Computer Algebra Methods for Holonomic and ∂-finite Functions

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Introductory Examples (1)

Task: Find a closed form for the sum

$$s(n) = \sum_{k=0}^{n} \frac{(-1)^k}{2^k} \binom{n}{k} \binom{2k}{k}.$$

$\longrightarrow$ Use Zeilberger's algorithm, e.g., the implementation `fastZeil` (by P. Paule and M. Schorn)!

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$$s(n) = \begin{cases} \frac{(n-1)!!}{n!!} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

Introductory Examples (2)

Task: Find a closed form for the double sum

$$s(m,n) = \sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{i+j}\binom{i+j}{i}\binom{m}{i}\binom{n}{j}$$

$\longrightarrow$ Use `MultiSum` (by K. Wegschaider)!

Introductory Examples (2)

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$\longrightarrow$ Use `MultiSum` (by K. Wegschaider)!

Solution: `-(n SUM[-1 + m, -1 + n]) + (1 - m + n)`
`SUM[-1 + m, n] + m SUM[m, n] == 0`

Introductory Examples (2)

Task: Find a closed form for the double sum

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$$s(m,n) = \delta_{m,n}$$

Introductory Examples (3)

Task: Prove

$$\sum_{j=-\infty}^{\infty}(-1)^j q^{4j^2-3j}\begin{bmatrix}2n+1\\ n+j\end{bmatrix}_2=(q^{2n+2};q^2)_{n+1}\sum_{j=0}^{\infty}\frac{q^{2j^2+2j}}{(-q;q^2)_{j+1}}\begin{bmatrix}n\\ j\end{bmatrix}_2.$$

$\longrightarrow$ Use `qZeil` (by A. Riese), `qGeneratingFunctions` (by C.K.)!

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$$\sum_{j=-\infty}^{\infty} (-1)^j q^{4j^2-3j} \begin{bmatrix} 2n+1 \\ n+j \end{bmatrix}_2 = (q^{2n+2};q^2)_{n+1} \sum_{j=0}^{\infty} \frac{q^{2j^2+2j}}{(-q;q^2)_{j+1}} \begin{bmatrix} n \\ j \end{bmatrix}_2 .$$

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Solution strategy:

- Find recurrences for both sides of the identity
- In this case they are different
- Compute a recurrence for the sum of both
- Check initial values

Main Topic

Generalization to

- non-hypergeometric multivariate functions
- both discrete and continuous variables
- mixed difference-differential equations
- handling of "standard" and q-problems in the same framework

The main ingredients to achieve this are

- translation to pure algebra, i.e., to operator algebras (Ore algebras)
- noncommutative Gröbner bases

$\longrightarrow$ We follow D. Zeilberger's "Holonomic Systems Approach" (1991) with extensions and refinements by F. Chyzak (1998)

Ore Algebra: Examples

Example 1: $\mathbb{K}[x][D_x; 1, D_x]$ is the Weyl algebra A_1.

Example 2: $\mathbb{K}[n][S_n; S_n, 0]$ is the first shift algebra.

Example 3: $\mathbb{K}(n)[S_n; S_n, 0]$

Holonomic functions

Definition is complicated (at least for the multivariate case)...
maybe later.

Closure properties:

- sum
- product
- integration
- ...

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Elimination property:

Given an ideal J in A_n s.t. A_n/J is holonomic; then for any choice of $n+1$ among the $2n$ generators of A_n there exists a nonzero operator in J that contains only these. In other words, we can eliminate $n-1$ variables.

∂-finite functions

Definition: Let $\mathbb{O}$ be an Ore algebra over some field $\mathbb{A}$, e.g., $\mathbb{A} = \mathbb{K}(\mathbf{x})$. A left ideal J in $\mathbb{O}$ is called ∂-finite w.r.t. $\mathbb{O}$, if $\mathbb{O}/J$ is a finite-dimensional vector space over $\mathbb{A}$.
A function f is called ∂-finite w.r.t. $\mathbb{O}$ if it is annihilated by a ∂-finite ideal. Further we have $\mathbb{O}/J \cong \mathbb{O} \bullet f$ when J is the annihilator of f in $\mathbb{O}$.

