + (-x^4 - N x^4 - 2 x^3 y - 2 N x^3 y - x^2 y^2 - N x^2 y^2 + x^4 N_R + 2 x^2 y N_R + 2 x^3 y N_R + 4 x y^2 N_R + N x y^2 N_R + x^2 y^2 N_R + x^2 y N_R^2) D_x + (x^3 y - N x^3 y + 5 x y^2 + N x y^2 + 2 x^2 y^2 - 2 N x^2 y^2 + 8 y^3 + 2 N y^3 + x y^3 - N x y^3 - N x^3 N_R + 2 x y N_R - 2 N x^2 y N_R + x^3 y N_R + 4 y^2 N_R + N y^2 N_R + 4 x y^2 N_R - N x y^2 N_R + 2 x^2 y^2 N_R + 4 y^3 N_R + N y^3 N_R + x y^3 N_R + x^3 N_R^2 + 3 x y N_R^2 + 2 x^2 y N_R^2 + 4 y^2 N_R^2 + N y^2 N_R^2 + 2 x y^2 N_R^2 + x y N_R^3) D_y\}$$

```
UnderTheStaircase[annc]
```

$$\{1, D_y, D_x, D_y^2, D_x^2, D_y^3, D_x^3\}$$

Example 3

```
Annihilator[Integrate[(n - 1) * Gamma[n] / Gamma[(n + 1) / 2]
    (1 - ρ^2) ^ (n / 2) * (1 - x^2) ^ ((n - 3) / 2) * (1 - ρ * x) ^ (-n + 1 / 2) *
    Hypergeometric2F1[1 / 2, 1 / 2, n + 1 / 2, (1 + ρ * x) / 2], {x, -1, r}], Der[ρ]] // Factor
```

$$\{(-1+\rho)^2 (1+\rho)^2 (-1 + r\rho) (1 + r\rho) D_\rho^3 + (-1+\rho)\rho(1+\rho)(-4 + 2n - r^2 - 2 n r^2 + 5 r^2 \rho^2) D_\rho^2 + (2 - n - r^2 + n^2 r^2 - 2\rho^2 + 3 n \rho^2 - n^2 \rho^2 - 2 r^2 \rho^2 - 2 n r^2 \rho^2 + 3 r^2 \rho^4) D_\rho\}$$

```
annf = Annihilator[(n - 1) * Gamma[n] / Gamma[(n + 1) / 2]
    (1 - ρ^2) ^ (n / 2) * (1 - r^2) ^ ((n - 3) / 2) * (1 - ρ * r) ^ (-n + 1 / 2) *
    Hypergeometric2F1[1 / 2, 1 / 2, n + 1 / 2, (1 + ρ * r) / 2], {Der[ρ], Der[r]}]
```

$$\{(\rho - r^2\rho - \rho^3 + r^2\rho^3) D_\rho + (-r + r^3 + r\rho^2 - r^3\rho^2) D_r + (3 r^2 - n r^2 + n\rho^2 - 3 r^2 \rho^2),\ (-1 + 2 r^2 - r^4 + r^2\rho^2 - 2 r^4 \rho^2 + r^6 \rho^2) D_r^2 + (6 r - 2 n r - 6 r^3 + 2 n r^3 + r\rho^2 + 2 n r \rho^2 - 8 r^3 \rho^2 - 2 n r^3 \rho^2 + 7 r^5 \rho^2) D_r + (3 - n - 6 r^2 + 5 n r^2 - n^2 r^2 + n^2 \rho^2 - 6 r^2 \rho^2 - 4 n r^2 \rho^2 + 9 r^4 \rho^2)\}$$

```
DFiniteSubstitute[annf, {r → x^2}, Algebra → OreAlgebra[Der[ρ], Der[x]]]
```

$$\{(2\rho - 2x^4\rho - 2\rho^3 + 2x^4\rho^3) D_\rho + (-x + x^5 + x\rho^2 - x^5\rho^2) D_x + (6x^4 - 2 n x^4 + 2 n \rho^2 - 6 x^4 \rho^2),\ (-x + 2x^5 - x^9 + x^5\rho^2 - 2x^9\rho^2 + x^{13}\rho^2) D_x^2 + (1 + 10 x^4 - 4 n x^4 - 11 x^8 + 4 n x^8 + x^4\rho^2 + 4 n x^4 \rho^2 - 14 x^8 \rho^2 - 4 n x^8 \rho^2 + 13 x^{12} \rho^2) D_x + (12 x^3 - 4 n x^3 - 24 x^7 + 20 n x^7 - 4 n^2 x^7 + 4 n^2 x^3 \rho^2 - 24 x^7 \rho^2 - 16 n x^7 \rho^2 + 36 x^{11} \rho^2)\}$$

```
intg= (n-1) *Gamma [n] / Gamma [(n+1) / 2] (1-ρ^2) ^ (n/2) * (1-x^2) ^ ((n-3) / 2) *
      (1-ρ*x) ^ (-n+1/2) *Hypergeometric2F1 [1/2, 1/2, n+1/2, (1+ρ*x) / 2];
{{P}, {Q}} = CreativeTelescoping[intg, Der[x], Der[ρ]]
```

$$\left\{\{D_\rho\},\left\{\frac{-1+x^2+x^2\rho^2-x^4\rho^2}{(-1+n)\,x\,(-1+\rho)\,(1+\rho)}D_\rho+\frac{(n-2\,n\,x^2+n\,x^4)\,\rho}{(-1+n)\,x\,(-1+\rho)^2\,(1+\rho)^2}\right\}\right\}$$

```
test = ApplyOreOperator[Q, intg];
inh= Limit[test, x→0] - FullSimplify[test /. x→-1, n>1]
```

$$-\frac{1}{(1+2\,n)\,\sqrt{\pi}\,(-1+\rho)\,(1+\rho)}2^{-\frac{3}{2}+n}\,(1-\rho^2)^{n/2}\,\text{Gamma}\left[\frac{n}{2}\right]\left((-1+4\,n^2)\,\text{Hypergeometric2F1}\left[\frac{1}{2},n,\frac{1}{2}+n,-1\right]+\text{Hypergeometric2F1}\left[\frac{3}{2},n,\frac{3}{2}+n,-1\right]\right)$$

```
minOp = First[Annihilator[inh, Der[ρ]]] ** P
```

$$(-1+\rho^2)\,D_\rho^2+(2\,\rho-n\,\rho)\,D_\rho$$

```
cfs = {-(-1+r^2) ^5,
        (-17+3n) r (-1+r^2) ^4,
        -(-1+r^2) ^3 (-28+84 r^2+n (4-36 r^2) +n^2 (-1+3 r^2)),
        (n-2) r (-1+r^2) ^2 (-22 n r^2+n^2 (-3+r^2) +66 (-1+r^2)),
        (-1+r^2) (3 n^4 r^2-48 (-1+r^2) ^2+
              n^2 (-3+30 r^2-39 r^4) +n^3 (3+5 r^4) +2 n (9-66 r^2+41 r^4)),
        -n r (-30+20 r^2+n^4 r^2-6 r^4+n^3 (3+2 r^2) -3 n^2 (-1+r^4) +
              n (-3-26 r^2+9 r^4))} /. r→ρ;
op = Total [MapThread[NonCommutativeMultiply , {Reverse[cfs], Der[ρ] ^Range[6]}]];
OreReduce[op, {minOp }]
```

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