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Examples:

- $\sin x$
- Legendre polynomials
- Fibonacci numbers

∂-finite functions

Closure properties:

- sum
- product
- application of Ore operators
- algebraic substitution
- subsequences

$\longrightarrow$ These closure properties can be executed effectively (using an extended version of the FGLM algorithm).

Examples for ∂-finite functions

The annihilator of a ∂-finite function is usually not too difficult to compute.

$\longrightarrow$ Use database and closure properties!

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Some functions that are not ∂-finite:

- $\tan x$
- $\dfrac{\ln x}{e^x + e^{-x} - 1}$
- $\ln \ln x$
- $\dfrac{x^2}{x^2 + \ln^2(2e^{-a} \cos x)}$
- $\sqrt{n}$ w.r.t. $\mathbb{Q}(n)[S_n; S_n, 0]$
- $x!$ w.r.t. $\mathbb{Q}(x)[D_x; 1, D_x]$

∂-finite vs. holonomic

Consider the function

$$f(k,n) = \frac{1}{k^2+n^2}.$$

$f(n,k)$ is ∂-finite w.r.t. $\mathbb{Q}(k,n)[S_k;S_k,0][S_n;S_n,0]$; the corresponding annihilating ideal is

$$J_1 = \langle (k^2+n^2+2n+1)S_n - (k^2+n^2), (k^2+2k+n^2+1)S_k - (k^2+n^2) \rangle.$$

$f(n,k)$ is also ∂-finite w.r.t. $\mathbb{Q}(k,n)[D_k;1,D_k][D_n;1,D_n]$; the corresponding annihilating ideal is

$$J_2 = \langle (k^2+n^2)D_n + 2n, (k^2+n^2)D_k + 2k \rangle.$$

Note: $f(k,n)$ regarded as a function in the continuous variables k and n is holonomic, but regarded as a sequence in the discrete variables k and n it is not holonomic!

Definite integration

Given: $\operatorname{Ann}_{\mathbb{O}} f$, the annihilator of a holonomic function $f(x, y)$ in the Ore algebra $\mathbb{O} = \mathbb{K}[x, y][D_x; 1; D_x][D_y; 1, D_y]$.

Find: The annihilator of $F(y) = \int_a^b f(x, y) \mathrm{d}x$

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Since f is holonomic, there exists $P(y, D_x, D_y) \in \operatorname{Ann}_{\mathbb{O}} f$ that does not contain x. Write

$$P(y, D_x, D_y) = Q(y, D_y) + D_x \cdot R(y, D_x, D_y)$$

Throwing the integral on $P \bullet f = 0$ gives

$$Q(y, D_y)F(y) + \Big[R(y, D_x, D_y)f(x, y)\Big]_a^b = 0$$

Definite integration with Takayama

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Find: The annihilator of $F(y) = \int_a^b f(x, y)\mathrm{d}x$

Find $P \in \operatorname{Ann}_{\mathbb{O}} f$ which can be written in the form

$$P(x, y, D_x, D_y) = Q(y, D_y) + D_x R(x, y, D_x, D_y)$$

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$$\begin{aligned} 0 &= \int_a^b P(x,y,D_x,D_y) f(x,y)\mathrm{d}x \\ &= \int_a^b Q(y,D_y) f(x,y)\mathrm{d}x + \int_a^b D_x R(x,y,D_x,D_y) f(x,y)\mathrm{d}x \end{aligned}$$

Hence $Q(y,D_y)F(y) = 0$ (in the case of "natural boundaries").

The operator Q can be computed with Takayama's algorithm (noncommutative Gröbner bases over modules). The theory of holonomy guarantees that such an operator exists.

Definite summation with Takayama

Given: $\operatorname{Ann}_{\mathbb{O}} f$, the annihilator of a holonomic sequence $f(k,n)$ in the Ore algebra $\mathbb{O} = \mathbb{K}[k,n][S_k; S_k, 0][S_n; S_n, 0]$.
Find: The annihilator of $F(n) = \sum_k f(k,n)$
Find $P \in \operatorname{Ann} f$ which can be written in the form

$$P(k,n,S_k,S_n) = Q(n,S_n) + \Delta_k R(k,n,S_k,S_n)$$

$$\begin{aligned} 0 &= \sum_k P(k,n,S_k,S_n) f(k,n) \\ &= \sum_k Q(n,S_n) f(k,n) + \sum_k \Delta_k R(k,n,S_k,S_n) f(k,n) \end{aligned}$$

Hence $Q(n,S_n)F(n) = 0$ (in the case of "natural boundaries").

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Example

Task: Compute the definite integral

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x)\, dx = 0$$

Solution: First verify that the integral has natural boundaries, i.e.,

$$\Big[P \bullet \Big(e^{-x^2} H_n(x)\Big)\Big]_{-\infty}^{\infty} = 0 \quad \forall P \in \mathbb{Q}(n,x)[S_n; S_n, 0][D_x; 1, D_x].$$

Then apply Takayama's algorithm!

Jacobi Polynomials (1)

The Jacobi polynomials are defined by

$$P_n^{(a,b)}(x) = \sum_{k=0}^{\infty} \frac{(a+1)_n(-n)_k(n+a+b+1)_k}{n!(a+1)_k k!} \left(\frac{1-x}{2}\right)^k$$

The summand is both hypergeometric and hyperexponential.

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The summand is both hypergeometric and hyperexponential.
Applying Takayama's algorithm gives an annihilator for $P_n^{(a,b)}(x)$:

$$\begin{aligned}
&\{(-2n^2 - 2an - 2bn - 4n - 2a - 2b - 2)S_n \\
&\quad +(ax^2 + bx^2 + 2nx^2 + 2x^2 - a - b - 2n - 2)D_x \\
&\quad +xa^2 + a^2 + na + 2bxa + 3nxa + 3xa + a - b^2 - b - bn \\
&\quad +b^2x + 2n^2x + 3bx + 3bnx + 4nx + 2x, \\
&\ (-a - b - n - 1)S_b + (x-1)D_x + (a + b + n + 1), \\
&\ (a + b + n + 1)S_a + (-x-1)D_x + (-a - b - n - 1), \\
&\ (1 - x^2)D_x^2 + (-xa - a + b - bx - 2x)D_x + (n^2 + an + bn + n)\}.
\end{aligned}$$

Jacobi polynomials (2)

Task: Prove (or even better: find!):

$$
\begin{aligned}
(2n+a+b)P_n^{(a,b-1)}(x) &= (n+a+b)P_n^{(a,b)}(x) \\
&\quad +(n+a)P_{n-1}^{(a,b)}(x), \\
(1-x)\frac{\mathrm{d}}{\mathrm{d}x}P_n^{(a,b)}(x) &= aP_n^{(a,b)}(x)-(n+a)P_n^{(a-1,b+1)}(x).
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Solution: Use Gröbner bases for elimination. We get:

$$(a+b+n+2)S_bS_n+(a+n+1)S_b-(a+b+2n+3)S_n,$$

$$(1-x)D_xS_a+(a+n+1)S_b-(a+1)S_a$$

Irresistible integral (Boros / Moll, 7.2.1)

Task: Compute a closed form for the definite integral

$$N_{0,4}(a,m) = \int_0^\infty \frac{\mathrm{d}x}{(x^4 + 2ax^2 + 1)^{m+1}}, \quad a \in \mathbb{C}, m \in \mathbb{N}$$

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Solution: Use e.g. Takayama's algorithm to obtain an annihilator for the integral:

$$\big\{(4m+4)S_m - 2aD_a - 4m - 3,\\ (4a^2-4)D_a^2 + (8ma + 12a)D_a + 4m + 3\big\}$$

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With Mathematica's `DSolve`) we get:

$$N_{0,4}(a,m) = -\frac{(1+i)(-i)^m 2^{-m-1}\left(a^2-1\right)^{-\frac{m}{2}-\frac{1}{4}}\sqrt{\pi}Q_m^{m+\frac{1}{2}}(a)}{\Gamma(m+1)}$$

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Solution of V. Moll: $N_{0,4} = \dfrac{\pi P_m^{\left(m+\frac{1}{2},-m-\frac{1}{2}\right)}(a)}{2^{m+\frac{3}{2}}(a+1)^{m+\frac{1}{2}}}$

Nicholson's integral

Task: Prove the following identity involving a Bessel function:

$$\int_0^\infty e^{-xt} I_a(t)\,\mathrm{d}t = \frac{\left(x - \sqrt{x^2-1}\right)^a}{\sqrt{x^2-1}}$$

where $\Re(x) > 1$.

Multiple integration / summation

Task: Prove

$$\int_{-\infty}^{\infty}\left(\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{H_l(x)H_m(x)H_n(x)r^l s^m t^n e^{-x^2}}{l!m!n!}\right)dx$$

$$= \quad \sqrt{\pi}e^{2(rs+rt+st)}.$$

Chyzak's algorithm

Given a function f that is ∂-finite w.r.t. an Ore algebra $\mathbb{O}$.
Any function in $\mathbb{O} \bullet f$ can be written in normal form

$$\left(\sum_{\boldsymbol{\alpha} \in V} \varphi_{\boldsymbol{\alpha}} \boldsymbol{\partial}^{\boldsymbol{\alpha}} \right) \bullet f.$$

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Task: Find an operator $Q \in \operatorname{Ann}_{\mathbb{O}} f$ with certain properties, e.g., such that $D_x \cdot Q - 1 = 0$ (indefinite integration).

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Algorithm:

- compute a Gröbner basis G for $\operatorname{Ann}_{\mathbb{O}} f$
- make an ansatz for Q with undetermined coefficients
- reduce the ansatz with G, i.e., compute the normal form
- all coefficients of the normal form must be zero
- solve the resulting system

Integrated Jacobi polynomials (1)

Define

$$\begin{aligned} p_n^a(x) &= \sum_{k=0}^{\infty} \frac{(a+1)_n(-n)_k(n+a+1)_k}{n!(a+1)_k k!} \left(\frac{1-x}{2}\right)^k, \\ \hat{p}_n^a(x) &= \int_{-1}^{x} p_{n-1}^a(y)\mathrm{d}y. \end{aligned}$$

Task: Express $\hat{p}_n^a(x)$ in terms of $p_{n-1}^a(x)$ and $p_n^{a-2}(x)$.

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Ansatz: $\hat{p}_{n+1}^{a+2}(x) = Q \bullet p_n^a(x)$ with $Q = \varphi_1(x)S_a^2 + \varphi_2(x)S_n$.

Integrated Jacobi polynomials (2)

Ansatz: $\hat{p}_{n+1}^{a+2}(x) = Q \bullet p_n^a(x)$ with $Q = \varphi_1(x)S_a^2 + \varphi_2(x)S_n$.

Solution:

- compute a Gröbner basis G for $\operatorname{Ann} p_n^a$
- $\frac{\mathrm{d}}{\mathrm{d}x}\hat{p}_{n+1}^{a+2} = p_n^{a+2}$ translates to $0 = D_x Q - S_a^2 =: Z$
- compute the normal form of Z by reducing it with G
- all coefficients of the normal form must be zero
- solve the system of coupled differential equations for rational solutions: use `OreSys` (by S. Gerhold) for uncoupling.

We find

$$(a+1)\hat{p}_{n+1}^{a+2}(x) = (1-x)p_n^{a+2}(x) + 2p_{n+1}^a(x).$$

Creative telescoping

Chyzak's ansatz can be extended in order to do definite summation and integration with creative telescoping!
$\longrightarrow$ Compare Zeilberger's extension of Gosper's algorithm!

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Example: Strang's integral

$$\int_{-1}^{1} \left(\frac{P_{2k+1}(x)}{x} \right)^2 dx = 2$$

Ansatz: $D_x \cdot Q + \sum_{i=0}^{d} \eta_i S_k^i$

Integral from Amdeberhan/Espinosa/Moll

We compute an annihilator for the integral

$$\int_0^x \frac{\ln t \, \mathrm{d}t}{(1+t^2)^{(n+1)}}.$$

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Let's prove the special case for $x = 1$:

$$\int_0^1 \frac{\ln t \, \mathrm{d}t}{(1+t^2)^{(n+1)}} = -2^{-2n}\binom{2n}{n}\left(G + \sum_{k=0}^{n-1} \frac{\frac{\pi}{4} + p_k(1)}{2k+1}\right)$$

where $p_k(1) = \displaystyle\sum_{j=1}^{k} \frac{2^j}{2j\binom{2j}{j}}$.

Thanks for your attention!